Chaos and Fractals

Math 320 - Dr. Monks

Contents

Part I

Fun Facts: Chaos and Fractals

This is **not** a complete set of lecture notes for Math 320, Chaos and Fractals. Additional material will be covered in class.

1 Logic

In this section we give an informal overview of logic and proofs. For a more formal introduction see my lecture notes for Math 299.

1.1 Variables, Expressions, and Statements in Mathematics

We begin with a review of some basic terminology we will use in this course.

Basic Terminology	
Term	Description
set	A *set* is a collection of items.
element	The items in a set are called its *elements* (or members).
expression	An *expression* is an arrangement of symbols which represents an element of a set
type	The set of elements that an expression can represent is called the *type* of the expression.
value	The element of the domain that the expression represents is called a *value* of that expression.
variable	A *variable* is an expression consisting of a single symbol
constant	A *constant* is an expression whose domain contains a single element.
statement	A *statement* (or *Boolean expression*) is an expression whose domain is $\{\text{true}, \text{false}\}$.
truth value	The value of a statement is called its *truth value*.
solve	To *solve* a statement is to determine the set of all elements for which the statement is true.
solution set	The set of all solutions of a statement is called the *solution set*.
equation	An *equation* is a statement of the form $A = B$ where A and B are expressions.
inequality	An *inequality* is a statement of the form $A \star B$ where A and B are expressions and $\star$ is one of $\leq$, $\geq$, $>$, $<$, or $\neq$.

Remarks:

- An element is either in a set or it is not in a set, it cannot be in a set more than once.
- It is not necessary that we know specifically which element of the domain an expression represents, only that it represents some unspecified element in that set.
- We do not have to know if a statement is true or false, just that it is either true or false.
- If a statement contains n variables, $x_1, \ldots x_n$, then to solve the statement is to find the set of all n-tuples $(a_1, \ldots, a_n)$ such that each a_i is an element of the domain of x_i and the statement becomes true when $x_1, \ldots, x_n$ are replaced by $a_1, \ldots, a_n$ respectively. In this situation, each such n-tuple is called a *solution* of the statement.
- In formal mathematics *true* means *provable*.

1.2 Propositional Logic

The Five Logical Operators

Definition 1.1. Let P, Q be statements. Then the expressions

1. $\neg P$
2. P and Q
3. P or Q
4. $P \Rightarrow Q$
5. $P \Leftrightarrow Q$

are also statements whose truth values are completely determined by the truth values of P and Q as shown in the following table

P	Q	$\neg P$	P and Q	P or Q	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Note: In compound statements we usually put parentheses around the statements φ or ψ involved. For instance if φ is the statement 'P or Q' and ψ is the statement 'R and S' then $\varphi \Rightarrow \psi$ should be written

$$(P \text{ or } Q) \Rightarrow (R \text{ and } S)$$

in order to avoid the confusion that 'P or $Q \Rightarrow R$ and S' might actually mean something like P or $(Q \Rightarrow (R \text{ and } S))$. In order to cut down on parentheses, we assign a **precedence** order for our operators, meaning we apply the operators in the following order (from highest to lowest).

Precedence of Notation
parentheses, brackets, $(), \{\}, []$ etc.
arithmetic operations* $\wedge, \cdot, +, \ldots$ etc.
set operations $\times, -, \cap, \cup, \ldots$ etc.
arithmetic and set relations $=, \subseteq, \leq, \neq, \ldots$ etc.
not
and , or
$\Rightarrow$
$\Leftrightarrow$
$\forall, \exists, \exists!$

* with the usual precedence among them

1.3 Rules of Inference and Proof

Definition 1.2. A **rule of inference** is a rule which takes zero or more statements (or other items) as input and returns one or more statements as output.

Notation 1.3. An expression of the form

Rule Name Here	
P_1	(show)
$\vdots$	
P_k	(show)
······························	
Q_1	(conclude)
$\vdots$	
Q_n	(conclude)

represents a rule of inference whose inputs are $P_1 \ldots P_k$ and outputs are $Q_1, \ldots, Q_n$.

Definition 1.4. A **formal logic system** consists of a set of statements and a set of rules of inference.

Definition 1.5. A **proof** in a formal logic system consists of a finite sequence of statements (and other inputs to the rules of inference) such that each statement follows from the previous statements in the sequence by one or more of the rules of inference.

1.4 Natural Deduction

The rules of inference we will use for propositional logic are as follows.

Propositional Logic

and +	
φ	(show)
ψ	(show)
······························	
φ and ψ	(conclude)

and −	
φ and ψ	(show)
······························	
φ	(conclude)
ψ	(conclude)

$\Rightarrow +$	
Assume φ	
ψ	(show)
$\leftarrow$	
······························	
$\varphi \Rightarrow \psi$	(conclude)

$\Rightarrow -$ (modus ponens)	
φ	(show)
$\varphi \Rightarrow \psi$	(show)
······························	
ψ	(conclude)

Propositional Logic (cont.)

$\Leftrightarrow +$

$\varphi \Rightarrow \psi$	(show)
$\psi \Rightarrow \varphi$	(show)
···	
$\varphi \Leftrightarrow \psi$	(conclude)

$\Leftrightarrow -$

$\varphi \Leftrightarrow \psi$	(show)
···	
$\varphi \Rightarrow \psi$	(conclude)
$\psi \Rightarrow \varphi$	(conclude)

or +

φ	(show)
···	
φ or ψ	(conclude)
ψ or φ	(conclude)

or − (proof by cases

φ or ψ	(show)
$\varphi \Rightarrow \rho$	(show)
$\psi \Rightarrow \rho$	(show)
···	
ρ	(conclude)

not + (proof by contradiction)

Assume φ	
$\rightarrow\leftarrow$	(show)
$\leftarrow$	
···	
$\neg\varphi$	(conclude)

not – (proof by contradiction)

Assume $\neg\varphi$	
$\rightarrow\leftarrow$	(show)
$\leftarrow$	
···	
φ	(conclude)

$\rightarrow\leftarrow +$

φ	(show)
$\neg\varphi$	(show)
···	
$\rightarrow\leftarrow$	(conclude)

copy

φ	(show)
···	
φ	(conclude)

Remarks:

- The symbol $\leftarrow$ is an abbreviation for "end assumption".
- The symbol $\rightarrow\leftarrow$ is called "contradiction" and represents the logical constant false.
- The word Assume is actually entered as part of the proof itself, it is not just an instruction in the recipe like '(show)' and '(conclude)'.
- The inputs Assume and "$\leftarrow$" are not themselves statements that you prove or are given, but rather are inputs to rules of inference that may be inserted into a proof at any time. There is no useful reason however, to insert such statements unless you intend to use one of the rules of inference that requires them as an input.
- The statement following an Assume is the same as any other statement in the proof and can be used as an input to a rule of inference.
- Statements in an Assume-$\leftarrow$ block can be used as inputs to rules of inference whose conclusion is also inside the same block only. Once a Assume is closed with a matching $\leftarrow$, only the entire block can be used as an input to a rule of inference. The individual statements within

a block are no longer valid outside the block. We usually indent and Assume-$\leftarrow$ block to keep track of what statements are valid under which assumptions.

Definition. A compound statement of propositional logic is called a *tautology* if it is true regardless of the truth values the atomic statements that comprise it. (Its "truth table" contains only T's.)

It can be shown that a statement can be proved with Propositional Logic if and only if the statement is a tautology.

Example 1.6. Prove $P \Rightarrow (P \text{ or } Q)$ and verify it with a truth table

Example 1.7. Prove $(P \text{ or } Q) \Rightarrow \neg(\neg P \text{ and } \neg Q)$ and verify it with a truth table

1.5 Predicate Logic

Quantifiers

Definition 1.8. The symbols $\forall$ and $\exists$ are **quantifiers**. The symbol $\forall$ is called "for all", "for every", or "for each". The symbol $\exists$ is called "for some" or "there exists".

Definition 1.9. If W is a statement and x is any variable then $\forall x, W$ and $\exists x, W$ are both statements.

Notation 1.10. If x is a variable, t an expression, and $W(x)$ a statement then $W(t)$ is the statement obtained by replacing every free occurrence of x in $W(x)$ with (t),

The rules of inference for these two quantifiers are as follows.

Predicate Logic*

$\forall+$	$\forall-$
Let s BE ARBITRARY (variable declaration) $\varphi(s)$ (SHOW) $\leftarrow$ $\forall x, \varphi(x)$ (CONCLUDE)	$\forall x, \varphi(x)$ (SHOW) $\varphi(t)$ (CONCLUDE)
$\exists+$	**$\exists-$**
$\varphi(t)$ (SHOW) $\exists x, \varphi(x)$ (CONCLUDE)	$\exists x, \varphi(x)$ (SHOW) For some c, (constant declaration) $\varphi(c)$ (CONCLUDE)

**Restrictions and Remarks* - there are restrictions on the rules of inference for quantifiers which are not listed above (see my Math 299 lecture notes for details). In most situations they are not a concern.

Example 1.11. Prove $(\exists x, P(x)) \Rightarrow \neg\forall y, \neg P(y)$

1.6 Equality

Definition 1.12. The equality symbol, =, is defined by the following two rules of inference.

Equality

Reflexivity	Substitution*
....	$x = y$ (SHOW)
$x = x$	φ (SHOW)
	....
	φ with any free occurrences of x replaced by y. (CONCLUDE)

Remark. Note that in the Reflexive rule there are no inputs, so you can insert a statement of the form $x = x$ into your proof at any time.

Example 1.13. Given $x = y$ and $y = z$, prove $x = z$.

Problems - Logic

1.1. Math Zoology 101: (1 point each) Let x, y be real numbers, f, g functions from the set of real numbers to the set of real numbers, and A, B sets. Classify each of the following expressions as either a number, statement, function, or set (assuming the expression is defined).

(a) $x^2 > 0$

(b) $x^2 + y^2$

(c) $A \cup B$

(d) $x \in A$

(e) $\{x\}$

(f) $f \circ g$

(g) 3

(h) $f(x) = 2x$

(i) g' (the derivative of g)

(j) $3 < 2$

(k) $A \subseteq B$

(l) $f(y)$

(m) $\sqrt{\frac{x+y}{2}}$

1.2. (1 point each) Let P, Q be statements. Use a truth table to show that each of the following is a tautology.

(a) $P \Rightarrow P$

(b) $(P \text{ and } \neg P) \Rightarrow Q$

(c) $P \Rightarrow (Q \Rightarrow P)$

(d) $((P \text{ or } Q) \text{ and } \neg Q) \Rightarrow P$

(e) $P \text{ or } \neg P$

(f) $\neg(\neg P) \Leftrightarrow P$

1.3. (2 points each) Use the rules of natural deduction to prove each of the tautologies in exercise 1.2.

1.4. (2 points each) Let $P(x)$ be a statement containing x and $Q(x, y)$ a statement containing x, y. Use the rules of natural deduction to prove the following.

(a) $(\exists x, P(x)) \Rightarrow (\exists y, P(y))$

(b) $(\exists y, \forall x, Q(x, y)) \Rightarrow (\forall x, \exists y, Q(x, y))$

(c) $\neg(\forall x, P(x)) \Rightarrow \exists x, \neg P(x)$

(d) $(\forall x, P(x) \Rightarrow Q(x))$ and $(\forall x, \neg Q(x)) \Rightarrow (\forall x, \neg P(x))$

(e) $x = y \Leftrightarrow y = x$

1.5. (1 point each) Give examples of particular statements $P(x)$ and $Q(x, y)$, and then translate each of the statements in exercise 1.4 into English statements using your examples.

2 Sets, Functions, Numbers

2.1 Some Definitions from Set theory

The symbol $\in$ is formally undefined, but it means "is an element of". Many of the definitions below are informal definitions that are sufficient for our purposes.

Set notation and operations

Set notation and operations	
Term	Definition
Finite set notation:	$x \in \{x_1, \ldots, x_n\} \Leftrightarrow x = x_1$ or $\cdots$ or $x = x_n$
Set builder notation:	$x \in \{\, y : P(y) \,\} \Leftrightarrow P(x)$
Cardinality:	$\#S$ = the number of elements in a finite set S
Subset:	$A \subseteq B \Leftrightarrow \forall x, x \in A \Rightarrow x \in B$
Set equality:	$A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$
Def. of $\notin$:	$x \notin A \Leftrightarrow \neg(x \in A)$
Empty set:	$\exists \emptyset, \forall x, x \notin \emptyset$
Relative Complement:	$x \in B - A \Leftrightarrow x \in B$ and $x \notin A$
Intersection:	$x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$
Union:	$x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$
Indexed Intersection:	$x \in \bigcap_{i \in I} A_i \Leftrightarrow \forall i, i \in I \Rightarrow x \in A_i$
Indexed Union:	$x \in \bigcup_{i \in I} A_i \Leftrightarrow \exists i, i \in I$ and $x \in A_i$
Two convenient abbreviations:	$(\forall x \in A, P(x)) \Leftrightarrow \forall x, x \in A \Rightarrow P(x)$ $(\exists x \in A, P(x)) \Leftrightarrow \exists x, x \in A$ and $P(x)$

Some Famous Sets	
Set	Definition
The Natural Numbers	$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$
The Integers	$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Some Famous Sets (cont.)

Set	Definition
The Rational Numbers	$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}, b \neq 0, \text{ and } \gcd(a,b) = 1 \right\}$
The Real Numbers	$\mathbb{R} = \{x : x \text{ can be expressed as a decimal number}\}$
The Complex Numbers	$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}$ where $i^2 = -1$
The positive real numbers	$\mathbb{R}^+ = \{x : x \in \mathbb{R} \text{ and } x > 0\}$
The negative real numbers	$\mathbb{R}^- = \{x : x \in \mathbb{R} \text{ and } x < 0\}$
The positive reals in a set A	$A^+ = A \cap \mathbb{R}^+$
The negative reals in a set A	$A^- = A \cap \mathbb{R}^-$
The first n positive integers	$\mathbb{I}_n = \{1, 2, \ldots, n\}$
The first $n+1$ natural numbers	$\mathbb{O}_n = \{0, 1, 2, \ldots, n\}$

Cartesian Products

Name	Definition
Ordered Pairs:	$(x, y) = (u, v) \Leftrightarrow x = u \text{ and } y = v$
Ordered n-tuple:	$(x_1, \ldots, x_n) = (y_1, \ldots, y_n) \Leftrightarrow x_1 = y_1 \text{ and } \cdots \text{ and } x_n = y_n$
Cartesian Product:	$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$
Cartesian Product:	$A_1 \times \cdots \times A_n = \{(x_1, \ldots, x_n) : x_1 \in A_1 \text{ and } \cdots \text{ and } x_n \in A_n\}$
Power of a Set	$A^n = A \times A \times \cdots \times A$ where there are n " A's" in the Cartesian product

Functions and Relations

Name	Definition
Def of relation:	R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$
Def of function:	$f\colon A \to B \Leftrightarrow f \subseteq A \times B$ and $f \neq \emptyset$ and $(\forall x, \exists y, (x, y) \in f)$ and $(\forall x, ((x, y) \in f \text{ and } (x, z) \in f) \Rightarrow y = z)$
Alt function notation	$X \xrightarrow{Y} \Leftrightarrow f\colon X \to Y$
Def of $f(x)$:	$f(x) = y \Leftrightarrow f\colon A \to B \text{ and } (x, y) \in f$
Domain:	$\text{Domain}(f) = A \Leftrightarrow f\colon A \to B$
Codomain:	$\text{Codomain}(f) = B \Leftrightarrow f\colon A \to B$
Image:	$f(S) = \{y : \exists x, x \in S \text{ and } y = f(x)\}$
Range:	$\text{Range}(f) = f(\text{Domain}(f))$
Identity Map:	$\text{id}_A\colon A \to A \text{ and } \forall x, \text{id}_A(x) = x$
Composition:	$f\colon A \to B$ and $g\colon B \to C \Rightarrow (g \circ f)\colon A \to C$ and $\forall x, (g \circ f)(x) = g(f(x))$

Functions and Relations (cont.)

Name	Definition
Injective (one-to-one):	f is injective $\Leftrightarrow\ \forall x, \forall y, f(x) = f(y) \Rightarrow x = y$
Surjective (onto):	f is surjective $\Leftrightarrow f\colon A \to B$ and $(\forall y, y \in B \to \exists x, y = f(x))$
Bijective:	f is bijective $\Leftrightarrow f$ is injective and f is surjective
Inverse:	$f^{-1}\colon B \to A \Leftrightarrow f\colon A \to B$ and $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$
Inverse Image:	$f\colon A \to B$ and $S \subseteq B \Rightarrow f^{\mathrm{inv}}(S) = \{x \in A : f(x) \in S\}$

Example 2.1. Prove that if $A \subseteq B$ then $A \cap B = A$.

Example 2.2. (left cancellation for injective functions) Let X, Y, Z be sets and $f\colon Y \to Z$. Show that if f is injective then for any functions $g, h\colon X \to Y$

$$(f \circ g = f \circ h) \Rightarrow g = h$$

Problems - Sets

In the following problems, let A, B, C, W, X, Y, Z be sets.

2.1. (2 points) Prove that $A \cap B \subseteq A \cup B$.

2.2. (2 points) Prove that if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

2.3. (2 points) Let $A \xrightarrow{B}$. Prove that $f^{\mathrm{inv}}(B) = A$.

2.4. (3 points) Let $A \xrightarrow{f} B$. Prove $f(A) = B$ if and only if f is surjective.

2.5. (3 points) Prove id_A is bijective.

2.6. (2 points) Let $f\colon X \to X$. Prove that $\mathrm{id}_X \circ f = f \circ \mathrm{id}_X = f$.

2.7. (2 points) Prove that composition of functions is associative, i.e., if $f\colon Z \to W$, $g\colon Y \to Z$, and $h\colon X \to Y$ then $f \circ (g \circ h) = (f \circ g) \circ h$.

2.8. (3 points) Let $A \xrightarrow{f} A \times A$ by $f(x) = (x, x)$ for all $x \in A$. Prove f is injective.

2.9. (2 points) Let t be the infinite sequence

$$1, 2, 4, 7, 11, 16, 22, 29, 37, 46, 56, 67, 79, 92, 106, 121, \ldots$$

whose sequence of consecutive differences is arithmetic. What is the 1000000^{th} term of t? What is t_{t_4} (if the first term is t_1)?

2.10. (2 points) (*left cancellation law for injective functions*) Let $Y \xrightarrow{f} Z$. Prove that f is injective if and only if for all nonempty sets X and all functions $g, h\colon X \to Y$

$$(f \circ g = f \circ h) \Rightarrow g = h$$

2.11. (4 points) (*right cancellation law for surjective functions*) Let $X \xrightarrow{f} Y$. Prove that f is surjective if and only if for all nonempty sets Z and all functions $g, h \colon Y \to Z$

$$(g \circ f = h \circ f) \Rightarrow g = h$$

2.12. Math Zoology: (1 point each) Let $x \in \mathbb{R}$, $f \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, and $A, B \subseteq \mathbb{R}$. Classify each of the following expressions as either a number, statement, function, ordered pair, or set (assuming the expression is defined).

(a) $f(A \times A)$

(b) $Range(f)$

(c) f^{-1}

(d) $f^{-1}(x, x)$

(e) $f(\{(x, x)\})$

(f) $f(x, x)$

(g) id_A

(h) $\mathrm{Domain}(f) = \mathbb{R} \times \mathbb{R}$

(i) $A \times B$

(j) $f(\mathbb{I}_4 \times \mathbb{I}_4)$

2.2 Sequences

Definition 2.3. A **finite sequence** is a function $t \colon \mathbb{I}_n \to A$ where n is a natural number and A is a set. An **infinite sequence** is a function $t \colon \mathbb{N}^+ \to A$ where A is a set. In either case, $t(k)$ is called the k^{th} term of the sequence.

Remark. It is often convenient to say that t is a finite (resp infinite) sequence if $t \colon \mathbb{O}_n \to A$ (resp. $t \colon \mathbb{N} \to A$). In this case we say that $t(k)$ is the $k + 1^{\text{st}}$ term of the sequence.

Notation 2.4. If $t \colon \mathbb{I}_n \to A$ is a finite sequence we write

$$t_1, t_2, t_3, \ldots, t_n$$

as another notation for t, where $t_k = t(k)$ for all $k \in \mathbb{I}_n$. Similarly if $t \colon \mathbb{N}^+ \to A$ we write

$$t_1, t_2, t_3, \ldots$$

for t where $t_k = t(k)$ for all $k \in \mathbb{N}^+$.

Remark. Sometimes for readability we might want to enclose a sequence in parenthesis. For example, we might write " Let $t = (1, 2, 3, 4)$" instead of " Let $t = 1, 2, 3, 4$" . In this sense there is really no distinction between n-tuples and finite sequences.

Notation 2.5. We use an overbar to indicate an infinite repeating sequence, i.e.,

$$t_0, t_1, \ldots, t_{k-1}, \overline{t_k, \ldots, t_{k+n-1}}$$

denotes the sequence infinite sequence t such that $t_i = t_{k+((i-k) \bmod n)}$ for all $i \geq k + n$.

Example 2.6. Write the first five terms of the sequence $\mathbb{N} \xrightarrow{a} \mathbb{N}$ by $a_n = n^2 + 1$.

Example 2.7. What is the 1000^{th} term in the sequence

$$9, 0, 8, \overline{3, 2, 4, 1, 5, 7, 6}$$

Example 2.8. Write the first five terms of the sequence $\mathbb{N} \xrightarrow{a} \mathbb{N}$ given by

$$a(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot a(n-1) & \text{otherwise} \end{cases}$$

2.3 Some Facts from Number Theory

Theorem 2.9 (Math Induction). *Let $P(n)$ be any statement about a natural number variable n. Then*

$$(P(0) \text{ and } \forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1)) \Rightarrow \forall n \in \mathbb{N}, P(n)$$

Theorem 2.10 (Division Algorithm). *Let $a, b \in \mathbb{Z}$ and $b > 0$. Then there exist unique integers $q, r \in \mathbb{Z}$ such that*

$$a = qb + r \text{ and } 0 \leq r < b.$$

Remark. In this theorem the number q is called the **quotient** and r is called the **remainder** when a is divided by b.

Definition 2.11. Let $a, b \in \mathbb{Z}$ with $b > 0$. Then $a \bmod b$ is the remainder when a is divided by b. The quotient can be written as $\left\lfloor \frac{a}{b} \right\rfloor$ where $\lfloor x \rfloor$ is the greatest integer less than or equal to a real number x.

Definition 2.12. Let $a, b \in \mathbb{Z}$. We say a **divides** b if $ak = b$ for some integer k. If a divides b we write $a \mid b$.

Definition 2.13. Let $a, b \in \mathbb{Z}$ with $a \neq 0$ or $b \neq 0$. Then $\gcd(a, b)$ is the greatest positive integer which divides both a and b.

Example 2.14. What is the quotient and remainder when $2^{1000} + 1$ is divided by 32?

Example 2.15. What is the quotient and remainder when -100 is divided by 7?

Example 2.16. True or False:

(a) $14 \mid 7$ (b) $7 \mid 14$ (c) $7 \mid -14$ (d) $7 \mid 0$

Example 2.17. What is the $\gcd(72, 60)$? $\gcd(295927, 304679)$?

Problems - Functions and Sequences

2.13. (1 point) What is the quotient and remainder when -371 is divided by 17?

2.14. Consider the function $\mathbb{N} \times \mathbb{N} \xrightarrow{B} \mathbb{N}$ given by

$$B(m,n) = \begin{cases} 1 & \text{if } m = 0 \text{ or } n = 0 \\ B(m-1,n) + B(m,n-1) & \text{otherwise} \end{cases}$$

(a) (1 point) Make a 10×10 table showing the values of $B(m,n)$ for $0 \leq m,n$ with $m+n \leq 10$.

(b) (1 point) Do the numbers look familiar? Give a closed formula for $B(m,n)$ in terms of well-known mathematical functions.

2.15. Consider the function $\mathbb{N} \times \mathbb{N} \xrightarrow{A} \mathbb{N}$ given by

$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ A(m-1,1) & \text{if } n = 0 \text{ and } m > 0 \\ A(m-1, A(m,n-1)) & \text{otherwise} \end{cases}$$

(a) (1 point) Compute $A(2,2)$

(b) (2 points) Find an explicit formula for $A(1,n)$ ("explicit" means that the formula does not contain the symbol A).

(c) (3 points) Find an explicit formula for $A(2,n)$ in terms of n.

(d) (4 points) Find an explicit formula for $A(3,n)$.

(e) (4 points) Compute $A(4,2)$. Can you come up with an explicit formula for $A(4,n)$?

(f) (1 point) Write an essay discussing just how big $A(5,1)$ is!

3 Iteration

3.1 Discrete Dynamical Systems

Definition 3.1. Let X be any nonempty set. Any function $f\colon X \to X$ is called a **set theoretic discrete dynamical system** (or simply **discrete dynamical system**).

Definition 3.2. Let X be a set and $f\colon X \to X$. Define $f^0 = \mathrm{id}_X$ and for all $k \geq 1$ define

$$f^k = f \circ f^{k-1}$$

Example 3.3. Let $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x+1$. Find a nonrecursive formula for $f^k(x)$ for $k \geq 0$.

Theorem 3.4 (Power Theorem)**.** *Let* $f\colon X \to X$. *For any* $k, n \in \mathbb{N}$,

$$f^{k+n} = f^k \circ f^n$$

and

$$f^{kn} = \underbrace{f^n \circ f^n \circ \cdots \circ f^n}_{k \text{ terms}} = (f^n)^k$$

Definition 3.5. Let $f\colon X \to X$ and $x \in X$. The sequence

$$x, f(x), f^2(x), f^3(x), \ldots$$

is called the f-**orbit** of x. The first term, x, is called the **seed** of the orbit. The $k+1^{\text{st}}$ term is called the k^{th} f**-iterate** of x (or k^{th} **iterate** or k^{th} **iteration**). We write $\operatorname{Orb}_f(x)$ for the f-orbit of x.

Remark. $\operatorname{Orb}_f(x)\colon \mathbb{N} \to X$ and $\operatorname{Orb}_f(x)(n) = f^n(x)$ for all $n \in \mathbb{N}$.

Example 3.6. Find the complete f-orbit of 5 for $\mathbb{C} - \{0,1\} \xrightarrow{f} \mathbb{C} - \{0,1\}$ by $f(z) = \frac{1}{1-z}$. What is the f-orbit of 3? How about a?

Definition 3.7. Let X be a set, $x \in X$, and $f\colon X \to X$. Then the set of terms in the f-orbit of x is denoted $\mathcal{O}_f(x)$, i.e.,

$$\mathcal{O}_f(x) = \left\{ f^k(x) : k \in \mathbb{N} \right\}$$

We call $\mathcal{O}_f(x)$ **the set of** f**-iterates of** x (or simply **the set of terms in the** f**-orbit of** x).

Example 3.8. What is $\mathcal{O}_f(5)$ in Example 3.6? How many elements are in $\mathcal{O}_f(5)$?

3.2 Types of Orbits

Definition 3.9. Let $f\colon X \to X$ and $x \in X$. The f-orbit of x is **cyclic** if $f^n(x) = x$ for some $n \geq 1$. In this situation we say that x is a **cyclic point** (or **periodic point**) for f.

Example 3.10. Is $\operatorname{Orb}_f(5)$ cyclic in Example 3.6?

Definition 3.11. Let $f\colon X \to X$ and $x \in X$. If $f^n(x) = x$ for some $n \geq 1$, we say x has **period** n. If in addition $f^k(x) \neq x$ for all $1 \leq k < n$ then we say x has **minimum period** n. If x has period 1 we say x is a **fixed point** of f. If x has period n we also say that $\operatorname{Orb}_f(x)$ has **period** n and if x has minimum period n we also say $\operatorname{Orb}_f(x)$ has **minimum period** n as well.

Example 3.12. What is the minimum period of 5 in Example 3.6?

Example 3.13. Does f have any fixed points in Example 3.6?

Lemma 3.14. *Let* $f\colon X \to X$, $x \in X$, *and* $n \in \mathbb{N}^+$. *If* x *has minimum period* n *then* $\#\mathcal{O}_f(x) = n$.

Example 3.15. Why isn't it if and only if?

Definition 3.16. Let $f\colon X \to X$ and $x \in X$. The f-orbit of x is **eventually cyclic** if $f^n(x) = f^m(x)$ for some n, m with $n \neq m$. In this situation we also say that x is an **eventually cyclic point** (or **eventually periodic point**) for f.

Definition 3.17. Let $f\colon X \to X$ and $x \in X$ periodic point with period n. We say that $O_f(x)$ is an n**-cycle** if and only if $\mathrm{Orb}_f(x)$ is cyclic with minimum period n.

Definition 3.18. Let $f\colon X \to X$ and $x \in X$. The f-orbit of x is **acyclic** if it is not eventually cyclic.

Example 3.19. Can you come up with examples of each of these?

Problems - Orbits

3.1. (3 points) Let $A \xrightarrow{P} A$, $s_0 \in A$, and $s = (s_0, s_1, s_2, \ldots)$ the P-orbit of s_0. Show that if s is cyclic with period n then it is also cyclic with period kn for any positive integer k. [Hint: Use induction on k.]

3.2. (3 points) Let $A \xrightarrow{P} A$, $s_0 \in A$, and $s = (s_0, s_1, s_2, \ldots)$ the P-orbit of s_0. Show that if $s_n = s_m$ then $s_{n+k} = s_{m+k}$ for any positive integer k.

3.3. (4 points) Let $A \xrightarrow{P} A$, $s_0 \in A$, and $s = (s_0, s_1, s_2, \ldots)$ the P-orbit of s_0. Show that if s is both cyclic with period n and cyclic with period m, then s is cyclic with period $\gcd(n, m)$. [Hint: Use the fact that $\gcd(n, m) = sn + tm$ for some integers s, t.]

3.4. (3 points) *(Fun with composition!)* Let $A \xrightarrow{f} A$ and $A \xrightarrow{g} A$. Show that if $f \circ g \circ f = g$ and $g \circ f \circ f = f$ then $g = f$.

3.3 The Digraph

Definition 3.20. A **directed graph** (or **digraph**) is a pair (V, E) where V is a set of elements called the **nodes** and $E \subseteq V \times V$ is the set of **directed edges**.

Definition 3.21. Let $X \xrightarrow{f} X$ be a discrete dynamical system. The **digraph** of f is the directed graph (X, S) where $S = \{\,(a, f(a)) : a \in X\,\}$, i.e., the nodes are the elements of the domain and the directed edges connect each element a in the domain to $f(a)$.

Problems - Digraphs

3.5. (1 point each) Draw the directed graph of the following discrete dynamical systems.

(a) $\mathbb{O}_{11} \xrightarrow{f} \mathbb{O}_{11}$ by $f(n) = (n+3) \bmod 12$.

(b) $\mathbb{O}_{11} \xrightarrow{f} \mathbb{O}_{11}$ by $f(n) = (n+7) \bmod 12$.

(c) $\mathbb{O}_{11} \xrightarrow{f} \mathbb{O}_{11}$ by $f(n) = n^6 \bmod 12$.

(d) $\mathbb{O}_6 \xrightarrow{f} \mathbb{O}_6$ by $f(n) = n^6 \ \mathrm{Mod}\ 7$.

3.6. (1 point each) Give an example of, and draw the directed graph of a dynamical system $X \xrightarrow{f} X$ which has the following properties.

(a) f is neither injective nor surjective

(b) f is bijective

(c) f is injective and not surjective

(d) f is surjective and not injective

4 Examples of Iteration

4.1 The Collatz Conjecture

Definition 4.1. Define $T\colon \mathbb{Z} \to \mathbb{Z}$ by $\forall x \in \mathbb{Z}$

$$T(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{3x+1}{2} & \text{if } x \text{ is odd} \end{cases}$$

Conjecture 4.2 (Collatz)**.** For all $n \in \mathbb{N}^+$, $\exists k \geq 0, T^k(n) = 1$, i.e., the T-orbit of any positive integer contains one.

Remark. Note that $\mathrm{Orb}_T(1) = \overline{1, 2}$ so that the conjecture is equivalent to saying that the T-orbit of any positive integer is eventually periodic and enters the 2-cycle $\{1, 2\}$.

4.2 Sumerian Method for Computing Square Roots

Claim 4.3. *Let $a \in \mathbb{R}^+$ and $\mathrm{Root}_a(x) = \frac{1}{2}(x + \frac{a}{x})$. For any $x \in \mathbb{R}^+$, the Root_a-orbit of x converges to $\sqrt{a}$.*

Example 4.4. Find a fraction and a decimal that are a good approximation to $\sqrt{2}$ and $\sqrt{3}$ by the Sumerian Method.

4.3 Multiple Inputs: The Euclidean Algorithm

Remark. If $f\colon A \times B \to C$ and $a \in A$, $b \in B$, we usually abbreviate $f((a, b))$ as $f(a, b)$.

Claim 4.5. *Define $\mathrm{Euc}\colon \mathbb{N}^+ \times \mathbb{N} \to \mathbb{N}^+ \times \mathbb{N}$ by*

$$\mathrm{Euc}(n, m) = \begin{cases} (m, n) & \text{if } n < m \\ (n, 0) & \text{if } m = 0 \\ (m, n \bmod m) & \text{otherwise} \end{cases}.$$

for any $(n, m) \in \mathbb{N}^+ \times \mathbb{N}$. Then the Euc-orbit of any (n, m) is eventually fixed and contains the fixed point $(\gcd(n, m), 0)$.

Remark. This method of computing $gcd(n, m)$ is called the **Euclidean algorithm**.

Example 4.6. Reduce the fraction $\frac{295927}{304679}$ by hand.

4.4 Non-numeric Inputs: Post's Tag Problem

Definition 4.7. Let S be a set and let S^* be the set of words (finite sequences) which can be made from the alphabet S, i.e.,

$$S^* = \bigcup_{n=1}^{\infty} \left\{ f : \mathbb{I}_n \xrightarrow{f} S \right\}$$

If $x \in S^*$ then $\#x$ is the number of letters in the word x. If $x, y \in S^*$ then $x \cdot y$ is the word formed by concatenating the words x and y. If $x = x_1 x_2 \cdots x_n \in S^*$ then $x[a \ldots b]$ is the word $x_a x_{a+1} x_{a+2} \cdots x_{b-1} x_b$ and $x[a] = x_a$.

Example 4.8. What is $\{ \bigstar \}^*$? $\{a, b\}^*$?

Definition 4.9. Define Tag: $\{a, b\}^* \to \{a, b\}^*$ as follows. If $x \in \{a, b\}^*$ and $n = \#x$ then

$$\mathrm{Tag}(x) = \begin{cases} x & \text{if } n < 3 \\ x[4..n] \cdot aa & \text{if } x[1] = a \\ x[4..n] \cdot bbab & \text{if } x[1] = b \end{cases}$$

In other words, if a word is less than three letters long, Tag returns it unchanged, if it is 3 or more letters and begins with the letter a then Tag deletes the first three letters and appends aa on the right, and if it is 3 or more letters and begins with b then Tag deletes the first three letters and appends $bbab$.

Problem 4.10 (TAG)**.** (Emil Post 1921) Are there any Tag-orbits which are not eventually cyclic?

Example 4.11. What is the Tag-orbit of a? $baba$? $bbbaa$?

Problems - Iteration

4.1. Verify the Collatz conjecture for the first 500 positive integers by doing the following.

  (a) (2 points) Computing the T-orbit of n for $1 \leq n \leq 50$. Note that you should compute the complete orbit of each n, indicating any repeating parts with an overbar or other appropriate notation. You can do this by hand or by computer or calculator (but not the internet), its your choice.

  (b) (2 points) Define the *total stopping time* of n to be the number of iterations of T required for the orbit of n to reach 1 For example, the total stopping time of 1 is 0, the total stopping time of 2 is 1, and the total stopping time of 3 is 5. Using your results from the first part, compute the total stopping time of the integers from 1 to 50.

4.2. (3 points) The Collatz function T is defined for all integers, since odd and even are defined for any integer. Compute the T-orbit if n for all integers n satisfying $-50 \leq n \leq 0$. List all of the disjoint cycles you find and state their minimum period. You can do this by hand or by computer or calculator (but not the internet), its your choice.

4.3. In Post's tag problem, verify that each of the following seeds have eventually cyclic orbits by listing the orbits, and determine the number of iterations required before the orbit becomes cyclic. (1 point each except for part (d).)

(a) aabb

(b) aaaab

(c) baabaa

(d) *The "Whose is Longest" Contest*: Find a seed whose Tag-orbit is cyclic. You must hand in the complete orbit and must indicate the number of terms in the cycle itself. The student who has the longest minimum cycle length will receive a bonus of 3 points added to their homework grade. In the event of a tie no bonus points will be awarded to any student.

4.4. (3 points) Use the Sumerian method for computing square roots to compute the $\sqrt{7}$ accurate to five digits (counting the leading 2 as one of the five digits). Use 1 for the value of the seed. Give the approximations both as fractions and as decimals. How many iterations are required?

4.5. (3 points) Use the Euclidean algorithm to reduce the fraction

$$\frac{498672943}{520221547}$$

You must do this by hand, not by Maple and show your work. You can use a calculator to do the division and remainder computations.

4.6. (4 points) Define a Fibonacci-like sequence

$$F(a,b) = x_0, x_1, x_2, \ldots$$

as follows:

$$\begin{aligned} x_0 &= a, \\ x_1 &= b, \\ x_n &= x_{n-1} + x_{n-2} \text{ for } n > 1 \end{aligned}$$

Compute the first ten terms for the sequences, $F(1,1)$, $F(2,1)$, $F(-1,1)$, $F(-1,-3)$ and $F(\frac{1}{3},\frac{1}{2})$. Then describe an iterative process (i.e. a discrete dynamical system) that would compute this sequence.

4.5 Stick Figure Fractals

Definition 4.12. Let A, B be any distinct points in the plane. Then $\overline{AB}$ denotes the line segment with endpoints A and B (i.e., the set of all points in the plane which are on the line containing A and B and are either between A and B or are equal to A and B). The **directed segment from** A **to** B is a pair $(\overline{AB}, A)$, and is denoted $\overrightarrow{AB}$. In this case we say $\overline{AB}$ is the segment associated with $\overrightarrow{AB}$ (and can think of a directed segment as being a set of points in the plane in this sense).

Remark. We use $\overrightarrow{AB}$ to denote a directed segment, not a ray from Euclidean plane geometry.

Notation 4.13. If $A = (a_1, a_2), B = (b_1, b_2)$ then $\overrightarrow{AB}$ can be written dseg$([a_1, a_2], [b_1, b_2])$ and $\overline{AB}$ can be written seg$([a_1, a_2], [b_1, b_2])$.

Remark. A directed line segment can be thought of as a line segment with an arrow drawn on it in one of the two possible directions. Note that $\overline{AB} = \overline{BA}$ but $\overrightarrow{AB} \neq \overrightarrow{BA}$. A directed line segment can also be thought of as a set of points since the line segment associated with it is a set of points. So if we talk about a point being "on a directed segment" we mean that it is on the line segment associated with the directed segment and so on.

Definition 4.14. A **stick** is either a line segment or a directed line segment. A **stick figure** is a finite set of sticks. Let U_{sf} be the set of all stick figures.

Remark. Note that we can also consider a stick figure to be a set of points in the plane by considering the union of the points in the line segments and (line segments associated with) directed segments.

Definition 4.15. Let $s = \text{dseg}([a, b], [c, d])$. Define $T_s \colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T_s(x, y) = ((c - a)x + (b - d)y + a, (d - b)x + (c - a)y + b)$$

T_s is called the **affine map induced by** s.

Remark. We will show how to derive this map later in the course. Intuitively, it is the map that sends the directed segment from $(0, 0)$ to $(1, 0)$ to the directed segment $s = \text{dseg}([a, b], [c, d])$, and the directed segment from $(0, 0)$ to $(0, 1)$ to the directed segment obtained by rotating s by 90° counterclockwise about (a, b).

Example 4.16. Find the affine map induced by the directed segment from $(1, 1)$ to $(2, 2)$.

Lemma 4.17. *Let s be a directed segment and t a line segment. Then $T_s(t)$ is a line segment.*

Definition 4.18. Let s, t be directed segments with $t = \overrightarrow{AB}$. Then $T_s(t) = (T_s(\overline{AB}), T_s(A))$.

Definition 4.19. If G is a stick figure and s a directed segment then $T_s(G)$ is the stick figure $\bigcup_{x \in G} \{ T_s(x) \}$.

Definition 4.20. For each stick figure G define a dynamical system $\gamma_G \colon U_{sf} \to U_{sf}$ as follows. Let $S \in U_{sf}$ be a stick figure. For each $x \in S$, define

$$g(x) = \begin{cases} \{ x \} & \text{if } x \text{ is a line segment} \\ T_x(G) & \text{if } x \text{ is a directed segment} \end{cases}$$

Then $\gamma_G(S) = \bigcup_{x \in S} g(x)$. The dynamical system γ_G is called the **stick figure iterator associated with** G. The figure G is called the **generator** for the stick figure iterator.

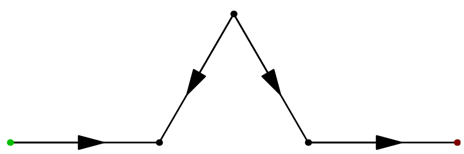
Claim 4.21. *In many cases the γ_G-orbit of a seed S converges to a fractal shape.*

Problems - Stick Figures

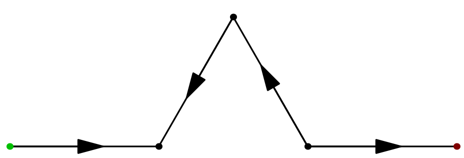
4.7. (1 point each) Below is the generator of a stick figure iterator that replaces directed line segments with the indicated collection of line segments. The seed is a single directed line segment directed from $(0,0)$ to $(1,0)$. The green dots are at $(0,0)$ and the red dots are at $(1,0)$ so the stick figure shown is the first iteration. Draw the next two iterations. Be accurate. You can assume all directed line segments in the figures are congruent, and angles are what they appear to be (integer multiples of 30 degrees).

You must draw these by hand, no computers allowed. Tip: Draw a large copy of the image below on a piece of paper, then put a second piece on top of the first as if you are going to trace the first and us it as a guide to draw the second. When finished you can then draw/trace over the second iteration in the same manner to produce the third.

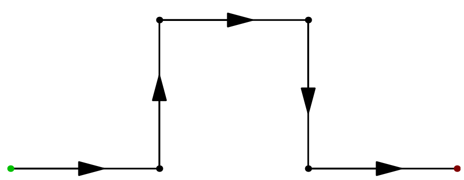
(a)



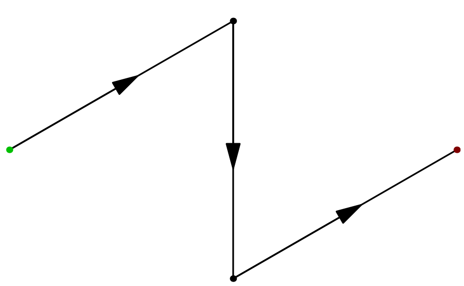
(b)



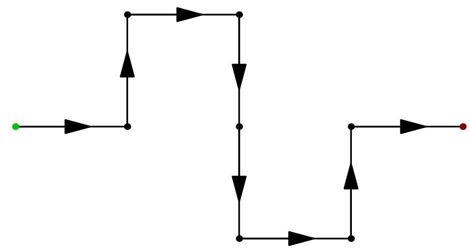
(c)



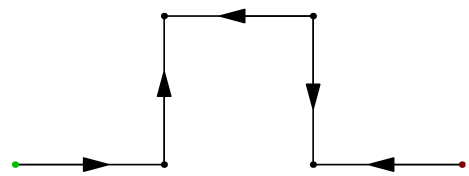
(d)



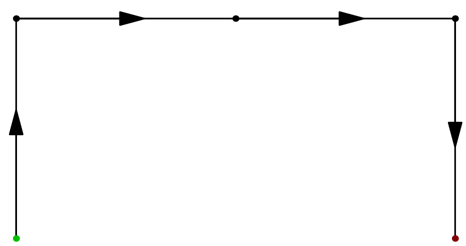
(e)



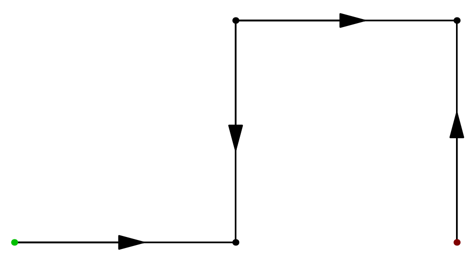
(f)



(g)



(h)



4.8. (1 point) Let n be a positive integer. How many directed sticks are in the n^{th} iteration of the stick figure fractal system defined in part (c) of 4.7 above?

4.9. Consider the directed segment $s = \text{dseg}([1,2],[3,4])$.

(a) (1 point) Find the affine transformation, T_s, induced by s.

(b) (1 point) Let $A = T_s(0,0)$, $B = T_s(1,0)$, and $C = T_s(0,1)$. Compute the coordinates of A, B, C and plot dseg(A,B) and dseg(A,C) (clearly label everything).

(c) (1 point) Now replace s with dseg$([3,4],[1,2])$ and answer the previous two questions again.

4.10. (2 points) **Stick Figure Art Contest**: Make up your own stick figure generator and use dseg$([0,0],[1,0])$ as you seed. Draw a minimum of two iterations of your stick figure fractal (but you can draw more iterations for a higher quality picture). The most interesting result will be awarded an extra three bonus points toward their homework grade.

4.6 GeeBees (Grid Based Fractals)

Algorithm 4.22. Let $n \in \mathbb{N}^+$ and $a_1, \ldots, a_k \in \mathbb{I}_{n^2}$. Define a dynamical system $GB(n; a_1, \ldots, a_k)$ as follows. Let the seed be a set containing one uncolored square. The process is:

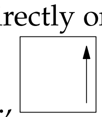
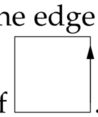
1. Subdivide each uncolored square in the input set into an $n \times n$ grid of congruent subsquares and number these subsquares from 1 to $n^2$from left to right and bottom to top, starting in the lower left corner.
2. Color the subsquares numbered $a_1, \ldots, a_k$.
3. Output the set of colored and uncolored subsquares.

Claim 4.23. *The background (uncolored subsquares of the original square) of a GB converges to a fractal shape.*

Example 4.24. Plot the first few iterations of $GB(3; 2, 4)$.

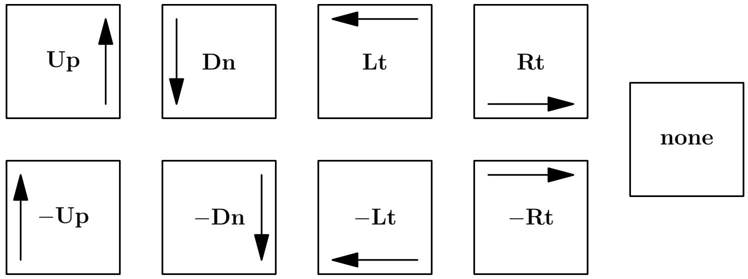
4.7 HeeBGB's

Definition 4.25. A **directed square** is a pair $(S, \overrightarrow{AB})$ where S is a square in the plane and $\overrightarrow{AB}$ is a directed segment whose associated line segment is a side of S.

Notation 4.26. When drawing a picture of a directed square we will draw the directed segment inside the square next to the edge instead of directly on top of the edge to avoid confusion when two directed squares share a common edge, i.e.,  instead of .

Remark. The arrow is not part of the square S associated with the directed square.

Definition 4.27. A **labeled square** is a member of the following 9 families:



Each labeled square must have one of the orientations shown above, but can have any position or size. Let U_{LS} be the set of all labeled squares.

Remark. Notice that every labeled square is a directed square except for the members of the family labeled none. Each labeled directed square is either positive or negative (the negative ones have the arrow on the left when viewed with the arrow pointing upwards).

Definition 4.28. Define the **mirror image** of each labeled square a to be $\overleftrightarrow{a}$ as follows:

a	Up	−Up	Dn	−Dn	Lt	−Lt	Rt	−Rt	none
$\overleftrightarrow{a}$	−Up	Up	−Dn	Dn	−Rt	Rt	−Lt	Lt	none

i.e., the sign always changes and left and right are interchanged.

Remark. Note that this is what is obtained if each of the images in Figure 1 above are reflected about the vertical line through the center of the square.

Definition 4.29. A **GB figure** is a finite set of labeled squares. Let U_{GB} be the set of all GB figures.

Definition 4.30. Let $n \in \mathbb{N}^+$ and $a_1, \ldots, a_{n^2} \in \{\mathrm{Up}, \mathrm{Dn}, \mathrm{Lt}, \mathrm{Rt}, -\mathrm{Up}, -\mathrm{Dn}, -\mathrm{Lt}, -\mathrm{Rt}, \mathrm{none}\}$ (the label set). Define a dynamical system $\mathrm{HeeBGB}(a_1, \ldots, a_{n^2}) : U_{\mathrm{GB}} \to U_{\mathrm{GB}}$ as follows. First define $g\colon U_{\mathrm{LS}} \to U_{\mathrm{GB}}$ as follows.

For each $x \in U_{\mathrm{LS}}$,

1. If x is labeled 'none' then $g(x) = \{x\}$.
2. If x is a directed square then
   (a) Rotate x so its arrow points upwards.
   (b) Subdivide x into an $n \times n$ grid of congruent subsquares.
      i. if x is positive, label these subsquares from a_1 to a_{n^2} from left to right and bottom to top, starting in the lower left corner.
      ii. if x is negative, label these subsquares from $\overleftrightarrow{a_1}$ to $\overleftrightarrow{a_{n^2}}$ from right to left and bottom to top, starting in the lower right corner.
   (c) Undo the rotation from step number 1 to return the square (and all its new subsquares) to the original position and orientation. $g(x)$ is the set of these subsquares.

(d) Now, let $S \in U_{\text{GB}}$. Define $\text{HeeBGB}(a_1, \ldots, a_{n^2})(S) = \bigcup_{x \in S} g(x)$.

Algorithm 4.31. To draw a HeeBGB fractal starting with a seed consisting of a set containing a single labeled square whose label is Up and a choice of labels $a_1, \ldots, a_{n^2}$. Compute the $\text{HeeBGB}(a_1, \ldots, a_{n^2})$-orbit of the seed, but color the squares labeled 'none' as you iterate. The uncolored portion of the HeeBGB-orbit of this seed that is contained in the original square always converges to a fractal image (i.e., you are coloring the background, not the fractal).

Example 4.32. Draw the first few iterations of HeeBGB(Up, Up, Up, none).

Example 4.33. Draw the first few iterations of HeeBGB(Up, −Dn, Lt, none).

Example 4.34. Draw the first few iterations of HeeBGB(Dn, −Dn, Rt, none).

Problems - HeeBGBs

4.11. (2 points each) Draw the third iteration of the following grid based fractal constructions. You must draw these by hand and you must use graph paper of the correct size or my online Fractal Coloring Book. Your choice.

(a) GB(2; 3)

(b) GB(3; 1, 3, 7, 9)

(c) GB(4; 2, 3, 6, 11, 14, 15)

(d) Make up one of your own (a new one, not one from class, the book, or listed above).

4.12. (3 points each) Use the coloring technique I showed you in class to draw the given iteration of the specified HeeBGB. You must use the specified graph paper and use an approved highlighter marker or use my online Fractal Coloring Book. Your choice. You must do these by hand, although you can check your answers with the `DrawDetIFS()` command in my Maple `chaos` library if you wish.

(a) HeeBGB(−Lt, Lt, Lt, none), 5$^{\text{th}}$ iteration on 32x32 graph paper.

(b) HeeBGB(Up, none, −Dn, −Rt, Up, none, none, Dn, −Lt), 3$^{\text{rd}}$ iteration on 27x27 graph paper

(c) Make up your own HeeBGB of the form HeeBGB(a_1, a_2, a_3, none) and color the 5$^{\text{th}}$ iteration on 32x32 graph paper or my online Fractal Coloring Book. None of your choices for a_1, a_2, a_3 should be *Up* or *none*, and at least one of them must be negative.

4.8 Newton's Method

Definition 4.35. Let $R \subseteq \mathbb{R}$, $f: R \to R$, and $r \in R$. We say r is a **root** of f if $f(r) = 0$.

Definition 4.36. Let $R \subseteq \mathbb{R}$ and $f: R \to R$ a differentiable function. Define $\text{Newt}_f : \widetilde{R} \to \mathbb{R}$ by

$$\text{Newt}_f(x) = x - \frac{f(x)}{f'(x)}$$

for all $x \in \widetilde{R}$ where $\widetilde{R} = \{\, x \in R : f'(x) \neq 0 \,\}$.

Remark. $\mathrm{Newt}_f(a)$ is the x coordinate of point where the tangent line to the graph of f at $(a, f(a))$ meets the x-axis.

Theorem 4.37 (Newton's Method)**.** *Let $R \subseteq \mathbb{R}$, $f\colon R \to R$ a differentiable function, and $r \in R$ a root of f. If $f'(r) \neq 0$ then there exists and interval $I \subseteq R$ such that $r \in I$, $\mathrm{Newt}_f\colon I \to I$ and for all $x \in I$ the Newt_f-orbit of x converges to r.*

Example 4.38. Where does $\cos(x) = x$?

Problems - Newton's Method

4.13. Let $f\colon \mathbb{R} \to \mathbb{R}$ with $f(x) = 2 - x^2$ for all $x \in \mathbb{R}$.

(a) (1 point) Find $f^2(x)$. Write your answer as an expanded polynomial.

(b) (1 point) Find $f^3(x)$. Write your answer as an expanded polynomial.

(c) (1 point) Make three plots, one showing the graphs of f and $\mathrm{id}_{\mathbb{R}}$, another showing the graphs of f^2 and $\mathrm{id}_{\mathbb{R}}$, and third showing the graphs of f^3 and $\mathrm{id}_{\mathbb{R}}$.

Hints: The following Maple command plots the graphs of sin and cos on the same graph. Imitate this command to make your plots. You can also plot these by hand or using some other software like Geogebra if you prefer.)

```
plot({sin(x),cos(x)},x=-3..3,view=[-3..3,3..3],numpoints=1000);
```

(d) (1 point) Explain how you can tell from the plots in part (c) how many fixed points, 2-cycles, and 3-cycles the function f has.

(e) (2 points) Use algebra to find all of the fixed points of f. Do this calculation entirely by hand and show your work. No credit will be given for estimates, decimal approximations, use of Maple, estimating from the graphs, or educated guesses. Exact answers only.

(f) (4 points) Use algebra to find all of the disjoint cycles of minimum period 2 for f. Do this calculation entirely by hand and show your work. No credit will be given for estimates, decimal approximations, use of Maple, estimating from the graphs, or educated guesses. Exact answers only.

Hint: If $p(x)$ is a polynomial function with real coefficients then $p(r) = 0$ if and only if $(x - r)$ is a factor of the polynomial $p(x)$.

(g) (1 point) Use the `fsolve();` command in Maple (or a graphing calculator, Geogebra, or some other software) to find decimal approximations to all disjoint cycles of minimum period three for f.

(h) (4 points) Use Newton's method to find a decimal approximation to one of the points of minimum period three that you found in part (g). Use -1.2 for the seed. How many iterations does it take to find the point accurate to within 10^{-8}?

4.14. Newton's method doesn't always work as expected. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function.

(a) (2 points) Give an example where Newton's method fails to produce a root because $f'(x) = 0$ for some x in the orbit of the chosen seed. Also describe geometrically why this fails.

(b) (3 points) Assume $f'(p) \neq 0$ and $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at p. Prove that p is a zero of f if and only if p is a fixed point of Newt_f.

4.15. The Sumerian method for computing square roots is just a special case of Newton's method.

(a) (2 points) Show that the function used in the Sumerian method is just the Newton's method iterator for finding the solutions of $x^2 = a$.

(b) (2 points) Derive a formula for a function analogous to the Sumerian method function, whose orbits converge to $\sqrt[3]{a}$ instead of $\sqrt{a}$.

(c) (2 points) Use your formula from part (b) to compute $\sqrt[3]{5}$ to six digits of accuracy. Give your approximations as both exact fractions and decimal approximations.

(d) (2 points) Generalize the results of part (a) to derive a formula for a function whose orbits converge to $\sqrt[n]{a}$.

4.9 Changing Integer Base

Theorem 4.39 (base b representation). *Let $b, n \in \mathbb{N}$, $b > 1$. There are unique integers $d_0, d_1, d_2, \ldots \in \mathbb{O}_{b-1}$ such that*

$$n = \sum_{i=0}^{\infty} d_i b^i$$

Definition 4.40. The sequence $\ldots d_2 d_1 d_0$ is called the **base b representation** of n. If b is not clear from context we may write $\ldots d_2 d_1 d_{0(b)}$ to indicate the base.

Remark. Since the sum is finite, there must exist $k \in \mathbb{N}$ such that $d_i = 0$ for all $i > k$. Thus we often abbreviate $d_0 d_1 d_2 \ldots$by $d_0 d_1 \ldots d_k$ (i.e., drop the trailing zeros).

Definition 4.41. Let $b \in \mathbb{N}$ and $b > 1$. Define $\mathrm{Base}_b\colon \mathbb{N} \to \mathbb{N}$ by

$$\mathrm{Base}_b(x) = \frac{x - (x \bmod b)}{b}$$

Example 4.42. If $b = 2$ then

$$\mathrm{Base}_2(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x-1}{2} & \text{if } x \text{ is odd} \end{cases}$$

Example 4.43. If $b = 3$ then

$$\mathrm{Base}_3(x) = \begin{cases} \frac{x}{3} & \text{if } x \bmod 3 \text{ is } 0 \\ \frac{x-1}{3} & \text{if } x \bmod 3 \text{ is } 1 \\ \frac{x-2}{3} & \text{if } x \bmod 3 \text{ is } 2 \end{cases}$$

Theorem 4.44 (base conversion). *Let $n \in \mathbb{N}$. If each term in the* Base_b*-orbit of n is replaced by its value* $\bmod b$*, the sequence produced will be the base b representation of n (with the digits listed from left to right from least significant to most significant).*

Example 4.45. What happens if we apply this to a base ten number?

Example 4.46. Convert 314 to base 2 by this method.

Problems - Iteration II

4.16. Define $\mathbb{Q}_{\text{odd}}$ to be the set of all reduced fractions having an odd denominator, i.e.,

$$\mathbb{Q}_{\text{odd}} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}, b \text{ is odd, and } \gcd(a,b) = 1 \right\}$$

Notice that $\mathbb{Z} \subseteq \mathbb{Q}_{\text{odd}}$. We say that such a reduced fraction $\frac{a}{b}$ is even if a is even and odd if a is odd. For example, $\frac{2}{3}$ is even whereas $\frac{5}{7}$ is odd. With these definitions we can extend the Collatz function T from the integers to a function from $\mathbb{Q}_{\text{odd}}$ to itself, i.e., we can consider $T\colon \mathbb{Q}_{\text{odd}} \to \mathbb{Q}_{\text{odd}}$.

(a) (1 point) Find all fixed points of T in $\mathbb{Q}_{\text{odd}}$.

(b) (2 points) Find all disjoint cycles of minimum period 2 for T.

(c) (3 points) Find all disjoint cycles of minimum period 3 for T.

(d) (4 points) Find an explicit piecewise linear formula for $T^2(x)$.

4.17. (1 point each) Use the iterative method shown in class to convert 1234 to base

(a) 2

(b) 3

(c) 4

(d) 5

4.10 Conway's Fractran

Definition 4.47. A **Fractran** program consists of a finite sequence of positive rational numbers

$$F = [r_1, r_2, \ldots, r_k]$$

with r_k an integer. Each such sequence defines a dynamical system $f_F\colon \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$f_F(n) = r_i n$$

where $i = \min\left\{ j : r_j n \in \mathbb{Z} \right\}$ i.e., f_F multiplies n by the first rational number in the sequence for which the product is an integer.

Remark. To do computations with a Fractran program we simply compute the f_F-orbit of some seed and look for certain terms in the orbit for the answers. For example, we might look at the exponents of the powers of two that appear in the orbit for the Fractan program's output.

Example 4.48. (Conway) Let

$$\text{PrimeGame} = \left[\frac{17}{91}, \frac{78}{85}, \frac{19}{51}, \frac{23}{38}, \frac{29}{33}, \frac{77}{29}, \frac{95}{23}, \frac{77}{19}, \frac{1}{17}, \frac{11}{13}, \frac{13}{11}, \frac{15}{2}, \frac{1}{7}, 55\right]$$

and define $f = f_{\text{PrimeGame}}$. The powers of 2 which occur in the f-orbit of 2 are

$$2^2, 2^3, 2^5, 2^7, 2^{11}, 2^{13}, 2^{17}, 2^{19}, \ldots$$

in exactly that order, i.e., PrimeGame computes the prime numbers in order.

Example 4.49. (Monks) Let

$$\text{CollatzGame} = \left[\frac{1}{11}, \frac{136}{15}, \frac{5}{17}, \frac{4}{5}, \frac{26}{21}, \frac{7}{13}, \frac{1}{7}, \frac{33}{4}, \frac{5}{2}, 7\right]$$

and define $f = f_{\text{CollatzGame}}$. The powers of 2 which occur in the f-orbit of 2^n are

$$2^n, 2^{T(n)}, 2^{T^2(n)}, 2^{T^3(n)}, \ldots$$

in exactly that order, i.e., CollatzGame computes the Collatz orbits of natural numbers.

Example 4.50. (Conway) Let

$$\text{PolyGame} = \begin{array}{l} \left[\frac{583}{559}, \frac{629}{551}, \frac{437}{527}, \frac{82}{517}, \frac{615}{329}, \frac{371}{129}, \frac{1}{115}, \frac{53}{86}, \frac{43}{53}, \frac{23}{47}, \frac{341}{46}, \right. \\ \left. \quad \frac{41}{43}, \frac{47}{41}, \frac{29}{37}, \frac{37}{31}, \frac{299}{29}, \frac{47}{23}, \frac{161}{15}, \frac{527}{19}, \frac{159}{7}, \frac{1}{17}, \frac{1}{13}, \frac{1}{3}\right] \end{array}$$

and define $f = f_{\text{PolyGame}}$. Define $f_c(n) = m$ if the f_{PolyGame}-orbit of $c2^{2^n}$ stops at 2^{2^m} and otherwise leave $f_c(n)$ undefined. Then every computable function appears among $f_0, f_1, f_2, \ldots$.

Problems - Fractran

4.18. (4 points total) Let s be the last two digits of your phone number. Use my CollatzGame Fractran program to compute the T-orbit of $s + 10$, where T is the $3x + 1$ function. State clearly

- what integer you want to compute the T-orbit of,
- what seed you are using for the $f_{\text{CollatzGame}}$-orbit in order to accomplish that,
- how, mathematically, you are obtaining the T orbit of your number from the $f_{\text{CollatzGame}}$-orbit.

(a) Do the first ten iterations (of the $f_{\text{CollatzGame}}$-orbit, not the T-orbit) by hand and show your work.

[Hint: When computing iterations by hand, leave your integers factored into their prime factorization rather than expanding them in the base-ten representation. Also factor the numerators and denominators of the fractions that are used to define the Fractran program function CollatzGame first, and then use that to do the iterations by hand. It's MUCH easier that way.]

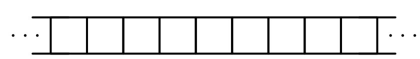
(b) The rest of the orbit you can compute by Maple, using the syntax shown in the Lecture-Examples worksheet. You do not have to print the whole orbit if it the orbit is very large (many are!), just have Maple count how many iterations were required before a cycle was reached.

4.11 Cellular Automata

Definition 4.51. An **n-dimensional k-state cellular automaton** is a discrete dynamical system $G \xrightarrow{c} G$ where G is the set of all functions from $\mathbb{Z}^n$ to $\mathbb{O}_{k-1}$. Each element of $\mathbb{Z}^n$ is called a **cell**. Each element $f \in G$ is called a **state** and its value on a particular cell is called the **state** of that cell. The set G is called the **state space**. To each cell we assign a finite **neighborhood** of cells such that the neighborhood of the translation of a cell is the translation of the neighborhood of the original cell. The map c must be completely determined by a single rule that determines $c(f)(p)$ from the values of $f(q)$ for all q in the neighborhood of p, i.e., the state of a cell after iterating is completely determined by the states of its neighbors before iterating.

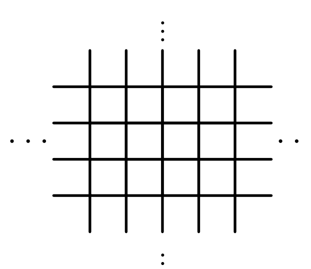
Remark. A cellular automaton (CA) is usually represented by a grid of squares (or n-dimensional cubes), where each square in the grid represents a cell, and the states of each cell are represented by colors.

Example 4.52. A one dimensional CA can be represented as a row of cells.

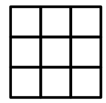


The states of the cells can be represented by coloring the cells different colors corresponding to the current state of that cell. The most common neighborhood to consider for a cell consists of the cell itself, the cell immediately to its left, and the cell immediately to its right (though others are possible).

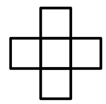
Example 4.53. A two dimensional CA can be represented as a grid of cells:



Definition 4.54. There are two commonly used neighborhoods. The **Moore neighborhood** is a square shaped neighborhood centered at the cell. The most commonly used one consists of a cell all all of the cells that share a boundary point in common with that cell:



The **von Neumann neighborhood** consists of a diamond shaped neighborhood centered at the cell. The most commonly used one consists of the cell and its neighbors immediately to the left, right, above, and below the cell:

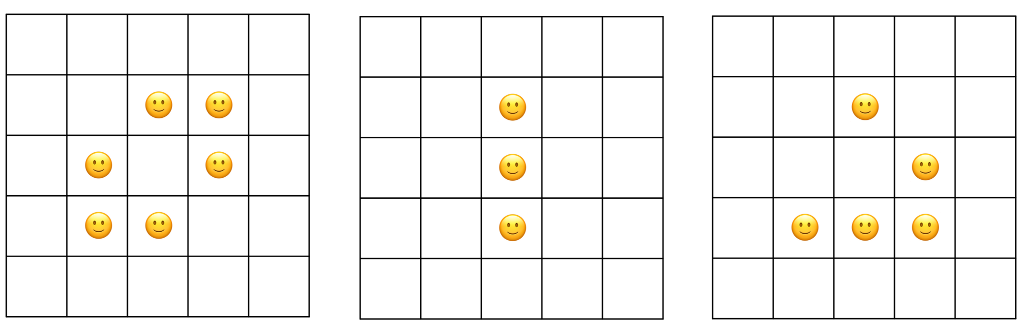


Definition 4.55. An **outer totalistic** (or simply **totalistic**) CA is one whose rule (map) is completely determined by the sum of the state values of the neighbors of each cell.

Definition 4.56. **Binary cellular automata** are those with only two states for each cell. In this situation we say that a cell is **alive** if its state is 1 and **dead** if its state is 0.

Example 4.57. The most famous CA is **Conway's Game of Life**. It is a 2-dimensional binary (2-state) CA whose rule is given as follows. A dead cell becomes alive if exactly three of its Moore neighbors are alive, and a live cell stays alive if either two or three of its Moore neighbors (other than itself) are alive. Otherwise the cell becomes dead.

Example 4.58. Compute the orbits of the following seed states in Conway's Game of Life if empty cells are dead and cells with faces are alive (and assuming that all cells other than the ones shown are dead as well).



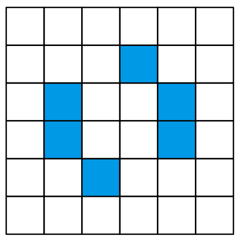
Definition 4.59. A fixed point of the Game of Life cellular automaton is called a **Still Life**.

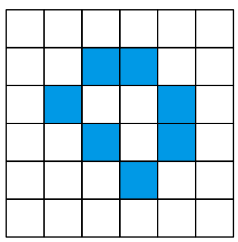
Example 4.60. See the program *Golly* for interesting examples.

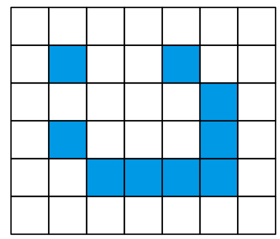
Example 4.61. In 1999 Paul Rendell impelemented a Turing Machine in Life. In 2002, Paul Chapman extended this to construct a universal Turing Machine in Life.

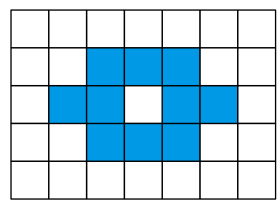
Problems - Life

4.19. (2 points each) Compute the entire orbit of each of the following seeds in Conway's Game of Life. All living cells in each seed are colored blue. In each case compute the first iteration by hand, showing the values (neighbor counts) of each cell that you used. Also in each case classify the orbit as cyclic, eventually cyclic but not cyclic, or acyclic and for eventually cyclic orbits state the period of the cycle that is obtained and the number of iterations that were required before a cyclic point was reached. You may use Golly to compute the orbits.

(a)


(b)


(c)


(d)


4.20. (3 points) Use your first name as the seed for a Game of Life, that is, in the Golly program, draw your name in the grid of cells by hand. Hand in the seed that you used by drawing it on a grid. Then compute the Game of Life orbit of your first name. Describe what happens. What kind of orbit is it? Eventually fixed? Eventually cyclic? Acyclic? How many iterations before things stabilize, if ever? Do any cells survive or do they all eventually die? Are any gliders produced? Do the gliders live forever? Save a copy of your Golly file format in Dropbox.

5 Metric Spaces

Definition 5.1. A **metric space** is a pair (X, d) where X is a set and $d\colon X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$,

1. $d(x, y) \geq 0$
2. $d(x, y) = 0 \Leftrightarrow x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) + d(y, z) \geq d(x, z)$

In this situation, d is called a **metric** (or distance function) on X, and the elements of X are called the **points** in the metric space.

5.1 Examples of Metric Spaces

Example 5.2. $(\mathbb{R}, d_{\text{Euc}})$ is a metric space where $d_{\text{Euc}}(x, y) = \left|x - y\right|$ for all $x, y \in \mathbb{R}$.

Notice this is just a special case of the more general theorem:

Theorem 5.3. *$(\mathbb{R}^n, d_{\text{Euc}})$ is a metric space where*

$$d_{\text{Euc}}((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

*d_{Euc} is called the **Euclidean metric** on $\mathbb{R}^n$.*

Definition 5.4. Let $d_{\text{Taxi}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$d_{\text{Taxi}}((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \sum_{i=1}^{n}\left| x_i - y_i \right|$$

The map d_{Taxi} is called the **lattice metric**, the **Manhattan metric**, or the **taxicab metric**.

Definition 5.5. Let $d_{\max} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$d_{\max}((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \max\left\{ \left| x_i - y_i \right| : i \in \{1,\ldots,n\} \right\}$$

The map $d_{\max}$ is called the **maximum metric**.

Definition 5.6. The set of **2-adic integers**, denoted $\mathbb{Z}_2$, is the set of all infinite sequences of 0's and 1's, i.e.,

$$\mathbb{Z}_2 = \{ (s_0, s_1, \ldots) : \forall i \in \mathbb{N}, s_i \in \{0,1\} \}$$

Definition 5.7. Let $d_2 : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{R}$ by

$$d_2((s_0,s_1,\ldots),(t_0,t_1,\ldots)) = \frac{1}{2^k}$$

where $k = \min\{ i : s_i \neq t_i \}$ if $(s_0,s_1,\ldots) \neq (t_0,t_1,\ldots)$ and

$$d_2((s_0,s_1,\ldots),(t_0,t_1,\ldots)) = 0$$

if $(s_0,s_1,\ldots) = (t_0,t_1,\ldots)$. The map d_2 is called the **2-adic metric**.

Theorem 5.8. *$(\mathbb{R}^n, d_{\text{Taxi}})$, $(\mathbb{R}^n, d_{\max})$, and $(\mathbb{Z}_2, d_2)$ are metric spaces.*

Remark. The metric space $(\mathbb{Z}_2, d_2)$ cannot be embedded in $(\mathbb{R}^n, d_{\text{Euc}})$ for any n. The 2-adic metric is simple to compute and work with, but the geometry of $(\mathbb{Z}_2, d_2)$ is very strange.

5.2 Properties of Metric Spaces

Definition 5.9. Let (X, d) be a metric space, $\delta \in \mathbb{R}^+$, and $x \in X$. Then define

$$\begin{aligned} B(x;\delta) &= \{\, y \in X \mid d(x, y) < \delta \,\} \text{ and} \\ \overline{B}(x;\delta) &= \{\, y \in X : d(x, y) \leq \delta \,\} \end{aligned}$$

The set $B(x;\delta)$ is called the **open ball of radius** δ centered at x, and $\overline{B}(x;\delta)$ is called the **closed ball of radius** δ centered at x.

Definition 5.10. Let (X, d) be a metric space and $U \subseteq X$. Then U is **open** if and only if

$$\forall x \in U, \exists \delta \in \mathbb{R}^+ \text{ such that } B(x;\delta) \subseteq U$$

Definition 5.11. Let (X, d) be a metric space and $U \subseteq X$. Then U is **closed** if and only if $X - U$ is open.

Remark. There are sets which are neither open nor closed.

Definition 5.12. Let (X, d) be a metric space and $U \subseteq X$. Then U is **bounded** if and only if $\exists \delta \in \mathbb{R}^+, \exists x \in X$, such that $U \subseteq B(x;\delta)$.

Definition 5.13. Let $x_0, x_1, x_2, \ldots \in X$ and (X, d) a metric space. Let $x \in X$. Then

$$\lim_{n\to\infty} x_n = x \Leftrightarrow \forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{N}^+ \text{ such that } \forall n \in \mathbb{N}, n > N \Rightarrow d(x_n, x) < \varepsilon$$

In this case we say that the sequence $x_0, x_1, x_2, \ldots$ **converges** to the limit x in (X, d).

Definition 5.14. Let $x_0, x_1, x_2, \ldots \in X$ and (X, d) a metric space. Then the sequence $x_0, x_1, x_2, \ldots$ is called a **Cauchy Sequence** ifand only if

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{N}^+ \text{ such that } \forall i > N, \forall j > N, d(x_i, x_j) < \varepsilon$$

i.e., the terms of the sequence get arbitrarily close to each other.

Definition 5.15. Let (X, d) be a metric space. Then (X, d) is a **complete** metric space if and only if every Cauchy sequence in (X, d) converges to a limit $x \in X$.

Example 5.16. $(\mathbb{R}, d_{\text{Euc}})$ is complete. In fact, this is one of the axioms that define the real numbers.

Example 5.17. $(\mathbb{Q}, d_{\text{Euc}})$ is not complete.

Definition 5.18. Let (X, d) be a metric space and $U \subseteq X$. We say U is **compact** if and only if every open cover has a finite subcover, i.e., whenever $\{\, U_i \,\}_{i\in I}$ satisfies $U \subseteq \bigcup_{i\in I} U_i$ and $\forall i \in I, U_i$ is open, then there exists $k \in \mathbb{N}^+$ and $i_1, i_2, \ldots, i_k \in I$ for which $U \subseteq U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_k}$.

Theorem 5.19. *(Heine-Borel) Let* $A \subseteq (\mathbb{R}^n, d_{\text{Euc}})$. *Then A is compact if and only if A is closed and A is bounded.*

5.3 Continuity

Definition 5.20. Let (X, d) and (Y, d') be metric spaces and $f\colon X \to Y$. Then f is **continuous** with respect to the metrics d and d' if and only if

$$\forall U \subseteq Y, U \text{ is open in } (Y, d') \Rightarrow f^{\text{inv}}(U) \text{ is open in } (X, d)$$

Remark. In other words a function between metric spaces is continuous if and only if the inverse image of every open set is open.

Theorem 5.21. *Let (X, d) be a complete metric space, $f\colon X \to X$ a continuous map, and $x_0, x_1, x_2, \ldots$ a convergent sequence in X with $\lim_{n\to\infty} x_n = x$. Then $f(x_0), f(x_1), f(x_2), \ldots$ is a convergent sequence and*

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$$

i.e., limits commute with continuous maps.

Problems - Metric Spaces

5.1. (1 point) Let $a = 1\overline{010}$ and $b = 10\overline{10}$ be 2-adic integers and d_2 the 2-adic metric. Compute $d_2(a, b)$.

5.2. (3 points) Let (X, d) be a metric space and $k \in \mathbb{R}^+$. Define $d_k\colon X \times X \to \mathbb{R}$ by $d_k(x, y) = kd(x, y)$ for all $x, y \in X$. Prove that (X, d_k) is a metric space.

5.3. (2 points) Complete the proof of the Euclidean Triangle Inequality given in the lecture notes by proving the cases where either $b = 0$ or $c = 0$.

5.4. (3 points) Prove that $(\mathbb{R}^2, d_{\text{Taxi}})$ is a metric space.

5.5. (3 points) Prove that $(\mathbb{R}^2, d_{\max})$ is a metric space.

5.6. (1 point each) Let d_{Taxi} be the taxicab metric on $\mathbb{R}^2$ and define the length of a line segment to be the d_{Taxi} distance between its endpoints. [Note: A subset $S \subseteq \mathbb{R}^2$ is a line segment in $(\mathbb{R}^2, d_{\text{Taxi}})$ if and only if S is a line segment in $(\mathbb{R}^2, d_{\text{Euc}})$.]

   (a) Find all equilateral triangles having the segment $\{\, (x, 0) : x \in [0..1] \,\}$ for one side.

   (b) Repeat part a, but this time use the maximum metric $d_{\max}$ instead of the taxicab metric.

5.7. (1 point each) Sketch the following subsets of $(\mathbb{R}^2, d_{\text{Euc}})$ and state if they are open, closed, bounded, or compact (state all the properties that apply to the given set).

   (a) $B((1, 1); 2)$

   (b) $\mathbb{R}^2 - B((0, 1); 1/2)$

   (c) $\left\{ z \in \mathbb{R}^2 : d_{\text{Euc}}(z, (2, 2)) \le 1 \right\} \cap \left\{ z \in \mathbb{R}^2 : d_{\text{Euc}}(z, (3, 2)) \le 1 \right\}$

   (d) $B((0, 0); 1) - \{\, (0, 0) \,\}$

   (e) $\left\{ (x, y) : \left| y \right| > |x| \right\}$

(f) $\{(n,n) : n \in \mathbb{N}\}$

(g) $\{(1/n, 1/n) : n \in \mathbb{Z}^+\}$

(h) the Middle Thirds Cantor set (i.e. the intersection of all of the iterates that produce the Cantor set).

5.8. (1 point each) Let (X,d) be a metric space. Prove the following.

(a) If $A, B \subseteq X$ are open, then $A \cup B$ is open.

(b) If $A, B \subseteq X$ are open, then $A \cap B$ is open.

(c) If $A, B \subseteq X$ are closed, then $A \cup B$ is closed.

(d) If $A, B \subseteq X$ are closed, then $A \cap B$ is closed.

5.9. (1 point) Show by example that the intersection of a collection of open subsets of a metric space can be closed and not open, and that the union of a collection of closed subsets can be open and not closed.

5.10. Prove the following.

(a) (2 points) $(\mathbb{Z}, d_{\text{Euc}})$ is a metric space.

(b) (2 points) For any $x \in \mathbb{Z}$, $\{x\}$ is open in $(\mathbb{Z}, d_{\text{Euc}})$ but closed in $(\mathbb{R}, d_{\text{Euc}})$.

5.11. (6 points) Let (X,d) be a complete metric space, $X \xrightarrow{f} X$ a continuous map, and $x_0, x_1, x_2, \ldots$ a convergent sequence in X with $\lim\limits_{n\to\infty} x_n = x$. Prove that $f(x_0), f(x_1), f(x_2), \ldots$ is a convergent sequence and that $\lim\limits_{n\to\infty} f(x_n) = f\left(\lim\limits_{n\to\infty} x_n\right)$, i.e. show that limits commute with continuous maps.

5.4 The Metric Space of Shapes

Definition 5.22. Let $n \in \mathbb{N}^+$. Define

$$K_n = \{ A \subseteq \mathbb{R}^n \mid A \text{ is compact} \}$$

Example 5.23. In particular, K_2 is the set of all compact subsets in the plane.

Definition 5.24. Let (X,d) be a metric space, $S \subseteq X$, and $\delta \in \mathbb{R}^+$. The **open collar of radius** δ about S is the set

$$B(S;\delta) = \bigcup_{\alpha \in S} B(\alpha;\delta)$$

and the **closed collar of radius** δ about S is the set

$$\overline{B}(S;\delta) = \bigcup_{\alpha \in S} \overline{B}(\alpha;\delta)$$

Example 5.25. Sketch $B(\text{MrFace};\delta)$ for various values of δ.

Definition 5.26. Let $S \subseteq \mathbb{R}$ and $t \in \mathbb{R} \cup \{\infty\}$. Then $t = \sup(S)$ if and only if $\forall x \in S, x \leq t$ and $\forall u \in \mathbb{R}$, if $\forall x \in S, x \leq u$ then $u \geq t$. The number $\sup(S)$ is called the **supremum** of the set S.

Example 5.27. It is the case that $\sup(0..1) = 1$, but $(0..1)$ has no maximum value.

Definition 5.28. Let $S \subseteq \mathbb{R}$ and $t \in \mathbb{R} \cup \{-\infty\}$. Then $t = \inf(S)$ if and only if $\forall x \in S, x \geq t$ and for all u, if $\forall x \in S, x \geq u$ then $u \leq t$. The number $\inf(S)$ is called the **infimum** of S.

Remark. If a set is closed and bounded, then $\sup(S) = \max(S)$ and $\inf(S) = \min(S)$.

Definition 5.29. Let $d_H\colon K_n \times K_n \to \mathbb{R}$ by $d_H(S,T) = \inf\left\{\delta : S \subseteq \overline{B}(T;\delta) \text{ and } T \subseteq \overline{B}(S;\delta)\right\}$. The function d_H is called the **Hausdorff metric**.

Example 5.30. Let $S = \{(x,x) : x \in [0..1]\}$ and $T = \{p : d_{\text{Euc}}(p,(1,0)) \leq 0.2\}$ be elements of K_2. Compute $d_H(S,T)$.

Theorem 5.31. (K_n, d_H) *(or the metric space where fractals live) is a complete metric space.*

Problems - Metric Space of Shapes

5.12. (6 points) Let

$$\begin{aligned} R &= \overline{B}((2,0);1) \\ S &= \{(0,1)\} \\ T &= \{(-x,0) : x \in [0..1]\} \\ U &= \{(x,y) : y \geq 0 \text{ and } x \leq 0 \text{ and } y \leq x+2\} \\ V &= \{z : d_{\text{Euc}}(z,(0,1)) = 1\} \end{aligned}$$

be elements of $(\mathcal{K}_2, d_H)$. Find the distances between all 25 pairs of these five elements and put your answers in a table of the form:

d_H	R	S	T	U	V
R	0				
S					
T					
U					
V					

(I filled out $d_H(R,R)$ to get you started. ☺) Show your work and sketch the regions.

5.13. (2 points) Let $z, w \in \mathbb{R}^2$. Prove that $d_H(\{z\},\{w\}) = d_{\text{Euc}}(z,w)$.

5.14. (3 points) Let $S = \{(x,y) : x \in [0..2\pi] \text{ and } y = 2\sin(x)\}$ and $T = \{(3,1)\}$. Compute $d_H(S,T)$. You do not have to give an exact answer but your answer must be accurate to within eight digits of accuracy. Hint: Time to break out Maple (or a graphing calculator) and your good old fashioned calculus knowledge!

6 Chaos

6.1 Dynamical Systems - Take 2

Definition 6.1. Let (X, d) be a metric space (or a topological space). Any function $f\colon X \to X$ is called a **discrete dynamical system**. To indicate the metric we sometimes write $f\colon (X, d) \to (X, d)$

Definition 6.2. Let $f\colon (X, d) \to (X, d)$, $g\colon (Y, d') \to (Y, d')$. We say the dynamical systems f, g are **conjugate** if and only if there exists $h\colon X \to Y$ such that

1. h is a **homeomorphism** (a continuous bijection with continuous inverse) and
2. $h \circ f = g \circ h$

In this situation h is called a **topological conjugacy** (or simply **conjugacy**) between f and g.

Remark. The study of discrete dynamical systems is the study of those properties which are preserved by conjugacy.

Definition 6.3. Let (X, d) be a metric space, $f\colon X \to X$, and let q be a fixed point of f. Then q is an **attracting fixed point** if and only if

$$\exists \delta \in \mathbb{R}^+, \forall x \in B(q; \delta), \lim_{n\to\infty} f^n(x) = q$$

i.e., if the f-orbit of every point in some ball centered at q converges to q.

Definition 6.4. Let (X, d) be a metric space, $f\colon X \to X$, and let q be a fixed point of f. Then q is a **repelling fixed point** if and only if

$$\exists \delta \in \mathbb{R}^+, \forall x \in B(q; \delta) - \{\, q \,\}, \exists N \in \mathbb{N}, f^N(x) \notin B(q; \delta)$$

i.e., if the f-orbit of every point other than q in some ball centered at q contains a point outside the ball.

Definition 6.5. Let (X, d) be a metric space, $f\colon X \to X$, and let q be a periodic point of f with period n. We say the n-cycle containing q is an **attracting cycle** (resp. **repelling cycle**) if and only if q is an attracting (resp. repelling) fixed point of f^n.

Example 6.6. Classify the fixed points of $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = 3x$.

Example 6.7. Classify the fixed points of $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{1}{2}x$.

Theorem 6.8. *Attracting and repelling fixed points are presevered by topological conjugacy.*

6.2 Graphical Analysis and Time Series Plots

Definition 6.9. Let $f\colon \mathbb{R} \to \mathbb{R}$, and $x \in X$. Then the **time series plot** of the orbit of x is the graph of the points

$$(0, x), (1, f(x)), (2, f^2(x)), \ldots, (k, f^k(x)), \ldots$$

Remark. Sometimes we connect the points with line segments to make them more visible.

Definition 6.10. Let $f\colon \mathbb{R} \to \mathbb{R}$, and $x \in X$. Then the **graphical analysis** of the orbit of x is the graph consisting of:

a) the graph of f

b) the graph of $y = x$

c) a line segment connecting (x_k, x_{k+1}) to (x_{k+1}, x_{k+1}) for each $k \in \mathbb{N}$

d) a line segment connecting to (x_{k+1}, x_{k+1}) to (x_{k+1}, x_{k+2}) for each $k \in \mathbb{N}$

where $x_k = f^k(x)$.

Remark. Usually we connect these line segments in order starting from $k = 0$, drawing the segment in part (c) before part (d). It is often customary to add the segment from $(x, 0)$ to (x, x_1) as an initial segment.

Example 6.11. Draw the graphical analysis for the f-orbit of seeds 0.23 and 0.230001 for

a) $f(x) = \frac{1-x}{2}$

b) $g(x) = 2x$

c) $h(x) = x^2 - 2$

6.3 Devaney's Definition of Chaos

Definition 6.12. A dynamical system $f\colon (X, d) \to (X, d)$ is said to be **transitive** if and only if

$$\forall x, y \in X, \forall \varepsilon \in \mathbb{R}^+, \exists z \in B(x; \varepsilon), \exists k \in \mathbb{N}, f^k(z) \in B(y; \varepsilon)$$

i.e., for any $\varepsilon \in \mathbb{R}^+$ and for any two points in X there is a third point whose orbit passes within ε of both points.

Remark. Sometimes this property is called **mixing**.

Definition 6.13. A dynamical system $f\colon (X, d) \to (X, d)$ is said to have **sensitive dependence on initial conditions** if and only if

$$\exists \delta \in \mathbb{R}^+, \forall x \in X, \forall \varepsilon \in \mathbb{R}^+, \exists y \in B(x; \varepsilon) - \{\, x \,\}, \exists k \in \mathbb{N}, d(f^k(x), f^k(y)) > \delta$$

i.e., there is a positive constant so that for any x there is a point y arbitrarily close to x such that the orbits of x and y will eventually be separated by at least the constant.

Definition 6.14. Let (X, d) be a metric space and $A \subseteq X$. Then A is **dense** in X if and only if

$$\forall x \in X, \forall \varepsilon \in \mathbb{R}^+, B(x; \varepsilon) \cap A \neq \emptyset$$

i.e., A is dense if every open ball contains a point of A.

Definition 6.15 (Devaney)**.** A discrete dynamical system is **chaotic** if and only if

1. it has dense periodic points,
2. it is transitive, and
3. it has sensitive dependence on initial conditions.

Remark. Chaotic maps give us a model for unpredicatable deterministic systems.

6.4 Touhey's Definition

Theorem 6.16 (Touhey 1997)**.** *A discrete dynamical system (on an infinite set) is chaotic if and only if every finite collection of open sets shares infinitely many periodic orbits.*

6.5 Chaotic Maps

The following are examples of chaotic maps:

1. **Quadratic maps**

   For each $c \in \mathbb{C}$ define $Q_c(x) = x^2 + c$.

   (a) $Q_{-2}(x) = x^2 - 2$ is chaotic on $[-2..2]$.

   (b) $Q_0(z) = z^2$ is chaotic on the unit circle.

   (c) Q_c is chaotic on a fractal set called J_c (more later).

   (d) The Logistic Map $Q(x) = 4x(1-x)$ is chaotic on $[0..1]$.

2. **The Doubling Map**

$$D(x) = \begin{cases} 2x & \text{if } x \in [0..1/2) \\ 2x-1 & \text{if } x \in [1/2..1] \end{cases}$$

   is chaotic on $[0..1]$.

3. **The Tent Map**

$$T(x) = 1 - |2x-1|$$

   is chaotic on $[0..1]$.

4. (J. Joseph) **The Extended Collatz Map**

$$T(z) = \begin{cases} \frac{1}{2}z & \text{if } z \underset{2}{\equiv} 0 \\ \frac{3z+1}{2} & \text{if } z \underset{2}{\equiv} 1 \\ \frac{3z+i}{2} & \text{if } z \underset{2}{\equiv} i \\ \frac{3z+1+i}{2} & \text{if } z \underset{2}{\equiv} 1+i \end{cases}$$

   is chaotic on $\mathbb{Z}_2[i]$.

Problems - Chaos

6.1. (3 points each) Let $(X, d), (Y, d')$ be metric spaces and $f\colon X \to X$, $g\colon Y \to Y$. Let h be a topological conjugacy between f and g.

(a) Prove that q is an attracting fixed point of f if and only if $h(q)$ is an attracting fixed point of g.

(b) Prove that q is a repelling fixed point of f if and only if $h(q)$ is a repelling fixed point of g.

(c) Prove that q is a term in an attracting (resp. repelling) n-cycle if and only if $h(q)$ is a term in an attracting (resp. repelling) n-cycle of g.

6.2. (1 point) Give five different examples of subsets of $\mathbb{R}^2$ which are dense in $(\mathbb{R}^2, d_{\text{Euc}})$.

6.3. (3 points) Prove that $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x$ has sensitive dependence on initial conditions, but is not chaotic.

6.4. (4 points) The function $Q(x) = x^2 - 2$ is chaotic on the interval $[-2..2]$, therefore, in any open subinterval $(a..b) \subseteq [-2..2]$ there must be a periodic point. Let $I_k = (0.2k..0.2(k+1))$ for $k \in \{0, 1, 2, \ldots 9\}$. For each such k, find a periodic point $p_k \in I_k$, and state its minimum period, n_k. List all of your points and their periods in a table and verify that they are periodic by listing their Q-orbit. You may use decimal approximations, but they should be accurate to at least 10 digits.

6.5. (1 point each) Draw the time series plot and the graphical analysis for the first twenty iterations starting with seed 1.5 for each of the following functions from $\mathbb{R}$ to $\mathbb{R}$.

(a) arctan

(b) f where $f(x) = 2\sin(x) + x$

(c) your own favorite nonlinear function

6.6. (4 points each) Give an example of differentiable functions f and g from $\mathbb{R} \to \mathbb{R}$ such that f has an attracting two cycle (that is not a fixed point) and a repelling fixed point and g has an attracting three cycle (that is not a fixed point) and a repelling fixed point. In each case

(a) explain why your function has the required properties

(b) draw the time series plots for appropriate seeds to illustrate that it behaves as claimed

(c) draw an animated graphical analysis in Maple for several appropriate seeds to illustrate that it behaves as claimed and

(d) explain how your time series and graphical analysis plots illustrate the required properties. You can do all of your work and write your explanations for this problem in Maple.

6.7. (3 points) Let $n \in \mathbb{N}^+$, $\mathbb{R} \xrightarrow{f} \mathbb{R}$ a differentiable function, and $x_0, x_1, x_2, \ldots$ the f-orbit of a cyclic point $x_0 \in \mathbb{R}$ with minimum period n. Derive a formula for $(f^n)'(x_0)$ in terms of $f'(x_0), f'(x_1), \ldots, f'(x_{n-1})$. Use this to prove that if one point in the n-cycle is attracting (resp. repelling), then they all are.

7 Contraction Mappings

7.1 The Contraction Mapping Theorem

Definition 7.1. Let (X, d) be a metric space and $f\colon X \to X$. Then f is called a **contraction mapping** if and only if $\exists s \in (0..1), \forall x, y \in X, d(f(x), f(y)) \leq s d(x, y)$. In this situation s is called a **contraction factor** of f.

Theorem 7.2. *Every contraction mapping is continuous.*

Theorem 7.3 (The Derivative Test)**.** *Let $I = (a..b) \subseteq \mathbb{R}$ and $f\colon I \to I$ differentiable on I. If $\exists s \in (0\ldots1)$ such that $\forall x \in I, \left|f'(x)\right| \leq s < 1$, then f is a contraction mapping with contraction factor s.*

Theorem 7.4 (The Contraction Mapping Theorem)**.** *Let $f\colon X \to X$ be a contraction mapping on a complete metric space (X, d) with contraction factor s.*

1. *The map f has a unique fixed point, q.*
2. *The f-orbit of every element of X converges to q (i.e., $\forall x \in X, \lim_{n\to\infty} f^n(x) = q$).*
3. *If $x_0, x_1, x_2, \ldots$ is the f-orbit of $x_0 \in X$ then*

$$d(x_n, q) \leq \frac{s^n}{1-s} d(x_0, x_1)$$

for all $n \in \mathbb{N}$.

Remark. Every contraction map has an attracting fixed point.

7.2 Hutchinson Operators

Definition 7.5. Let $w_0, w_1, \ldots, w_k$ be contraction mappings on $\mathbb{R}^n$ with contraction factors $c_0, c_1, \ldots, c_k$ respectively and define $W\colon K_n \to K_n$ by

$$W(A) = w_0(A) \cup w_1(A) \cup \cdots \cup w_k(A)$$

Then W is called the **Hutchinson operator** associated with $w_0, w_1, \ldots, w_k$ and we write $W = \mathrm{Hutch}(w_0, w_1, \ldots, w_k)$.

Theorem 7.6 (Hutchinson)**.** *A Hutchinson operator W is a contraction mapping on (K_n, d_H) with contraction factor $c = \max\{c_0, c_1, \ldots, c_k\}$.*

Definition 7.7. If W is a Hutchinson operator then the unique fixed point of W is called the **attractor** of W and is denoted F_W.

Remark. Applying the three parts of the contraction mapping theorem to W gives us a lot of information about producing fractals with Hutchinson operators.

Problems - Contraction Mapping Theorem

7.1. (1 point each) Give examples of contraction maps $\mathbb{R} \xrightarrow{f} \mathbb{R}$ which satisfy the given condition and explain why your function has the desired properties. Plot the graph of each of your functions.

(a) f is bijective and strictly decreasing

(b) f is injective but not surjective

(c) f is surjective but not injective

(d) f is neither surjective nor injective

7.2. Let $I \subseteq \mathbb{R}$ be an open interval and $f\colon I \to \mathbb{R}$ a twice differentiable function with $f'(x) \neq 0$ for any $x \in I$. Let $N = \mathrm{Newt}_f$ be the Newton's method iterator function for f.

(a) (1 point) Let $p \in I$ be a zero of f. Show that $N'(p) = 0$.

(b) (4 points) Let p be a zero of f in I. Prove there is an open interval $U \subseteq I$ containing p such that the N-orbit of x_0 converges to p for any $x_0 \in U$. Thus conclude that Newton's method is guaranteed to work if $f'(p) \neq 0$ and your initial guess is close enough to p.

7.3. (2 points each except as noted)

(a) Let $S\colon \mathbb{R}^+ \to \mathbb{R}^+$, be the function used in the Sumerian method for finding $\sqrt{2}$. Find the largest open interval $I \subseteq \mathbb{R}^+$ such that $|S'(x)| \leq 0.5$ for all $x \in I$.

(b) Prove that S is a contraction mapping on I. [Hint: you *must* show that $S\colon I \to I$ first, then use part (a).]

(c) (1 point) Let $x \in \mathbb{R}^+ - I$. Show that $S(x) \in I$.

(d) Prove that the Sumerian method for computing $\sqrt{2}$ works for *any* choice of seed in $\mathbb{R}^+$. [Hint: use parts (b) and (c).]

(e) Use the convergence estimate given by the contraction mapping theorem to compute the number of iterations required to compute $\sqrt{2}$ to six digits of accuracy starting with seed 1000000. Then compute the actual number of iterations required and show that it is less than or equal to your estimate.

8 Iterated Function Systems

8.1 Complex Numbers

Definition 8.1. Let $\mathbb{C} = \mathbb{R}^2$. For each $(x, y) \in \mathbb{C}$ we formally write $(x, y) = x + yi$. This form, $x + yi$, is called the **standard form** of the complex number (x, y).

Definition 8.2. Let $x + yi, a + bi \in \mathbb{C}$, then:

1. $\overline{x+yi} = x - yi$. (This is called the **complex conjugate**.)
2. $\left|x+yi\right| = \sqrt{x^2+y^2}$. (This is called the **complex norm**.)
3. $\operatorname{Arg}(x+yi) =$ the angle in $[0..2\pi)$ of (x,y) in polar form (not defined for $x=y=0$). (This is called the **Argument** of $x+yi$.)
4. $\operatorname{Re}(x+yi) = x$. (This is called the **real part** of $x+yi$.)
5. $\operatorname{Im}(x+yi) = y$. (This is called the **imaginary part** of $x+yi$.)
6. $(x+yi)+(a+bi) = (x+a)+(y+b)i$. (This is the definition of **addition** in $\mathbb{C}$.)
7. $(x+yi)(a+bi) = (xa-yb)+(ya+xb)i$. (This is the definition of **multiplication** in $\mathbb{C}$.)

Notation 8.3. We can abbreviate $0+yi$ as yi, $x+0i$ as x, $x+1i$ as $x+i$, and $x-1i$ as $x-i$ with no ambiguity in the above definitions. With this notation $i=(0,1)$ and $i^2=-1$. It is easy to verify that the usual properties of addition and multiplication (associative, commutative, distributive, identity, etc.) hold for the complex numbers as well.

Definition 8.4. Let $\theta \in \mathbb{R}$. Then $e^{i\theta} = \cos\theta + i\sin\theta$.

Definition 8.5. Let $x+yi \in \mathbb{C}-\{0\}$. The **standard polar form** of $x+yi$ is $re^{i\theta}$ where $r = \left|x+yi\right|$ and $\theta = \operatorname{Arg}(x+yi)$.

Theorem 8.6. $e^{i\pi}+1=0$ *(The most beautiful theorem in mathematics?)*

Theorem 8.7. *Let $\theta,\gamma \in \mathbb{R}$.*

1. $e^{i\theta}e^{i\gamma} = e^{i(\theta+\gamma)}$.
2. $\left|e^{i\theta}\right| = 1$.
3. $\overline{e^{i\theta}} = e^{i(-\theta)}$.

Theorem 8.8. *Let $z, z_1, z_2 \in \mathbb{C}$.*

1. $|z_1z_2| = |z_1||z_2|$
2. $d_{\text{Euc}}(z_1,z_2) = |z_2-z_1|$
3. $\overline{z_1z_2} = \overline{z_1}\,\overline{z_2}$ *i.e., the conjugate of a product is the product of conjugates.*
4. $\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$ *i.e.,the conjugate of a sum is the sum of the conjugates.*
5. $z\,\overline{z} = |z|^2$
6. $|z| = \left|\overline{z}\right|$
7. *If $z = re^{i\theta}$ in polar form, then $\overline{z} = re^{i(-\theta)}$*

Problems - Complex Numbers

8.1. (2 points each) Let $z, w \in \mathbb{C}$ and $r, \theta, \gamma \in \mathbb{R}$. Prove the following.

(a) $\left|re^{i\theta}\right| = |r|$

(b) $\bar{r} = r$

(c) $|z| \geq 0$

(d) $\left|e^{i\theta}\right| = 1$

(e) $e^{i\theta}e^{i\gamma} = e^{i(\theta+\gamma)}$

(f) $|zw| = |z||w|$

(g) $d_{\text{Euc}}(z, w) = |z - w|$

(h) $\overline{zw} = \bar{z}\,\overline{w}$

(i) $\overline{z + w} = \bar{z} + \overline{w}$

(j) $z\bar{z} = |z|^2$

(k) $|z| = |\bar{z}|$

(l) $\overline{e^{i\theta}} = e^{-i\theta}$

8.2. (1 point each) Convert the following complex numbers to polar form.

(a) $1 + i$

(b) $-i$

(c) -2

(d) 5

(e) $5 + 6i$

8.3. (1 point each) Let $z = 1 + i$, $w = -2 + i$, and $v = -i$. Convert the following expressions to a complex number in standard form.

(a) zw

(b) $z + w$

(c) $w^2 + 3w + 1$

(d) $\overline{w}^2 + 3\overline{w} + 1$

(e) $w\overline{w}$

(f) $\overline{5}$

(g) $3e^i$

(h) e^z

(i) $v^{1000001}$

8.4. (1 point) Find a complex number α in standard form, such that multiplication of any complex number z by α has the effect of rotating z about the origin by $20°$ clockwise.

8.2 Affine Maps

Definition 8.9. Define $M_{m,n}(\mathbb{R})$ to be the set of all $m \times n$ matrices with real number entries. Let $A \in M_{m,n}(\mathbb{R})$ and $i \in \mathbb{I}_m$, $j \in \mathbb{I}_n$. Then $A\langle i, j\rangle$ is the entry in the i^{th} row and j^{th} column of A. If $c \in \mathbb{R}$ then $cA \in M_{m,n}(\mathbb{R})$ and

$$(cA)\langle i, j\rangle = c(A\langle i, j\rangle)$$

If $B \in M_{m,n}(\mathbb{R})$ then $A + B \in M_{m,n}(\mathbb{R})$ and

$$(A + B)\langle i, j\rangle = A\langle i, j\rangle + B\langle i, j\rangle$$

If $B \in M_{n,p}(\mathbb{R})$ then $AB \in M_{m,p}(\mathbb{R})$ and for all $i \in \mathbb{I}_m$, $j \in \mathbb{I}_p$

$$AB\langle i, j\rangle = \sum_{k=1}^{n} A\langle i, k\rangle B\langle k, j\rangle$$

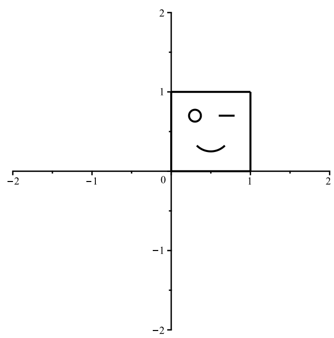
Remark. We often identify elements of $\mathbb{R}^n$ with elements in $M_{1,n}(\mathbb{R})$ and $M_{n,1}(\mathbb{R})$ when appropriate.

Definition 8.10. The function $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is called an **affine map** or **affine transformation** if and only if $T(x) = Mx + B$ for some $n \times n$ matrix M and $B \in \mathbb{R}^n$.

Remark. We will mostly restrict our attention to affine maps on $\mathbb{R}^2$ in this course.

Theorem 8.11. *An affine transformation on $\mathbb{R}^2$ is completely determined by where it maps 3 non-collinear points.*

In particular, it will be useful to visualize the effect of an affine transformation on the plane by seeing where it maps the image of MrFace:



Since MrFace contains at least three non-collinear points, an affine map is completely determine by where it sends MrFace by theorem 8.11.

Representations of Affine Maps on $\mathbb{R}^2$

Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an affine map, and let $p \in \mathbb{R}^2$. Let $x, y \in \mathbb{R}$ such that $p = (x, y)$ and let $z = x + yi$. Then there exist a 2×2 real matrix M, $B \in \mathbb{R}^2$, $\alpha, \beta, \gamma \in \mathbb{C}$, and $a, b, c, d, e, f, r, s, \theta, \phi \in \mathbb{R}$ such that

Form Name	**Math Notation**
Matrix	$T(p) = Mp + B$
Standard	$T(x, y) = (ax + by + e, cx + dy + f)$
Geometric	$T(x, y) = (r\cos(\theta)x - s\sin(\phi)y + e, r\sin(\theta)x + s\cos(\phi)y + f)$
Complex	$T(z) = \alpha z + \beta\overline{z} + \gamma$

These are expressed in the *chaos* Maple package in the following notation

Form Name	**Maple Notation**
Matrix	affineM(M, B)
Standard	affine(a, b, c, d, e, f)
Geometric	Affine$(r, s, \theta, \phi, e, f)$
Complex	affineC(α, β, γ)

In Maple an IFS (see below) is a Maple list of one or more of these affine maps. Note that in the Geometric Form, the Maple program assumes that θ, ϕ are in degrees, not radians.

Each form has its own advantages. We can convert from any form to any other form. It suffices to give the formulas for converting between any form and standard form. Thus, if you are given Matrix form and want to convert to Complex form, first convert to standard and then to Complex.

Theorem 8.12. *In the above notation:*

Converting Standard form to Matrix form and vice-versa:

$\text{affine}(a,b,c,d,e,f) = \text{affineM}(M,B) \Leftrightarrow$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} e \\ f \end{pmatrix}$$

Converting Standard form to Complex form and vice-versa:

$\text{affine}(a,b,c,d,e,f) = \text{affineC}(A+Bi,\ C+Di,\ E+Fi) \Leftrightarrow$

$$\begin{aligned} a &= A + C & \textit{and} \quad A &= \tfrac{1}{2}(a+d) \\ b &= D - B & B &= \tfrac{1}{2}(c-b) \\ c &= B + D & C &= \tfrac{1}{2}(a-d) \\ d &= A - C & D &= \tfrac{1}{2}(c+b) \\ e &= E & E &= e \\ f &= F & F &= f \end{aligned}$$

Converting Standard form to Geometric form and vice-versa:

$\text{affine}(a,b,c,d,e,f) = \text{Affine}(r,s,\theta,\phi,E,F) \Leftrightarrow$

$$\begin{aligned} a &= r\cos(\theta) & \textit{and} \quad r &= \sqrt{a^2+c^2} \\ b &= -s\sin(\phi) & s &= \sqrt{b^2+d^2} \\ c &= r\sin(\theta) & \theta &= \arctan(\tfrac{c}{a}) \\ d &= s\cos(\phi) & \phi &= \arctan(\tfrac{d}{b}) - 90^\circ \\ e &= E & E &= e \\ f &= F & F &= f \end{aligned}$$

The effect of the affine map Affine$(r, s, \theta, \phi, E, F)$ on a geometric figure is as follows.

Geometric Form	
parameter	**geometric effect**
r	scales the figure horizontally by a factor of $\lvert r \rvert$
	(if r is negative, it also reflects the figure across the y-axis)
s	scales the figure vertically by a factor of $\lvert s \rvert$
	(if s is negative, it also reflects the figure across the x-axis)
θ	rotates horizontal lines θ degrees CCW about their y-intercept
ϕ	rotates vertical lines ϕ degrees CCW about their x-intercept
e	translates the figure horizontally by an amount e
f	translates the figure vertically by an amount f

Note that if $\theta = \phi$, then the effect of both numbers combined is to rotate the entire figure about the origin by an angle θ counterclockwise (CCW). Negative angles rotate clockwise (CW) instead of counterclockwise. Also note that Affine$(r, s, \theta, \phi, e, f)$ always sends the origin, $(0, 0)$,to the point (e, f).

Contraction Factor for Affine Maps

Theorem 8.13. *Let $\alpha, \beta, \gamma \in \mathbb{C}$ and $c = |\alpha| + \left|\beta\right|$. Then the map $T = \mathrm{affineC}(\alpha, \beta, \gamma)$ is a contraction mapping if and only if $c < 1$. Further if T is a contraction mapping then c is a contraction factor for T.*

Problems - Affine Maps

In what that follows when we refer to affine maps, we are referring to affine maps on $\mathbb{R}^2$ unless specifically stated otherwise.

8.5. (1 point each) Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 0 \end{pmatrix}, B = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}, b = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Compute each of the following.

(a) $A + B$

(b) AB

(c) BA

(d) $3Ab$

(e) $Ab + 2b$

8.6. (2 points) Let $M \in M_{2,2}(\mathbb{R})$. Prove that for any $x, y \in \mathbb{R}^2$ and any $a, b \in \mathbb{R}$,

$$M(ax + by) = aMx + bMy$$

8.7. (1 point each) For each T and p below, compute $T(p)$. Do it by hand and then check your answer in Maple using my *chaos* library.

(a) $T = \text{affine}(2, 1, 30°, 45°, -1, 2)$, $p = (3, 2)$

(b) $T = \text{affine}(2, 1, -1, 3, 0, -1)$, $p = (-1, 3)$

(c) $T = \text{affineC}(2 + i, 1 - i, i)$, $p = -1 + 3i$

(d) $T = \text{affineM}\left(\begin{pmatrix} 2 & 0.3 \\ -\frac{1}{2} & -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$, $p = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

8.8. (4 points) Derive the conversion formulas given in lecture for converting between the four forms of affine maps: standard, matrix, geometric and complex. Show your work, don't use Maple.

8.9. Recall that there is a unique affine map that sends any three noncollinear points to any other three points.

(a) (2 points) Find an affine map T such that $T(0) = i$, $T(i) = 1 + i$, and $T(2 + i) = -1$. Do this by hand.

(b) (3 points) Use Maple to convert your answer into standard, matrix, complex, and geometric form. (Note: there is a command to do this in my chaos library).

(c) (1 point) Use Maple to plot T(MrFace).

8.10. Let $T = \text{affineC}(\alpha, \beta, \gamma)$ and $C_T = |\alpha| + \left|\beta\right|$. By the theorem proved in the Appendix, T is a contraction mapping if and only if $C_T < 1$.

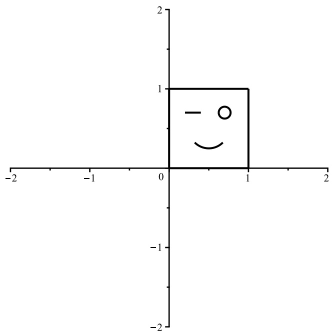
(a) (2 points) Suppose $T = \text{affine}(a, b, c, d, e, f)$. Find a formula for C_T in terms of a, b, c, d, e, f.

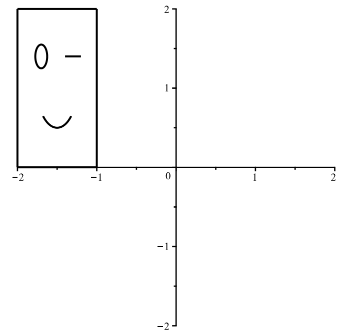
(b) (2 points) Suppose $T = \text{Affine}(r, s, \theta, \phi, e, f)$. Find a formula for C_T in terms of r, s, θ, ϕ, e, f.

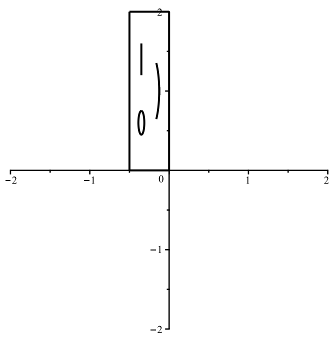
8.11. (2 points) Show that if $|r| < 1$, $|s| < 1$, and $\theta = \phi$ then $\text{Affine}(r, s, \theta, \phi, e, f)$ is a contraction mapping.

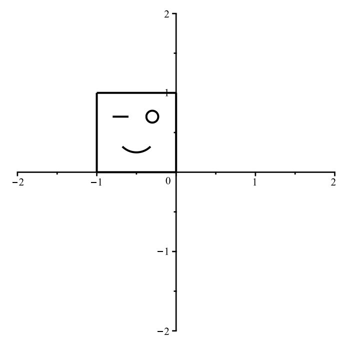
8.12. (4 points) Prove that affine maps send line segments to line segments (or single points, which can be thought of as very short line segments. :)), i.e., if T is an affine map and S is a line segment in the plane, show that $T(S)$ is a line segment.

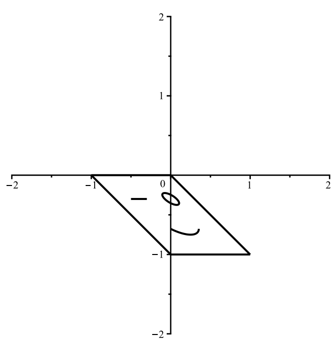
8.13. (1 point each) For each image A below, find an affine map $T = \text{Affine}(r, s, \theta, \phi, e, f)$ in geometric form so that $A = T(\text{MrFace})$. Do the first six by hand and the remaining two any way you like.

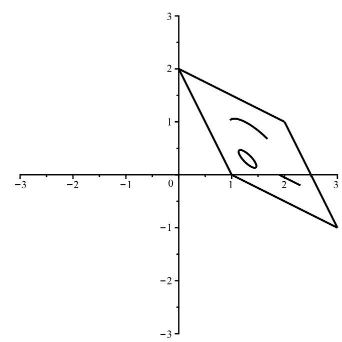
(a) 

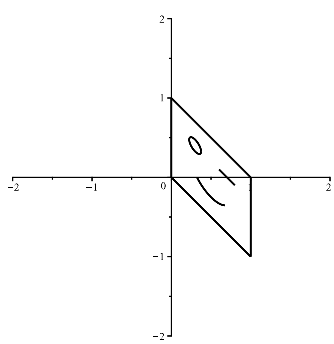
(e) 

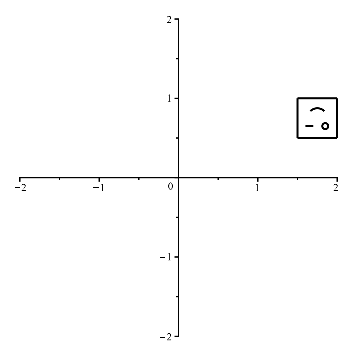
(b) 

(f) 

(c) 

(g) 

(d) 

(h) 

8.3 IFS's

Definition 8.14. A Hutchinson operator $W = \text{Hutch}(w_0, w_1, \dots, w_k)$ such that $w_0, w_1, \dots, w_k$ are all affine maps is called an **iterated function system** or IFS. We write $W = [w_0, w_1, \dots, w_k]$ in this case.

Remark. Every affine contraction map is an affine map, but there are many contraction mappings that aren't affine. For example, $f(x) = \frac{1}{2}\cos(x)$ on $\mathbb{R}$, or $f(x) = \sqrt{x}$ on $[1..\infty)$.

Remark. A stick figure generator that contains finitely many directed segments whose associated affine maps are contraction maps (and no ordinary segments) is an example of a IFS. The attractor of the IFS can be obtained by iterating the stick figure dynamical system starting with a single directed segment as the seed.

Remark. Both HeeBGB's and GB's are IFS's which map the original square onto the appropriate subsquares in the given manner. Coloring a HeeBGB is just producing the attractor of the IFS by the Deterministic Method (see below) where we color the background, not the image itself.

8.4 The Deterministic Method

Algorithm 8.15. To draw the attractor of an IFS, W, simply compute the terms in the W-orbit of any seed in K_n until the image is as close to the attractor as you desire.

Remark. By the contraction mapping theorem, the IFS W has an attractor, no matter what shape we start with for a seed, its orbit will converge to the attractor, and we can compute the number of iterations required to obtain an image that is within any desired accuracy of the attractor.

8.5 Guess My IFS

Remark. Let $W = [w_0, w_1, \dots, w_k]$ be an IFS. By the contraction mapping theorem, F_W is the unique fixed point of W, i.e., F_W is the only element of K_n such that

$$W(F_W) = F_W$$

By the definition of W this means that F_W is the unique solution A in K_n to the equation:

$$A = w_0(A) \cup w_1(A) \cup \cdots \cup w_k(A)$$

In particular,

$$F_w = w_0(F_w) \cup w_1(F_w) \cup \cdots \cup w_k(F_w)$$

so that the attractor is a union of finitely many affine images of itself (each of which must therefore be a finite union of strictly smaller affine images of itself, and so on ad infinitum).

Thus, given the attractor of an IFS, we can determine an IFS that produces it (not unique!) by identifying a finite number of strictly smaller affine images if the attractor whose union is the entire shape.

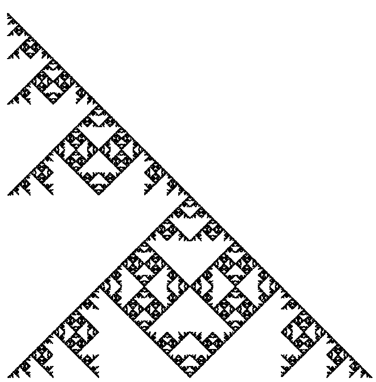
Problems - Return of the Heebie GB's

8.14. (1 point each fractal) Plot the fractals (i.e. the attractors of the IFS's) in problems 3.13, 3.17, 3.18 in the **Iteration** section above using the `DrawIFS` command in my Maple `chaos` library. Use at least 30000 points. Tips: More points will produce a better picture but will take longer and may crash Maple if you run out of memory.

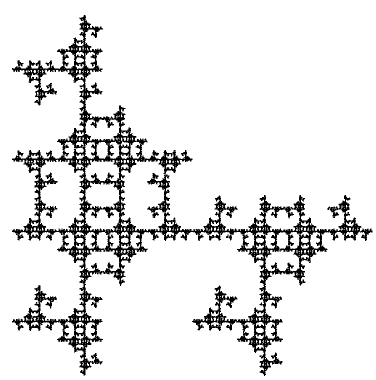
8.15. *Heebie Geebie Free Three for Three Contest*: Try to create the most interesting or beautiful fractal you can using the `DrawIFS` command in my Maple `chaos` library. Your fractal does not have to be the attractor of a `HeeBGB`, but can be the attractor of any IFS you like. You can also make an image that consists of a collage of more than one image using the Maple `display()` command in `plots` package. If you want to add text or a title to your Fractal Work of Art, you can do so with the Maple `textplot()` command in the Maple `plots` package. I will share all entries (anonymously) and they will be judged by a panel of "experts" of my choosing (contest participants will not be eligible for judging). The top three winners will receive a bonus of three free points added directly to their homework grade. If there are three or fewer contestants they will win by default. In the case of duplicate entries which are winners, the three point bonus will be evenly divided among those having the duplicate entries. The judges will be asked to judge the fractals using two criteria: how aesthetically pleasing the image is and how mathematically interesting it appears. You can change the color of the image if you wish or even use multiple colors (if you can figure out how to do it). Be sure you show the Maple commands you used to produce the image in your worksheet along with the image itself.

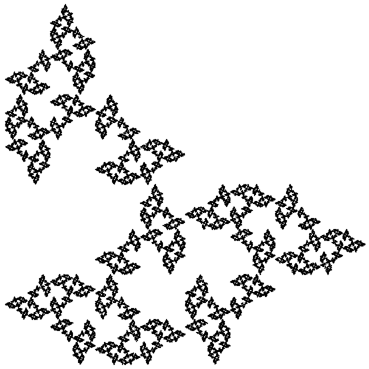
8.16. (2 points each) Play *Guess My IFS* with each of the following IFS attractors by drawing (or plotting) the first iteration of the IFS applied to MrFace as a seed. Then use the DrawIFS command in Maple to confirm your guess by plotting the attractor and hand in your plots. Hint: (a), (b), (c), (e), (g) are grid based IFS's, the other three are not. You should be able to do all of these "by eye", i.e. without resorting to any calculations or measurments.

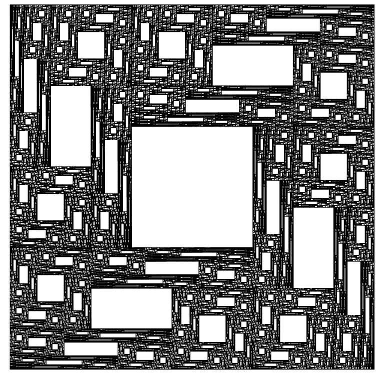
(a)

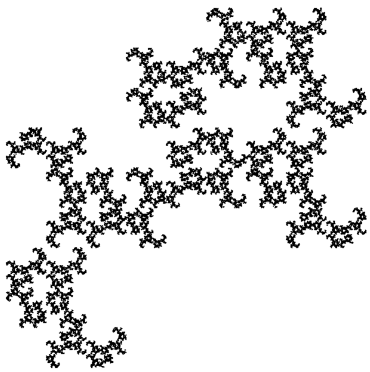


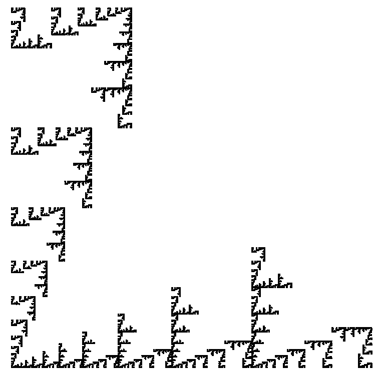
(b)

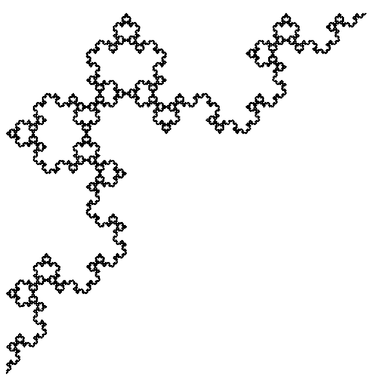


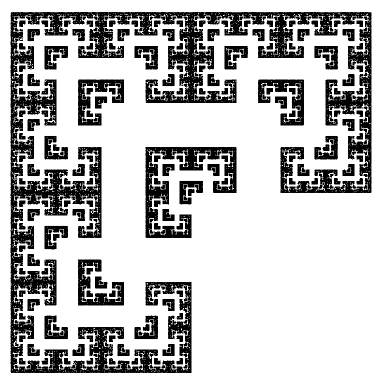
(c) 

(d) 

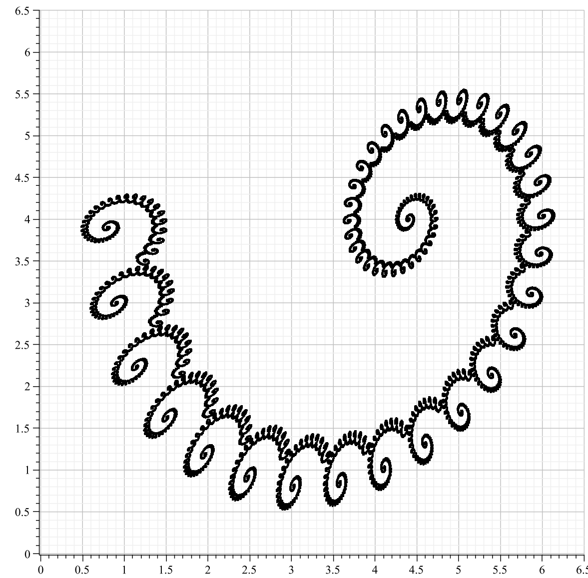
(e) 

(f) 

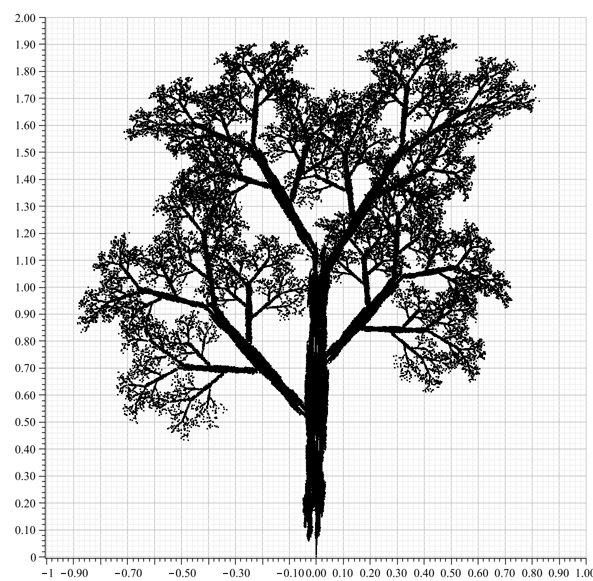
(g) 

(h) 

8.17. (5 points) Find an IFS whose attractor is the following figure. Draw the first iteration of the IFS applied to MrFace using Maple and then plot the attractor as you did in the previous problem. You may use Maple as much as you wish in this problem.



8.18. (5 points) Repeat the previous problem for the fractal tree image below. Notice that the method you are using to make a fractal which looks like this particular tree can be applied to making a fractal that looks like any shape you like. Thus you have learned how to make fractals which look like a given image you are trying to model. Hints: Many trees look alike, but are all unique to some extent... be sure your fractal has all of its branches connected in the same places as the tree below!



9 Address My IFS

Definition 9.1. Let $n \geq 2$. Define Σ_n to be the set of all infinite sequences whose terms are in $\{0, 1, 2, \ldots, n-1\}$, i.e.,

$$\sum_n = \{ s_0, s_1, s_2, \ldots : s_i \in \{0, 1, 2, \ldots, n-1\} \}$$

$\Sigma_n$is called the **sequence space on n letters**.

Example 9.2. Σ_2 is the set of 2-adic integers $\mathbb{Z}_2$.

Definition 9.3. Let $W = [w_0, w_1, \ldots, w_{n-1}]$ be an IFS and let F_w be the attractor of W. Define the **address map** $\Phi \colon \Sigma_n \to F_w$ by

$$\Phi(s_0 s_1 s_2 \ldots) = \bigcap_{i=0}^{\infty} w_{s_0} \circ w_{s_1} \circ \cdots \circ w_{s_i}(F_w)$$

Theorem 9.4. *Let $W = [w_0, w_1, \ldots, w_{n-1}]$ be an IFS.*

1. *$\Phi(s)$ is a single point in F_w for any $s = s_0 s_1 s_2 \ldots \in \sum_n$.*
2. *Φ is onto.*
3. $\lim_{i\to\infty} w_{s_0} \circ w_{s_1} \circ \cdots \circ w_{s_i}(F_w) = \{\Phi(s)\}$ *in* (K_n, d_H)
4. $\lim_{i\to\infty} w_{s_0} \circ w_{s_1} \circ \cdots \circ w_{s_i}(x) = \Phi(s)$ *for any* $x \in \mathbb{R}^m$ *in* $(\mathbb{R}^m, d_{\text{Euc}})$

Definition 9.5. The sequence s is called an **address** of the point $\Phi(s)$ in the attractor.

Definition 9.6. Let W be an IFS and F_w its attractor. Then W is said to be **totally disconnected** if and only if every point in F_w has a unique address, i.e., if and only if Φ is bijective.

How big is the Attractor?

Definition 9.7. Let $S \in K_2$ be a compact set. Then the **diameter** of S is the real number

$$\operatorname{diam}(S) = \sup \{ d_{\text{Euc}}(x, y) : x, y \in S \}$$

Example 9.8. The diameter of a circle is a special case of this definition.

Remark. You will prove the following for homework: Let $W = [w_0, \ldots, w_{n-1}]$ be an IFS and $c_0, \ldots, c_{n-1}$ the contraction factors of $w_0, \ldots, w_{n-1}$ respectively. Let $c = \max\{c_0, \ldots, c_{n-1}\}$ and let $a = \Phi(t_1, \ldots, t_m, t_{m+1}, \ldots)$ and $b = \Phi(t_1, \ldots, t_m, t'_{m+1}, \ldots)$ then

$$d_{\text{Euc}}(a, b) \leq c^m \operatorname{diam}(F_W)$$

i.e., if two points have addresses that agree in the first m digits, then they will be no further than $c^m \operatorname{diam}(F_W)$ apart.

Theorem 9.9 (Monks). *Let $W = [w_0, \ldots, w_n]$ be an IFS, $c_0, \ldots, c_n$ the contraction factors of $w_0, \ldots, w_n$ respectively, and $q_0, \ldots, q_n$ the fixed points of $w_0, \ldots, w_n$ respectively. Define $c = \max\{ c_0, \ldots, c_n \}$ and $r = \max\left\{ d(q_i, q_j) : i, j \in \mathbb{O}_n \right\}$. Then for any $a \in F_W$ and any $i \in \mathbb{O}_n$*

$$d_{\mathrm{Euc}}(a, q_i) \leq \frac{1}{1-c} r$$

Corollary 9.10. $F_W \subseteq \bigcap_{i=0}^{n} \overline{B}(q_i; \frac{r}{1-c})$

9.1 The Shift Map

Definition 9.11. Let $W = [w_0, \ldots, w_{n-1}]$ be a totally disconnected IFS. Define $\sigma \colon F_W \to F_W$ by $\sigma(\Phi(s_0 s_1 s_2 \ldots)) = \Phi(s_1 s_2 s_3 \ldots)$. Then σ is called the **shift map** on F_W.

Theorem 9.12. *A shift map is chaotic!*

9.2 The Address Method

Algorithm 9.13. To draw the attractor of an IFS, $W = [w_0, \ldots, w_n]$, start with a fixed point, q, of one of the maps $w_0, \ldots, w_n$. Use the preceeding theorem and remark to determine m such that any two points with addresses that agree on the first m digits will be less than a pixel width apart. For each finite sequence $t_1, \ldots, t_m$ in $\mathbb{O}_n^m$ plot the point

$$w_{t_1} \circ w_{t_2} \circ \cdots \circ w_{t_m}(q)$$

Remark. There are $(n+1)^m$ sequences in $\mathbb{O}_n^m$ so that sometimes this method may be limited by the number of points you can compute and plot.

9.3 The Random Iteration Method

Remark. If we choose $t_1, t_2, t_3, \ldots$ at random from $\mathbb{O}_n$, then it is very likely that every finite sequence of any given length will eventually occur as a subsequence of our choices.

Algorithm 9.14. To draw the attractor of an IFS, $W = [w_0, \ldots, w_n]$. Start with a fixed point, q, of one of the maps $w_0, \ldots, w_n$. This is your current point.

1. Choose a random number i from $\mathbb{O}_n$ and plot w_i of the current point. This becomes the new current point.
2. Iterate!

Remark. You can actually start with any point you like, not necessarily a fixed point, but starting with the fixed point guarentees that all points you plot will be in the attractor, not just near to it after a sufficient number of iterations.

Problems - Chaos Game

9.1. (2 points each) Let $W = [w_0, \ldots, w_{n-1}]$ be an IFS and $c_0, \ldots, c_{n-1}$ the contraction factors of $w_0, \ldots, w_{n-1}$ respectively. Let $a, b \in \mathbb{R}^k$ and $c = \max\{c_0, \ldots, c_{n-1}\}$. Let $t_1, \ldots, t_m \in \{0, 1, \ldots, n-1\}$ and define $J = w_{t_1} \circ w_{t_2} \circ \cdots \circ w_{t_m}$.

(a) Prove that $d_{\text{Euc}}(J(a), J(b)) \le c^m d_{\text{Euc}}(a, b)$.

(b) Now let $a = \Phi(t_1, \ldots, t_m, t_{m+1}, \ldots)$ and $b = \Phi(t_1, \ldots, t_m, t'_{m+1}, \ldots)$. Then

$$d_{\text{Euc}}(a, b) \le c^m \operatorname{diam}(F_W)$$

i.e. if two points have addresses that agree in the first m digits, then they will be no further than $c^m \operatorname{diam}(F_W)$ apart.

9.2. (1 point each) Define a *Generalized Chaos Game* as follows. Let $p_0, \ldots, p_{n-1}$ be points in $\mathbb{R}^2$ and $c_0, \ldots, c_{n-1}$ numbers in $(0..1)$. Then $\text{ChaosGame}([p_0, c_0], [p_1, c_1], \ldots, [p_{n-1}, c_{n-1}])$ represents the chaos game in which the current point is moved c_i of the distance towards a randomly selected goal point p_i. For example, in this notation the standard Chaos Game is $\text{ChaosGame}([(0,0), 0.5], [(0,1), 0.5], [(1,0), 0.5])$.

(a) Find an IFS that produces the same attractor as $\text{ChaosGame}([p_0, c_0], [p_1, c_1], \ldots, [p_{n-1}, c_{n-1}])$ (in terms of the p_i and c_i).

(b) Prove that p_i is the fixed point of the i^{th} affine map in the IFS you found in part a.

(c) Plot the attractor of $\mathit{ChaosGame}\left([e^{2\pi i}, 0.4], [e^{2\pi i/5}, 0.5], [e^{4\pi i/5}, 0.6], [e^{6\pi i/5}, 0.7], [e^{8\pi i/5}, 0.8]\right)$. Note: the ChaosGame command in my Maple package does not support multiple ratios c_i so it can't be used to do this.

(d) Find a generalized Chaos Game that produces the Sierpinski Carpet, i.e.,

$$\text{HeeBGB}(Up, Up, Up, Up, none, Up, Up, Up, Up)$$

and express your answer in the form $\text{ChaosGame}([p_0, c_0], [p_1, c_1], \ldots, [p_{n-1}, c_{n-1}])$.

(e) Use the ChaosGame in the chaos package to verify that your game works (note: the syntax is different... see the `chaos` package help file for details).

9.3. (1 point each) Let $W = [w_0, w_1, w_2] = \text{HeeBGB}(Rt, -Up, -Rt, none)$.

(a) Plot F_w. (Note: Turn on boxed style axes before printing.)

(b) Compute the fixed points of w_0, w_1, and w_2 analytically.

(c) Plot the fixed points found in part b on the plot of F_w.

(d) Compute the exact coordinates of $\Phi(01\overline{12})$.

(e) Plot $\Phi\left(01\overline{12}\right)$ on the plot of F_w.

(f) Indicate all of the points on the attractor which have an address starting with 201.

(g) Does $f\left(0.t_1 t_2 \ldots_{(3)}\right) = \Phi(t_1 t_2 \ldots)$ define a function for this IFS? If yes, explain why and state whether or not f is continuous. If no, find a number $r \in [0..1]$ such that $r = 0.r_1 r_2 \ldots_{(3)} = 0.s_1 s_2 \ldots_{(3)}$ but $\Phi(r_1 r_2 \ldots) \ne \Phi(s_1 s_2 \ldots)$.

9.4. Suppose your computer screen has a resolution of 800×600 pixels and you want to plot the attractor of

$$W = [\text{Affine}(0.9, 0.9, 45, 45, 0, 0), \text{Affine}(0.8, 0.8, 0, 0, 0.8, 0)]$$

(a) (3 points) What rectangular region of the plane should you display on the screen so that you are guaranteed that F_W will be entirely visible (and hopefully as large as possible)? Defend your answer.

(b) (2 points) Using any point on the screen as a seed and the Address Method for producing the fractal, what is the minimum length addresses should you use in order to be guaranteed that the results will be accurate to within the size of one pixel width?

(c) (1 point) Given your answer to part b, how many addresses would you need to generate and how many points would you have to plot in order to produce the attractor by the Address Method?

10 Applications

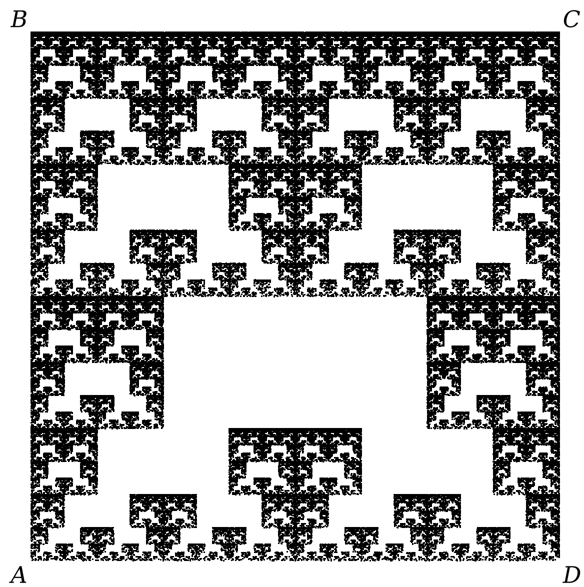
10.1 Fractal Randomness Testing

Algorithm 10.1. Given a sequence of values s whose terms are in $\mathbb{O}_3$, draw the attractor of HeeBGB(Up, Up, Up, Up) by the random iteration method, using s as the source of the "random" numbers.

Remark. Since the attractor of HeeBGB(Up, Up, Up, Up) is the filled-in unit square, if the sequence is missing any addresses there will be holes in the attractor where the points having those addresses would be.

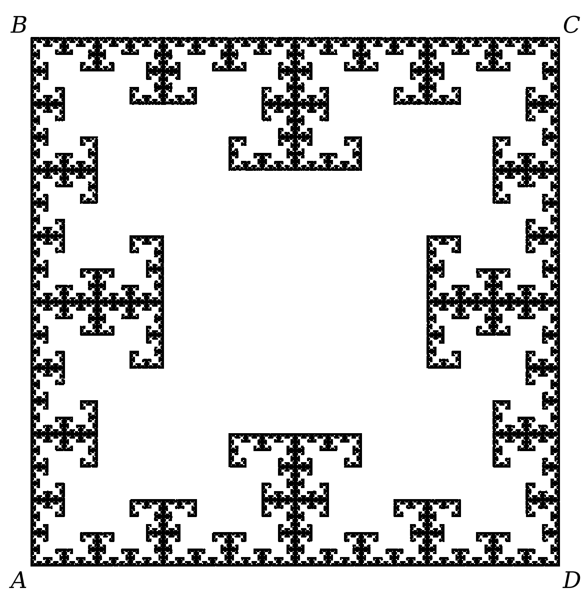
Problems - Fractal Data Analysis

10.1. (1 point) A radio astronomer (Jodie Foster) has a 10,000 character sequence of the letters A,B,C,D which she obtained from signals she received from outer space. The sequence appears to be random, but she decides to test it using the chaos game. She plots four points labeled A, B, C, and D at the four corners of a square and starting at the point A, she looks at the first letter in her sequence and moves half way towards the corner with that label. She then reads the next letter in the sequence and moves half way to the next corner, and so on in the usual manner for playing the chaos game. If the sequence was truly random, playing this game should fill in the square more or less uniformly. However, after plotting the 10,000 points from her sequence she obtained the following picture:



What does this picture indicate about the sequence of letters? In particular, what subsequences never appear in this sequence?

10.2. (1 point) Suppose you ran a fractal data analysis on your data and obtained the following image. What does that tell you about the data? In particular what finite addresses are missing from the data?



10.3. (up to bonus 3 points added to homework grade) Find some interesting source of data that can be grouped into four sets of data uniformly, and test it for randomness using the four corner chaos game. You should have at least 1000 data points to obtain a reasonable picture. Discuss your results intelligently.

10.2 Fractal Curves

Definition 10.2. Let b be an integer greater than 1. Then the **Base** b **Ruler IFS** is $W = [w_0, w_1, \ldots, w_{b-1}]$ where $w_i \colon \mathbb{R} \to \mathbb{R}$ by

$$w_i(x) = \frac{1}{b}x + \frac{i}{b}$$

for $i \in \mathbb{O}_{b-1}$.

Theorem 10.3. *The attractor of any Base b Ruler IFS is* $[0..1]$*. For each* $t \in [0..1]$*,* $t = 0.t_1t_2t_3\cdots_{(b)}$ *if and only if* $t = \Phi(t_1t_2t_3\cdots)$*, i.e., the digits in the base b representation of t and an address of t are the same sequence.*

Example 10.4. Determine the points in the Middle Thirds Cantor Set.

Example 10.5. Determine the coordinates of all points in the right Sierpinski triangle.

Theorem 10.6. *Let* $W = [w_0, \ldots, w_{n-1}]$ *be an IFS and define*

$$f_\Phi(0.t_1t_2t_3\cdots_{(n)}) = \Phi(t_1t_2t_3\cdots)$$

Then f_Φ *will be a function from* $[0..1] \to F_w$ *if and only if* $\Phi(s_1s_2\cdots) = \Phi(t_1t_2\cdots)$ *whenever* $0.s_1s_2s_3\cdots_{(n)} = 0.t_1t_2t_3\cdots_{(n)}$.

Theorem 10.7. *If* f_Φ *is a function, then it is continuous.*

Theorem 10.8. ***(Barnsley):*** *Let* $W = [w_0, \ldots, w_{n-1}]$ *be an IFS and* F_w *its attractor. If there exist distinct points* $\{(s_i, t_i) \in F_w : i \in \{0 \ldots n\}\}$ *such that for* $i \in \mathbb{O}_{n-1}$

1. $w_i(s_0, t_0) = (s_i, t_i)$ *and*
2. $w_i(s_n, t_n) = (s_{i+1}, t_{i+1})$

then the map f_Φ *is a continuous map from* $f \colon [0..1] \to F_w$.

Example 10.9. See Lecture Examples Maple worksheet.

Problems - Address my IFS

10.4. (1 point each)

(a) Convert $\frac{1}{5}$ to base 2.

(b) Plot $\frac{1}{5}$ on a base 2 ruler.

(c) How many different addresses does $\frac{1}{5}$ have with respect to the base 2 ruler IFS?

10.5. (1 point each) Determine if the following numbers are in the MTC by converting them to base 3. Explain.

(a) $\frac{2}{9}$

(b) $\frac{23}{27}$

(c) $\frac{13}{81}$

(d) 0.7

10.6. (2 points) Find an IFS that parameterizes the Sierpinski Triangle. Then use the DrawIFSCurve command in the Maple chaos package to sketch approximations to the curve using 3, 3^2, 3^3, and 3^4 subdivisions of the unit interval. [Hint: Map MrFace's "chin"to the left side of $w_0(F_W)$, the bottom of $w_2(F_W)$, and the "hypotenuse"of $w_1(F_W)$.]

10.7. (1 point) Let $P\colon [0..1] \to [0..1] \times [0..1]$ be the parameterization of the Peano curve. Find $P\left(\frac{1}{2}\right)$, $P\left(\frac{31}{81}\right)$, and $P\,(0.7)$.

10.8. (1 point each) Determine if the following points are in the Right Sierpinski Triangle by converting them to base 2. Explain.

(a) $\left(\frac{11}{32}, \frac{41}{64}\right)$

(b) $\left(\frac{1}{3}, \frac{1}{6}\right)$

(c) $\left(\frac{1}{3}, \frac{1}{5}\right)$

(d) $\left(\frac{1}{2}, \frac{1}{2}\right)$

(e) $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{4}\right)$

10.9. (1 point) Plot the points in the previous problem on the Sierpinski Triangle in Maple to verify your answers.

10.3 Fractal Interpolation

Definition 10.10. A set of **data** is a collection $\left\{ (x_i, y_i) \in \mathbb{R}^2 : i \in \{0, 1, \ldots, n\} \text{ and } x_0 < x_1 < \cdots < x_n \right\}$. An **interpolation function** for a given set of data is a continuous map $f\colon [x_0..x_n] \to \mathbb{R}$ such that $f(x_i) = y_i$ for all $i \in \{0, 1, \ldots, n\}$, i.e., its graph passes through all of the data points.

Example 10.11. Linear interpolation, cubic spline, polynomial, etc.

Definition 10.12. Let $(x_0, y_0), \ldots, (x_n, y_n)$ be a set of data with $n > 2$. Then a **Barnsley Interpolation Function** is an IFS $W = [w_0, w_2, \ldots, w_{n-1}]$ such that for all $i \in \mathbb{O}_{n-1}$,

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ c_i & d_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix}$$

and

$$w_i \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \text{ and } w_i \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix}$$

Remark. Simply put, we connect the data points with the chins of Mr Face, keeping the sides of his head vertical. The seed is the cousin of Mr Face whose chin connects the first and last data points. We choose the d_i's with $|d_i| < 1$ to vary the fractal dimension (ruggedness) of the interpolation graph. Choosing $|d_i| < 1$ guarentees that our function is a contraction mapping.

Theorem 10.13. *Let W be the IFS in the previous definition. Then for each $i \in \mathbb{O}_{n-1}$,*

$$w_i = \text{affine}\left(\frac{x_{i+1} - x_i}{x_n - x_0}, 0, \frac{y_{i+1} - y_i}{x_n - x_0} - d_i \frac{y_n - y_0}{x_n - x_0}, d_i, \frac{x_n x_i - x_0 x_{i+1}}{x_n - x_0}, \frac{x_n y_i - x_0 y_{i+1}}{x_n - x_0} - d_i \frac{x_n y_0 - x_0 y_n}{x_n - x_0}\right)$$

Furthermore, F_W is the graph of an interpolation function.

Problems - Fractal Interpolation

10.10. (3 points) Consider the data

$$\{(0,0), (2,4), (3,6), (5,3), (6,7)\}$$

Make a Barnsley interpolation function for the given data that has Hausdorff dimension:

(a) 1.0

(b) 1.2

(c) 1.4

(d) 1.6

(e) 1.9

and plot the graph of each function. Show your calculations which verify that the function has the requested dimension.

10.11. (8 bonus points) Write a Maple program that will compute the value of $f(x)$ for a given x, where f is a fractal interpolation function. (Note that this is not the same as computing $p(t)$ where p is a parameterization of the graph of f given by $p(0.t_1 t_2 \ldots_{(n)}) = \Phi(t_1 t_2 \ldots)$ as is done by my `IFSCurve` Maple procedure.) For example, if (x_i, y_i) is one of the original data points then $f(x_i) = y_i$ and your procedure should give the value of $f(x)$ for values of x which are in between the x values of data points for the given interpolation function as well. Give examples showing that your function actually works.

11 Dimension

11.1 Topological Dimension

Topological Background

Definition 11.1. Let (X, d) and (Y, d') be metric spaces. Then (X, d) is said to be **homeomorphic** to (Y, d') if $\exists, f : X \to Y$ such that f is bijective, continuous, and its inverse is continuous. In this case, f is said to be a **homeomorphism**.

Definition 11.2. Any property of a metric space which is preserved by homeomorphisms is called a **topological invariant**, i.e., if P is a topological invariant, and if (X, d) and (Y, d') are homeomorphic, then $P(X, d) \Leftrightarrow P(Y, d')$.

Definition 11.3. Let (X, d) be a metric space and $U \subseteq X$. Then the **interior** of U is

$$U^\circ = \{\, x \in U : \exists \delta > 0, B(x; \delta) \subseteq U \,\}$$

Also, the **boundary** of U is $\partial U = X - U^\circ - (X - U)^\circ$, i.e., we take away the interior of the set and the interior of the complement to get the boundary.

Theorem 11.4. *U° is open for any set.*

Theorem 11.5. *A set S is open $\Leftrightarrow S = S^\circ$.*

Definition 11.6. Let $U \subseteq \mathbb{R}^m$. Define the **topological dimension** of U, denoted $\dim_T(U)$, to be the integer given by

1. $\dim_T(\phi) = -1$ and ϕ is the only subset A of $\mathbb{R}^m$ for which $\dim_T(A) = -1$.
2. $\dim_T(U) \le n$ if and only if for any $x \in U$ and any open set $W \subseteq U$ containing x, there exists an open set V with $x \in V \subseteq W$ such that the topological dimension $\dim_T(\partial V) \le n - 1$. [Note: V, W must be open in U as a metric subspace of $\mathbb{R}^m$, but not necessarily open as a subset of $\mathbb{R}^m$. Similary, ∂V refers to the boundary of V in U, not in $\mathbb{R}^m$.]

Then $\dim_T(U) = n$ if and only if $\dim_T(U) \le n$ but $\dim_T(U) \nleq n - 1$.

Remark. The value of $\dim_T(U)$ is always an integer by definition.

11.2 Hausdorff Dimension

Example 11.7. Consider the following

Figure:	a point	unit segment	Sierpinski triangle	unit square	unit cube
# of pts:	1	∞	∞	∞	∞
Length:	0	1	∞	∞	∞
Area:	0	0	0	1	∞
Volume:	0	0	0	0	1

Notice that none of these standard measures give a nontrivial value for the Sierpinski triangle.

Definition 11.8. Let $\varepsilon \in \mathbb{R}^+, 0 \leq p < \infty$, and $A \subseteq \mathbb{R}^2$ a bounded subset of the plane. Then

$$M_p(A;\varepsilon) = \inf\left\{ \sum_{i=1}^{\infty} \operatorname{diam}(A_i)^p : A = \cup_{i=1}^{\infty} A_i \text{ and } \forall i, \operatorname{diam}(A_i) < \varepsilon \right\}$$

and

$$M_p(A) = \sup\left\{ M_p(A;\varepsilon) : \varepsilon \in \mathbb{R}^+ \right\}$$

We call $M_p(A)$ the Hausdorff p-measure of A.

Remark. The expression $M_p(A)$ can equal ∞. Furthermore, $M_p(A;\varepsilon)$ is obviously a nonincreasing function of ε, so $M_p(A)$ is the same as $\lim_{\varepsilon\to 0} M_p(A;\varepsilon)$.

Theorem 11.9. *For each A, $\exists d \in \mathbb{R}$ such that $M_p(A) = \infty$ for $p < d$ and $M_p(A) = 0$ for $p > d$.*

Definition 11.10. The number d in the previous theorem is called the **Hausdorff dimension of** A, and written $d = \dim_H(A)$.

Similarity and Congruence

Definition 11.11. Let (X,d) be a metric space. A **similitude** (or **similarity map**) is a surjective map $f\colon X \to X$ such that

$$\exists c \in \mathbb{R}^+, \forall x, y \in X, d(f(x), f(y)) = cd(x,y)$$

In this case c is called the **similarity factor** (or **scaling factor** or **ratio of similarity**).

Definition 11.12. Let (X,d) be a metric space and $A, B \subseteq X$. Then A is **similar** to B if $B = f(A)$ for some similitude f.

Theorem 11.13. *A function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a similitude with scaling factor c if and only if $f = \operatorname{Affine}(r,s,\theta,\phi,e,f)$ with $|r| = |s| = c$ and $\theta = \phi + \pi k$ for some $k \in \mathbb{Z}$.*

Theorem 11.14. *If f is a similitude with scaling factor c, then f is bijective and f^{-1} is a similitude with scaling factor $1/c$.*

Definition 11.15. Let (X, d) be a metric space. Then $f: X \to X$ is called an **isometry** if and only if f is a similitude with scaling factor equal to 1.

Remark. In other words f is an isometry if and only if

1. f is surjective and
2. $\forall x, y \in X, d(f(x), f(y)) = d(x, y)$,

i.e., f preserves all distances.

Definition 11.16. Let (X, d) be a metric space and $A, B \subseteq X$. Then A is **congruent** to B if $B = f(A)$ for some isometry f.

Corollary 11.17. *If f is an isometry, then f is bijective and f^{-1} is an isometry.*

Theorem 11.18. *A map $f: R^2 \to R^2$ is an isometry if and only if $f = \text{Affine}(r, s, \theta, \phi, e, f)$ with $|r| = |s| = 1$ and $\theta = \phi + \pi k$ for some $k \in \mathbb{Z}$.*

Self-similarity and Dimension

Definition 11.19. Let $W = [w_0, w_1, \ldots, w_{n-1}]$ be an IFS and F_w its attractor. Then W is said to be **just touching** if and only if it is not totally disconnected and there exists an open set $U \in \mathbb{R}^2$ such that

1. $W(U) \subseteq U$ and
2. $w_i(U) \cap w_j(U) = \phi$ for all $i, j \in [0, 1, \ldots, n-1]$ for $i \neq j$.

Definition 11.20. Let W be an IFS. Then W is said to be **overlapping** if and only if it is not totally disconnected and not just touching.

Example 11.21. The middle thirds Cantor set is totally disconnected.

Example 11.22. The Sierpinski triangle is just touching.

Example 11.23. A trivial example of an overlapping IFS is an IFS containing the same map twice.

Definition 11.24. Let $W = [w_0, w_1, \ldots, w_{n-1}]$ be an IFS. F_w is said to be **self-similar** if and only if F_w is not overlapping and each w_i is a similarity. F_w is **strictly self-similar** if and only if the similarity factors are all equal.

Definition 11.25. If F_w is a strictly self-similar attractor of a non-overlapping IFS $W = [w_1, \ldots, w_N]$, then the **similarity dimension** of F_w is defined to be the unique number d such that $N = (\frac{1}{r})^d$ where r is the similarity factor (= contraction factor for the affine maps). We write $\dim_S(F_w) = d$ in this case.

Remark. If we solve the defining equation for d we obtain

$$\dim_S(F_w) = \frac{\ln N}{\ln\left(\frac{1}{r}\right)}$$

Theorem 11.26. *For attractors of non-overlapping IFS's,* $\dim_S(F_w) = \dim_H(F_w)$.

Theorem 11.27. *(Moran) Let F_w be the attractor of a non-overlapping IFS, $W = [w_0, w_1, \ldots, w_{n-1}]$ such that each w_i is a similitude with similarity factor c_i respectively. Then* $\dim_H(F_w)$ *is the unique number d such that*

$$c_0^d + c_1^d + \cdots + c_{n-1}^d = 1$$

If W is overlapping then $\dim_H(F_w) \leq d$.

Theorem 11.28. *Let W be the Barnsley interpolation function IFS given in the definition. If $\sum_{k=0}^{n-1} |d_k| > 1$ and the interpolation points do not lie on a straight line, then the fractal dimension of F_W is the unique real number D such that*

$$\sum_{k=0}^{n-1} |d_k| a_k^{D-1} = 1$$

Approximating the Hausdorff Dimension

Grid Dimension

Algorithm 11.29. To estimate the Grid dimension of a shape:

1. Cover the shape with grids of size $l_1, l_2, \ldots, l_k$.
2. For each grid, count the number N_i of grid boxes whose interior intersects the shape.
3. Plot $\ln(N_i)$ vs. $\ln(\frac{1}{l_i})$ and compute the least squares linear regression line through the points $\left\{(\ln\left(\frac{1}{l_1}\right), \ln(N_1)), \ldots, (\ln\left(\frac{1}{l_k}\right), \ln(N_k))\right\}$.
4. The slope is an estimate of the Grid dimension (and is also an estimate of the Hausdorff dimension).

Remark. The slope is independent of the units used.

Problems - Dimension

11.1. (3 points) Prove that the topological dimension of a metric space having two points is zero.

11.2. (4 points) Prove that the Hausdorff dimension of a metric space having two points is zero.

11.3. (1 point each) What is the topological dimension of the following shapes. You don't have to prove your answer, but give an informal explanation.

(a) $\{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\}$.

(b) the Sierpinski Carpet (not triangle or gasket!)

(c) the attractor of GridIFS($Up, n, Up, n, n, n, Up, n, n$)

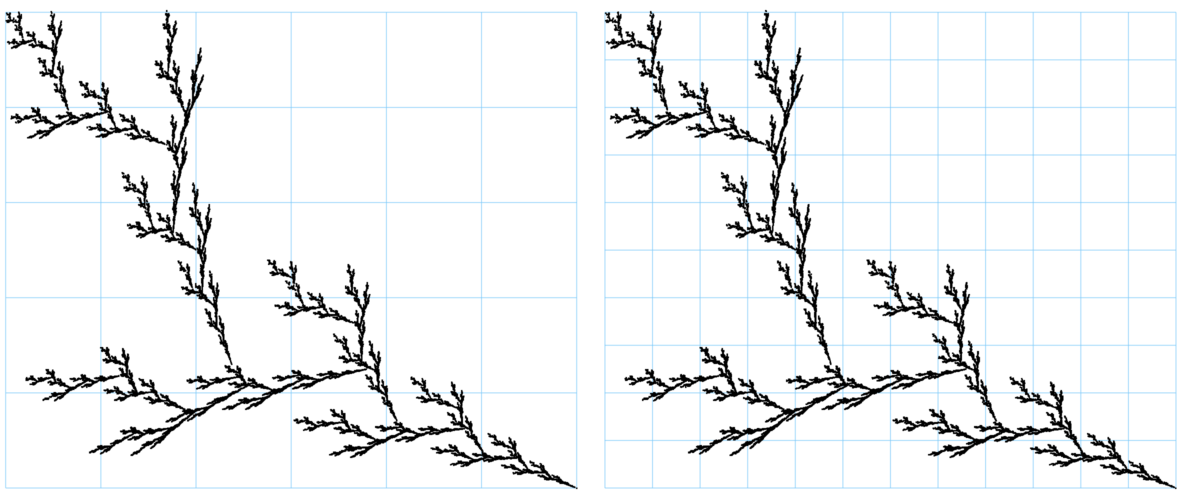
(d) the Peano curve

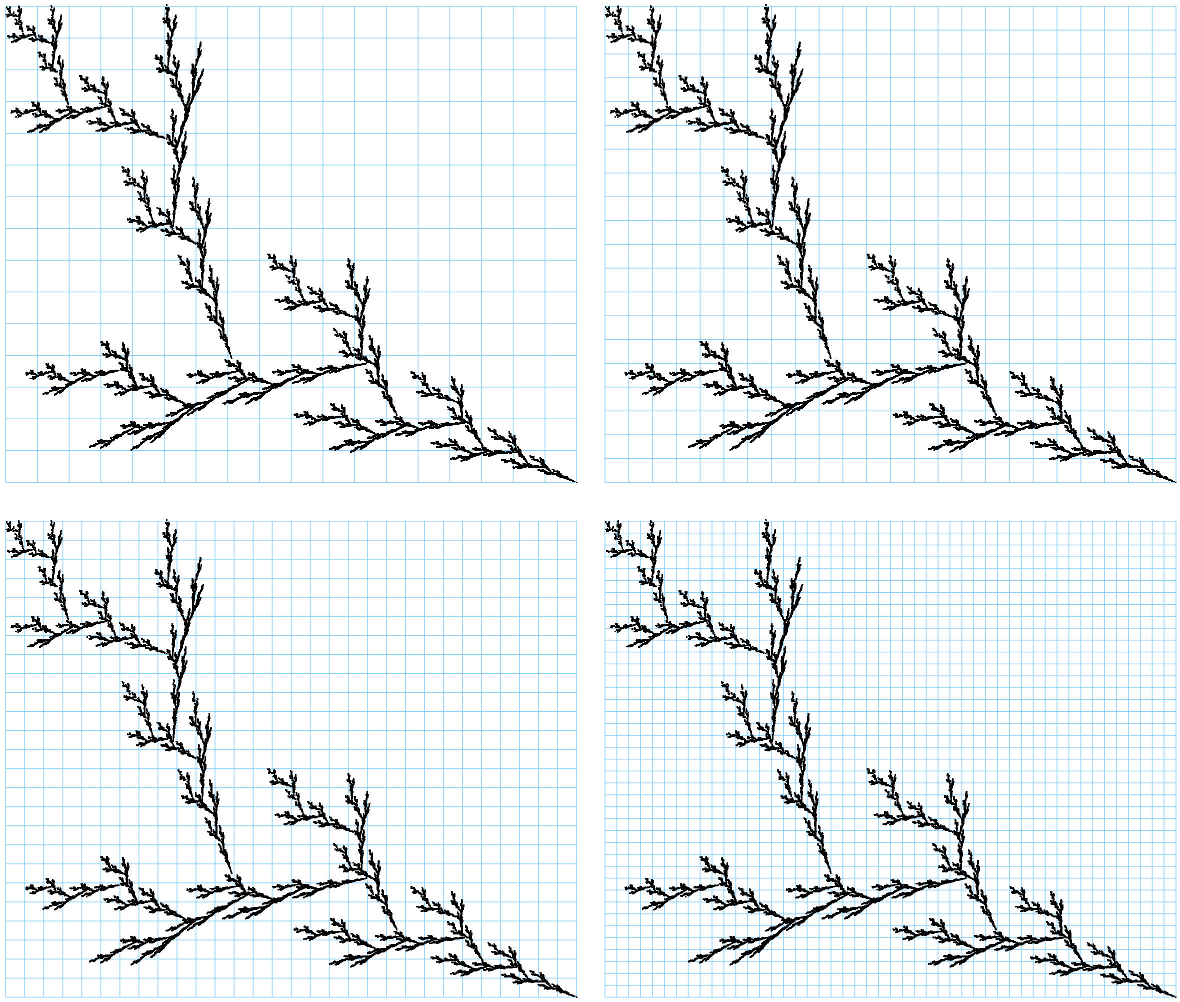
11.4. (4 points) Prove that every isometry is bijective and its inverse is also an isometry.

11.5. (3 points) Prove that affine$(r, s, \theta, \phi, e, f)$ is an isometry if and only if $|r| = |s| = 1$ and $\theta = \phi + \pi k$ for some $k \in \mathbb{Z}$.

11.6. (1 point each) Determine the similarity dimension of the fractals in problem 8.16 parts e, f, g, and h in the *Guess My IFS* section above. Give exact answers where possible, and in every case give a decimal approximation to at least 10 digits of accuracy.

11.7. (3 points) Compute the grid dimension of the following fractal. Use all of the data available with a least squares linear regression, don't just use two of the grids. Plot the data points and the regression line on the same set of axes using Maple. See ?stats[fit] and then click on leastsquare for examples of doing a least square regression analysis in Maple.





11.8. (4 points) Prove that $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a similitude with scaling factor c if and only if $f = \text{Affine}(r, s, \theta, \phi, e, f)$ with $|r| = |s| = c$ and $\theta = \phi + \pi k$ for some $k \in \mathbb{Z}$.

12 Complex Fractals

12.1 Julia Sets

Definition 12.1. If $z \in \mathbb{C}$ and $z = re^{i\theta}$ where $r = |z|$ and $\theta \in [0..2\pi]$ then $\sqrt{z} = z^{\frac{1}{2}}$ is $(re^{i\theta})^{\frac{1}{2}} = r^{\frac{1}{2}}e^{\frac{i\theta}{2}}$. Here $\sqrt{z}$ is the **principal square root** of a complex number.

Example 12.2. Let $c \in \mathbb{C}$. Let $w_0(z) = \sqrt{z-c}$, $w_1(z) = -\sqrt{z-c}$, and $W = [w_0, w_1]$. Since w_0, w_1 are not affine, W is not an IFS.

Is W a Hutchinson operator? Not quite. But it behaves like a Hutchinson Operator in the sense that $\exists \gamma \subseteq \mathbb{R}^2$ such that for any $A \in K_2$, $W^n(A \cap (\mathbb{R}^2 - \gamma))$ converges to a unique set F_w such that $W(F_w) = F_w$.

Definition 12.3. That unique set F_w is called the **Julia Set** associated with c and is denoted J_c.

Definition 12.4. Let $c \in \mathbb{C}$. Let $Q_c(z) = z^2 + c$. The **filled in Julia Set**, K_c, is

$$K_c = \{\, z : \text{the } Q_c \text{ orbit of } z \text{ is bounded } \}$$

Facts about Julia Sets:

1. $J_c = \partial K_c$
2. $J_c = K_c$ if J_c is totally disconnected
3. All Julia Sets fit into a closed ball of radius two centered at the origin, i.e., if $\left| Q_c^n(z) \right| > 2$ for any n then $z \notin K_c$.

Remark. For each $c \in \mathbb{C}$, there is a Julia Set J_c.

Theorem 12.5. *J_c is connected if and only if the Q_c orbit of 0 is bounded, i.e., if and only if 0 is in the filled in Julia Set K_c.*

12.2 The Mandelbrot Set

Definition 12.6. The **Mandelbrot Set**, M, is the set of all $c \in \mathbb{C}$ such that J_c is connected. Because of the previous theorem, we write

$$M = \{\, c \in \mathbb{C} : \text{the } Q_c\text{-orbit of 0 is bounded} \,\}$$

Remark. While there are infinitely many Julia Sets, there is only one Mandelbrot Set.

Facts about the Mandelbrot Set:

1. It is symmetric with respect to the x-axis.
2. It is connected.
3. Every open set containing any point on the boundary of M contains infinitely many "baby M's" . Note: The babies are not similiar to M.
4. K_c "looks like" M near c.
5. Every bulb, B, has the property that $\exists n \in \mathbb{N}^+$ such that $\forall c \in B$, the Q_c orbit of 0 converges to an n-cycle. The integer n is called the **period of the bulb**.

12.3 The Escape Time Algorithm

Algorithm 12.7. Assign to each screen pixel a representative complex number and color the pixel based on the number of iterations it required for a term in the Q_c-orbit to have absolute value greater than 2 (or some particular color if no such term is obtained after a predetermined number of iterations).

Problems - Complex Fractals

12.1. (1 point) For each of the following points $c \in \mathbb{C}$, determine if z is in the Mandelbrot set.

(a) 0.2

(b) $-0.75 - 0.1i$

(c) $-1 - i$

(d) $-i$

(e) $-0.919 + 0.248i$

12.2. (1 point each) For each of the values c in the previous problem, state whether or not the filled in Julia set K_c is connected. Plot K_c to verify your answer. Note: It takes a long time to plot these with Maple. They also appear rather small, so you might want to check the `File/Preferences/Plotting/Plot Display/Window` menu option before plotting to make them a little bigger. If you want to make really big ones you can export the image to a jpg or gif. See the examples at the bottom of the `?chaos` help screen if you want to do this. Do not send me enormous pictures via email! Print them out!

12.3. (4 points) Prove that the Mandelbrot set is symmetric with respect to the x-axis.

12.4. (8 points) *The Great Bulb Hunt*. Each of the "bulbs" (i.e. solid black regions) of the Mandelbrot set are characterized by the fact that there exists a positive integer n such that all points c in a given bulb produce a function $f(z) = z^2 + c$ having the property that the f-orbit of 0 converges to a cycle of period n. We say that this number n is the *period of the bulb*. The main cardioid is the only bulb of period 1. There is 1 bulb of period 2, 3 bulbs of period 3, 6 bulbs of period 4, 15 of period 5, and 27 bulbs of period 6. Your job is to find all the bulbs of period less than or equal to six. Find as many as possible, and your score will be determined by the number you find. Your answer should consists of

(a) A table listing a representative point from each bulb and the period of that bulb.

(b) Maple calculations showing that the given point does indeed produce a function for which the orbit of zero converges to a cycle of the stated period.

(c) A plot of the Mandelbrot set with the locations of the bulbs you found clearly marked similar to Figure 14.19 on page 809 in your book (second edition).

Note that you will only get credit for finding bulbs which are NOT listed on the diagram on page 809, although these should also be listed in your table. Also note that just randomly hunting for such points will take you a very very very very long time, but there is a direct way to find them. It is explained in Chapter 14.1 of the book (in particular on pages 792-796). So I am finally actually testing you on whether you read the book or not!

Part II

Appendix

13 Proofs

13.1 The Power Theorem

Theorem 13.1. *Let $f\colon X \to X$. For any $k, n \in \mathbb{N}$,*

$$f^{k+n} = f^k \circ f^n$$

and

$$f^{kn} = \underbrace{f^n \circ f^n \circ \cdots \circ f^n}_{k\ \text{terms}}$$

(where "0 terms" means the identity map).

Proof.

We proceed by induction on k for arbitrary n.

1.	Let $f\colon X \to X$	
2.	Let $n \in \mathbb{N}$	-

Base case:

3.	$f^{0+n} = f^n$	arithmetic
4.	$= \mathrm{id}_X \circ f^n$	(homework)
5.	$= f^0 \circ f^n$	definition of f^0

Inductive hypothesis:

6.	Let $k \in \mathbb{N}$	-
7.	Assume $f^{k+n} = f^k \circ f^n$	-
8.	$f^{k+1+n} = f^{1+k+n}$	arithmetic
9.	$= f \circ f^{k+n}$	def f^k
10.	$= f \circ (f^k \circ f^n)$	substitution
11.	$= (f \circ f^k) \circ f^n$	$\circ$ is associative (homework)
12.	$= f^{1+k} \circ f^n$	def f^k

This completes the inductive step.

13.	$\forall k \in \mathbb{N}, f^{k+n} = f^k \circ f^n$	by induction

14. $\forall n \in \mathbb{N}, \forall k \in \mathbb{N}, f^{k+n} = f^k \circ f^n$ — $\forall+$

The proof of the second equation is similar.

□

Lemma 13.2. *Let $f\colon X \to X$, $x \in X$, and $n \in \mathbb{N}^+$.*

$$x \text{ has minimum period } n \Rightarrow \#\mathcal{O}_f(x) = n$$

Proof.

1. Let $f\colon X \to X$, $x \in X$, and $n \in \mathbb{N}^+$

($\Rightarrow$)

2.	Assume x has minimum period n	-
3.	$f^n(x) = x$ and $\forall k \in \mathbb{I}_{n-1}, f^k(x) \neq x$	def min period
4.	Define $S = \left\{x, f(x), \ldots, f^{n-1}(x)\right\}$	

Let's show that $\mathcal{O}_f(x) = S$. First we show $\mathcal{O}_F(x) \subseteq S$.

5.	Let $y \in \mathcal{O}_f(x)$	-
6.	$y = f^j(x)$ for some $j \in \mathbb{N}$	def $\mathcal{O}$
7.	$j = nq + r$ and $0 \leq r < n$ for some $q, r \in \mathbb{N}$	division algorithm
8.	$y = f^j(x)$	line 6
9.	$= f^{nq+r}(x)$	substitution
10.	$= f^r(f^{nq}(x))$	Power Thm
11.	$= f^r(f^n(f^n(\cdots f^n(x))))$ (with q f^n's)	Power Thm
12.	$= f^r(x)$	substitution (q times)
13.	$\in S$	def S
14.	$\mathcal{O}_f(x) \subseteq S$	def subset

Now we show $S \subseteq \mathcal{O}_f(x)$

15.	Let $z \in S$	-
16.	$z = f^i(x)$ for some $i \in \mathbb{O}_{n-1}$	def S
17.	$\in \mathcal{O}_f(x)$	def $\mathcal{O}$
18.	$S \subseteq \mathcal{O}_f(x)$	def subset
19.	$\mathcal{O}_f(x) = S$	def set =

Now let's show that the n elements of S are distinct.

20.	Let $a, b \in \mathbb{O}_{n-1}$ and $a \geq b$	-

21.	Assume $f^a(x) = f^b(x)$	-
22.	$f^{a-b}(x) = f^{a-b}(f^n(x))$	substitution
23.	$= f^{n+a-b}(x)$	Power Thm
24.	$= f^{n-b}(f^a(x))$	Power Thm
25.	$= f^{n-b}(f^b(x))$	substitution
26.	$= f^n(x)$	Power Thm
27.	$= x$	substitution
28.	$a - b \notin \mathbb{I}_{n-1}$	by lines 3,22,27
29.	$a - b \in \mathbb{O}_{n-1}$	since $a, b \in \mathbb{O}_{n-1}$
30.	$a - b \in \mathbb{O}_{n-1} - \mathbb{I}_{n-1}$	def relative complement
31.	$= \{0\}$	def $\mathbb{O}_{n-1}$ and $\mathbb{I}_{n-1}$
32.	$a - b = 0$	def set notation
33.	$a = b$	arithmetic
34.	the elements of S are distinct	def distinct
35.	$\#S = n$	def #
36.	$\#\mathcal{O}_f(x) = n$	substitution
37.	x has minimum period $n \Rightarrow \#\mathcal{O}_f(x) = n$	$\Rightarrow +$

□

13.2 Change of Basis

Theorem 13.3. *Let $n \in \mathbb{N}$. If each term in the Base_b-orbit of n is replaced by its value $\bmod b$, the sequence produced will be the base b representation of n (with the least significant digit on the left).*

Proof.

We proceed by induction on n

1.	Let $b \in \mathbb{N}$ and $b > 1$	
2.	Define $\text{Orb} = \text{Orb}_{\text{Base}_b}$	

Base case:

3.	$\text{Base}_b(1) = \frac{1-(1 \bmod b)}{b} = \frac{1-1}{b} = 0$	def of Base_b
4.	$\text{Base}_b(1) = \frac{0-(0 \bmod b)}{b} = \frac{0-0}{b} = 0$	def of Base_b
5.	$\text{Orb}(1) = 1, 0, 0, 0, \ldots$	by definition of orbit

For any sequence s, define $s \bmod b$ to be the sequence whose i^{th} term is $s_i \bmod b$.

6.	$1 = 100000\ldots_{(b)}$	
7.	$= \text{Orb}(1) \bmod b$	-

Inductive hypothesis:

8.	Let $n \in \mathbb{N}^+$	-
9.	Assume $\text{Orb}(m) \bmod b = m_0 m_1 m_2 \ldots_{(b)} = m$ for all $m < n$	-
10.	$n = (n_0 n_1 \ldots n_k)_{(b)}$ for some $n_i \in \mathbb{O}_{b-1}$and some $k \in \mathbb{N}$	the representation theorem

Let's calculate $\text{Base}_b(n)$

11.	$\text{Base}_b(n) = \frac{n-(n \bmod b)}{b}$	
12.	$= \frac{(n_0 n_1 \ldots n_k)_{(b)} - n_0}{b}$	
13.	$= \frac{(n_0 n_1 \ldots n_k)_{(b)} - n_0}{b}$	
14.	$= \frac{(0 n_1 n_2 \ldots n_k)_{(b)}}{b}$	
15.	$= (n_1 n_2 \ldots n_k)_{(b)}$	
16.	$< n$	

Since its less than n the assumption holds for $\text{Base}_b(n)$

17.	$\text{Orb}(\text{Base}_b(n)) \bmod b = n_1 n_2 \ldots n_k \overline{0}$	
18.	$\text{Orb}(n) = n, \text{Orb}(\text{Base}_b(n))$	def of orbit
19.	$\text{Orb}(n) \bmod b = n \bmod b, \text{Orb}(\text{Base}_b(n)) \bmod b$	
20.	$= n_0, n_1 n_2 \ldots n_k \overline{0}$	
21.	$= n_0, n_1 n_2 \ldots n_k \overline{0}_{(b)}$	
22.	$= n$	
23.	$\forall n \in \mathbb{N}, \text{Orb}(n) \bmod b = n_0 n_1 n_2 \ldots_{(b)} = n$	

□

13.3 Triangle Inequality and the Euclidean Metric

Lemma 13.4 (Euclidean Triangle Inequality)**.** *Let $x, y, z \in \mathbb{R}^n$. Then*

$$d_{\text{Euc}}(x, z) \leq d_{\text{Euc}}(x, y) + d_{\text{Euc}}(y, z).$$

Proof.

1.	Let $x, y, z \in \mathbb{R}^n$	
2.	For some $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n \in \mathbb{R}$,	-
3.	$x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n), z = (z_1, \ldots, z_n)$	def of $\mathbb{R}^n$
4.	Define $a = d_{\text{Euc}}(x, z), b = d_{\text{Euc}}(x, y), c = d_{\text{Euc}}(y, z)$	

In this notation we are trying to show that $a \leq b + c$.

We will prove the case where $b \neq 0$ and $c \neq 0$. The other cases are for homework.

5. Define $a_i = x_i - z_i, b_i = x_i - y_i, c_i = y_i - z_i$ for each $i \in \mathbb{I}_n$
6. $\forall i \in \mathbb{I}_n, a_i = b_i + c_i$
7. $a, b, c \geq 0$
8. $a^2 = (d_{\text{Euc}}(x,z))^2$
9. $\quad = (\sqrt{\sum_{i=1}^{n}(x_i - z_i)^2})^2$
10. $\quad = \sum_{i=1}^{n}(x_i - z_i)^2$
11. $\quad = \sum_{i=1}^{n} a_i^2$

A similar argument shows $b^2 = \sum_{i=1}^{n} b_i^2$ and $c^2 = \sum_{i=1}^{n} c_i^2$.

12. $\quad = \sum_{i=1}^{n}(b_i + c_i)^2$
13. $\quad = \sum_{i=1}^{n}(b_i^2 + 2b_ic_i + c_i^2)$
14. $\quad = \sum_{i=1}^{n} b_i^2 + \sum_{i=1}^{n}(2b_ic_i) + \sum_{i=1}^{n} c_i^2$
15. $\quad = b^2 + c^2 + \sum_{i=1}^{n}(2b_ic_i)$
16. $\quad = b^2 + 2bc + c^2 + \sum_{i=1}^{n}(2b_ic_i) - 2bc$
17. $\quad = (b+c)^2 + \sum_{i=1}^{n}(2b_ic_i) - 2bc$
18. $\quad = (b+c)^2 + \sum_{i=1}^{n}(2b_ic_i) - (1+1)bc$
19. $\quad = (b+c)^2 + \sum_{i=1}^{n}(2b_ic_i) - (\frac{b^2}{b^2} + \frac{c^2}{c^2})bc$
20. $\quad = (b+c)^2 + \frac{bc}{bc}\sum_{i=1}^{n}(2b_ic_i) - (\frac{1}{b^2}\sum_{i=1}^{n} b_i^2 + \frac{1}{c^2}\sum_{i=1}^{n} c_i^2)bc$
21. $\quad = (b+c)^2 + (\sum_{i=1}^{n} \frac{2b_ic_i}{bc} - \sum_{i=1}^{n} \frac{b_i^2}{b^2} - \sum_{i=1}^{n} \frac{c_i^2}{c^2})bc$
22. $\quad = (b+c)^2 - (-\sum_{i=1}^{n} \frac{2b_ic_i}{bc} + \sum_{i=1}^{n} \frac{b_i^2}{b^2} + \sum_{i=1}^{n} \frac{c_i^2}{c^2})bc$
23. $\quad = (b+c)^2 - \sum_{i=1}^{n}(\frac{b_i^2}{b^2} - \frac{2b_ic_i}{bc} + \frac{c_i^2}{c^2})bc$
24. $\quad = (b+c)^2 - (\sum_{i=1}^{n}(\frac{b_i}{b} - \frac{c_i}{c})^2)bc$
25. $\quad \leq (b+c)^2$
26. $a^2 \leq (b+c)^2$
27. $a \leq b + c$

□

Theorem 13.5. *$(\mathbb{R}^n, d_{\text{Euc}})$ is a metric space.*

Proof.

1. $d_{\text{Euc}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$
2. Let $x, y, z \in \mathbb{R}^n$ -
3. For some $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n \in \mathbb{R}$, -
4. $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n), z = (z_1, \ldots, z_n)$

Show distances are nonnegative.

5. $d_{\text{Euc}}(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$
6. ≥ 0

Show distances are symmetric.

7. $d_{\text{Euc}}(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$
8. $= \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2}$
9. $= d_{\text{Euc}}(y,x)$

We proved the triangle inequality above.

10. $d_{\text{Euc}}(x,z) \leq d_{\text{Euc}}(x,y) + d_{\text{Euc}}(y,z)$

Show zero distances only occur between a point and itself.

11. $d_{\text{Euc}}(x,x) = \sqrt{\sum_{i=1}^{n}(x_i - x_i)^2}$
12. $= \sqrt{\sum_{i=1}^{n} 0^2}$
13. $= 0$
14. Assume $d_{\text{Euc}}(x,y) = 0$ -
15. $\sum_{i=1}^{n}(x_i - y_i)^2 = \left(\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}\right)^2$
16. $= (d_{\text{Euc}}(x,y))^2$
17. $= 0^2$
18. $= 0$
19. $(x_i - y_i)^2 = 0$ for all $i \in \mathbb{I}_n$
20. $x_i - y_i = 0$ for all $i \in \mathbb{I}_n$
21. $x_i = y_i$ for all $i \in \mathbb{I}_n$
22. $x = (x_1, \ldots, x_n)$
23. $= (y_1, \ldots, y_n)$
24. $= y$
25. $(\mathbb{R}^n, d_{\text{Euc}})$ is a metric space

□

13.4 Contraction Mappings are Continuous

Theorem 13.6. *Every contraction mapping is continuous.*

Proof.

1. Let (X,d) be a metric space and $f\colon X \to X$ a contraction mapping
2. f has contraction factor s for some $s \in (0..1)$

3. Let $U \in X$ be an open set -
4. Let $x \in f^{\text{inv}}(U)$ -
5. $f(x) \in U$
6. $B(f(x);\delta) \subseteq U$ for some $\delta \in \mathbb{R}^+$
7. Let $y \in B(x;\delta)$ -
8. $d(x,y) < \delta$
9. $d(f(x), f(y)) \leq sd(x,y)$
10. $< s\delta$
11. $< \delta$
12. $f(y) \in B(f(x);\delta)$
13. $f(y) \in U$
14. $y \in f^{\text{inv}}(U)$
15. $\forall y \in B(x;\delta), y \in f^{\text{inv}}(U)$
16. $B(x;\delta) \subseteq f^{\text{inv}}(U)$
17. $\exists\delta \in \mathbb{R}^+, B(x;\delta) \subseteq f^{\text{inv}}(U)$ -
18. $\forall x \in f^{\text{inv}}(U), \exists\delta \in \mathbb{R}^+, B(x;\delta) \subseteq f^{\text{inv}}(U)$
19. $f^{\text{inv}}(U)$ is open
20. The inverse image of every open set is open
21. f is continuous

□

13.5 The Derivative and Contraction Maps of $\mathbb{R}$

Theorem 13.7. *Let $I = (a..b) \subseteq \mathbb{R}$ and $f : I \to I$ differentiable on I. If there exists $s \in (0..1)$ such that $\forall x \in I, \left|f'(x)\right| \leq s < 1$, then f is a contraction mapping with contraction factor s.*

Proof.

1. Let $I = (a..b) \subseteq \mathbb{R}$ and $f : I \to I$ differentiable on I, and $d = d_{\text{Euc}}$
2. Assume for some $s \in (0..1), \forall x \in I, \left|f'(x)\right| \leq s < 1$ -
3. Let $x, y \in I$ -
4. $x = y$ or $x \neq y$

Case 1

5. Assume $x = y$ -
6. $d(f(x), f(y)) = d(f(x), f(x))$
7. $= 0$

8. $\leq s \cdot 0$
9. $= s \cdot d(x, x)$
10. $= s \cdot d(x, y)$

Case 2

11. Assume $x \neq y$ -
12. $f'(c) = \frac{f(x)-f(y)}{x-y}$ for some c between x and y
13. $\left| f'(c) \right| \leq s$
14. $d(f(x), f(y)) = \left| f(x) - f(y) \right|$
15. $= \left| f(x) - f(y) \right| \frac{|x-y|}{|x-y|}$
16. $= \left| \frac{f(x)-f(y)}{x-y} \right| \left| x - y \right|$
17. $= \left| f'(c) \right| \left| x - y \right|$
18. $\leq s \left| x - y \right|$
19. $= s \cdot d(x, y)$
20. $\exists s \in (0..1), \forall x, y \in I, d(f(x), f(y)) \leq s \cdot d(x, y)$
21. f is a contraction map with contraction factor s
22. If $\exists s \in (0..1), \forall x \in I, \left| f'(x) \right| \leq s < 1$, then f is a contraction mapping with contraction factor s.

□

13.6 Contraction Mapping Theorem

Theorem 13.8 (The Contraction Mapping Theorem). *Let $f: X \to X$ be a contraction mapping on a complete metric space (X, d) with contraction factor s. Then*

1. *f has a unique fixed point, q,*
2. *the f-orbit of every element of X converges to q (i.e., $\forall x \in X$, $\lim_{n\to\infty} f^n(x) = q$), and*
3. *if $x_0, x_1, x_2, \ldots$ is the f-orbit of $x_0 \in X$ then*

$$d(x_n, q) \leq \frac{s^n}{1-s} d(x_0, x_1)$$

for all $n \in \mathbb{N}$.

Proof.

1. Let (X, d) be a complete metric space.
2. Let $f: X \to X$ be a contraction mapping with contraction factor $s \in (0..1)$.

We want to show that the f-orbit of an arbitrary seed $x_0 \in X$ is a Cauchy sequence.

3. Let $x_0 \in X$ -

4. Define $x_i = f^i(x_0)$ for $i \in \mathbb{N}^+$ -

i.e., $x_0, x_1, x_2, \ldots$ is the f-orbit of x_0

5. Define $\alpha = d(x_0, x_1)$ -

Let's first prove that $d(x_i, x_{i+1}) \leq s^i\alpha$ for all i by induction on i.

Base Case:

6. $d(x_0, x_1) = \alpha \leq s^0\alpha$

Inductive step:

7. Let $i \in \mathbb{N}$ -
8. Assume $d(x_i, x_{i+1}) \leq s^i\alpha$ -
9. $d(x_{i+1}, x_{i+2}) = d(f(x_i), f(x_{i+1}))$
10. $\leq s \cdot d(x_i, x_{i+1})$
11. $\leq s \cdot s^i\alpha$
12. $= s^{i+1}\alpha$
13. $\forall i \in \mathbb{N}, d(x_i, x_{i+1}) \leq s^i\alpha$

That gives us the bound for how far apart consecutive terms in the sequence can be. Now let's apply that to get a bound for aribtrary pairs of terms.

14. Let $m, n \in \mathbb{N}$ with $m \leq n$ -

We will prove it by cases.

15. $m = n$ or $m < n$

Case 1

16. Assume $m = n$ -
17. $d(x_m, x_n) = d(x_m, x_m)$
18. $= 0$
19. $\leq \frac{s^m}{1-s}\alpha$

Case 2

20. Assume $m < n$ -
21. $d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n)$
22. $\leq s^m\alpha + s^{m+1}\alpha + \cdots + s^{n-1}\alpha$
23. $= s^m\alpha(1 + s + s^2 + s^3 + \cdots + s^{n-1-m})$
24. $\leq s^m\alpha(1 + s + s^2 + s^3 + \cdots)$
25. $= s^m\alpha\frac{1}{1-s}$

26. $= \frac{s^m}{1-s}\alpha$

So in both cases we have shown:

27. $d(x_m, x_n) \leq \frac{s^m}{1-s}\alpha$ -

But m, n are arbitrary, so

28. $\forall m, n \in \mathbb{N}, d(x_m, x_n) \leq \frac{s^m}{1-s}\alpha$

Using this, we can now show the sequence is Cauchy.

29. Let $\varepsilon \in \mathbb{R}^+$ -
30. $\lim_{m\to\infty} \frac{s^m}{1-s}\alpha = 0$
31. For some $N \in \mathbb{N}$, $\forall m \geq N$, $\frac{s^m}{1-s}\alpha < \varepsilon$
32. Let $m, n \geq N$ -
33. $d(x_m, x_n) \leq \frac{s^m}{1-s}\alpha$
34. $< \varepsilon$
35. $\exists N \in \mathbb{N}, \forall m, n \geq N, d(x_m, x_n) < \varepsilon$
36. $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N, d(x_m, x_n) < \varepsilon$
37. $x_0, x_1, x_2, \ldots$ is a Cauchy sequence

We can now use completeness to get q.

38. $\lim_{i\to\infty} x_i = q$ for some $q \in X$
39. f is continuous
40. $f(q) = f(\lim_{i\to\infty} x_i)$
41. $= \lim_{i\to\infty} f(x_i)$
42. $= \lim_{i\to\infty} f(f^i(x_0))$
43. $= \lim_{i\to\infty} f^{\,i+1}(x_0)$
44. $= \lim_{i\to\infty} x_{i+1}$
45. $= \lim_{i\to\infty} x_i$
46. $= q$
47. q is a fixed point of f
48. the f-orbit of every element of X converges to a fixed point

Let's show the fixed point is unique.

49. Let $p, q \in X$ -
50. Assume p, q are fixed points of f -
51. $f(p) = p$ and $f(q) = q$

52. $d(p,q) = d(f(p), f(q))$
53. $d(p,q) \le s \cdot d(p,q)$
54. Assume $d(p,q) \ne 0$ -
55. $1 \le s$
56. $1 \not\le s$
57. $\rightarrow\leftarrow$
58. $d(p,q) = 0$
59. $p = q$
60. f has at most one fixed point

Part (2) now follows immediatly from lines 48 and 60

61. $\forall x \in X, \lim_{i\to\infty} f^i(x) = q$

Finally, we will prove part (3) by contradiction.

62. Assume $d(x_n, q) > \frac{s^n}{1-s} d(x_0, x_1)$ for some $x_0 \in X$ and some $n \in \mathbb{N}$ -
63. $d(x_n, q) - \frac{s^n}{1-s} d(x_0, x_1) > 0$
64. Define $\varepsilon_1 = d(x_n, q) - \frac{s^n}{1-s} d(x_0, x_1)$ -
65. $\varepsilon_1 > 0$
66. For some $N_1 \in \mathbb{N}, \forall m > N_1, d(x_m, q) < \varepsilon_1$
67. $\forall m \in \mathbb{N}, d(x_n, x_m) \le \frac{s^n}{1-s} d(x_0, x_1)$
68. Let $m > N_1$ -
69. $d(x_m, q) < \varepsilon_1$ and $d(x_n, x_m) \le \frac{s^n}{1-s} d(x_0, x_1)$
70. $d(x_n, q) \le d(x_n, x_m) + d(x_m, q)$
71. $< \frac{s^n}{1-s} d(x_0, x_1) + \varepsilon_1$
72. $= \frac{s^n}{1-s} d(x_0, x_1) + d(x_n, q) - \frac{s^n}{1-s} d(x_0, x_1)$
73. $= d(x_n, q)$
74. $\rightarrow\leftarrow$
75. $\forall x_0 \in X, \forall n \in \mathbb{N}, d(x_n, q) \le \frac{s^n}{1-s} d(x_0, x_1)$

□

13.7 Hutchinson Operators are Contraction Maps

Theorem 13.9 (Hutchinson). *Let $w_0, w_1, \ldots, w_k$ be contraction mappings on $\mathbb{R}^n$ with contraction factors $c_0, c_1, \ldots, c_k$ respectively, and define $W \colon K_n \to K_n$ by*

$$W(A) = w_0(A) \cup w_1(A) \cup \cdots \cup w_k(A)$$

Then W is a contraction mapping on (K_n, d_H) with contraction factor $c = \max\{c_0, c_1, \ldots, c_k\}$.

Proof.

1. Let $w_0, w_1, \ldots, w_k$ be contraction mappings on $\mathbb{R}^n$
2. Let $c_0, c_1, \ldots, c_k$ be their respective contraction factors
3. Let $W\colon K_n \to K_n$ by $W(A) = w_0(A) \cup w_1(A) \cup \cdots \cup w_k(A)$
4. Define $c = \max\{c_0, c_1, \ldots, c_k\}$ -

show it moves arbitrary points closer together

5. Let $X, Y \in K_n$ -
6. Define $d = d_H$ -
7. Define $r = d(X, Y)$ -
8. $X \subseteq \overline{B}(Y; r)$ and $Y \subseteq \overline{B}(X; r)$

we want to show that $d(W(X), W(Y)) \leq cd(X, Y)$, so let's compute $d(W(X), W(Y))$

9. Let $x \in W(X)$ -
10. $x \in \bigcup_{i=0}^{k} w_i(X)$
11. $x \in w_j(X)$ for some $j \in \mathbb{O}_k$
12. $x = w_j(a)$ for some $a \in X$
13. $a \in \overline{B}(Y; r)$
14. $a \in \bigcup_{z \in Y} \overline{B}(z; r)$
15. $a \in \overline{B}(z; r)$ for some $z \in Y$
16. $d_{\text{Euc}}(a, z) \leq r$
17. $d_{\text{Euc}}(x, w_j(z)) = d_{\text{Euc}}(w_j(a), w_j(z))$
18. $\leq c_j d_{\text{Euc}}(a, z)$
19. $\leq c_j r$
20. $\leq cr$
21. $w_j(z) \in w_j(Y)$
22. $\subseteq \bigcup_{i=0}^{k} w_i(Y)$
23. $= W(Y)$
24. $x \in \overline{B}(w_j(z); cr)$
25. $\subseteq \bigcup_{\alpha \in W(Y)} \overline{B}(\alpha; cr)$
26. $= \overline{B}(W(Y); cr)$
27. $W(X) \subseteq \overline{B}(W(Y); cr)$
28. $W(Y) \subseteq \overline{B}(W(X); cr)$ (by a similar argument)

29. $d(W(X), W(Y)) \leq cr$
30. $= cd(X, Y)$
31. W is a contraction mapping on (K_n, d_H) with contraction factor c

□

13.8 Planar Affine Maps are Determined by 3 Points

Theorem 13.10. *An affine transformation on $\mathbb{R}^2$ is completely determined by where it maps any 3 non-collinear points.*

Proof.

1. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an affine map
2. Let Let $p_1, p_2, p_3 \in \mathbb{R}^2$ be noncollinear -
3. $p_1 = (x_1, y_1), p_2 = (x_2, y_2), p_3 = (x_3, y_3)$ for some $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$
4. $\forall x \in \mathbb{R}^2, T(x) = Mx + B$ for some $M \in M_{2,2}(\mathbb{R})$ and some $B \in \mathbb{R}^2$
5. Define $u = p_2 - p_1$ and $v = p_3 - p_1$ -
6. $u = (x_2 - x_1, y_2 - y_1)$ and $v = (x_3 - x_1, y_3 - y_1)$
7. Define $u_1 = x_2 - x_1, u_2 = y_2 - y_1, v_1 = x_3 - x_1, v_2 = y_3 - y_1$ -
8. Assume $u_1 v_2 - u_2 v_1 = 0$ -
9. $u_1 v_2 = u_2 v_1$
10. Assume $v_1 = v_2 = 0$ -
11. $x_3 - x_1 = 0$ and $y_3 - y_1 = 0$
12. $x_3 = x_1 = 0$ and $y_3 = y_1$
13. $(x_1, y_1) = (x_3, y_3)$
14. $p_1 = p_3$
15. p_1, p_2, p_3 are collinear
16. $\rightarrow\leftarrow$
17. $v_1 \neq 0$ or $v_2 \neq 0$

So we have two cases. We will prove the case where $v_1 \neq 0$ since the other case is similar.

18. Assume $v_1 \neq 0$ -
19. Define $c = \frac{u_1}{v_1}$ -
20. $u_1 = \frac{u_1}{v_1} v_1$
21. $= c v_1$
22. $u_2 = u_2 \frac{v_1}{v_1}$
23. $= \frac{u_2 v_1}{v_1}$

24. $= \frac{u_1 v_2}{v_1}$
25. $= \frac{u_1}{v_1} v_2$
26. $= c v_2$
27. $u = (u_1, u_2)$
28. $= (cv_1, cv_2)$
29. $= c(v_1, v_2)$
30. $= cv$
31. p_1, p_2, p_3 are collinear
32. $\rightarrow\leftarrow$
33. $v_1 \neq 0$
34. $v_2 \neq 0$ by a similar argument
35. $\rightarrow\leftarrow$
36. $u_1 v_2 - u_2 v_1 \neq 0$
37. Let $z \in \mathbb{R}^2$ -
38. $z = (z_1, z_2)$ for some $z_1, z_2 \in \mathbb{R}$

Note we should write a_z and b_z below but omit the subscript where possible to avoid clutter

39. Define $a = \frac{z_1 v_2 - z_2 v_1}{u_1 v_2 - u_2 v_1}, b = \frac{u_1 z_2 - u_2 z_1}{u_1 v_2 - u_2 v_1}$ -
40. $z = (z_1, z_2)$
41. $= (\frac{z_1 v_2 - z_2 v_1}{u_1 v_2 - u_2 v_1} u_1 + \frac{u_1 z_2 - u_2 z_1}{u_1 v_2 - u_2 v_1} v_1, \frac{z_1 v_2 - z_2 v_1}{u_1 v_2 - u_2 v_1} u_2 + \frac{u_1 z_2 - u_2 z_1}{u_1 v_2 - u_2 v_1} v_2)$
42. $= (au_1 + bv_1, au_2 + bv_2)$
43. $= (au_1 + bv_1, au_2 + bv_2)$
44. $= au + bv$
45. $Mz = M(au + bv)$
46. $= aMu + bMv$
47. $= aM(p_2 - p_1) + bM(p_3 - p_1)$
48. $= a(Mp_2 - Mp_1) + b(Mp_3 - Mp_1)$
49. $= a(Mp_2 + B - Mp_1 - B) + b(Mp_3 + B - Mp_1 - B)$
50. $= a((Mp_2 + B) - (Mp_1 + B)) + b((Mp_3 + B) - (Mp_1 + B))$
51. $= a(T(p_2) - T(p_1)) + b(T(p_3) - T(p_1))$
52. $\forall z \in \mathbb{R}^2, Mz = a_z(T(p_2) - T(p_1)) + b_z(T(p_3) - T(p_1))$
53. $M\binom{1}{0} = \frac{v_2}{u_1 v_2 - u_2 v_1}(T(p_2) - T(p_1)) + \frac{-u_2}{u_1 v_2 - u_2 v_1}(T(p_3) - T(p_1))$
54. $M\binom{0}{1} = \frac{-v_1}{u_1 v_2 - u_2 v_1}(T(p_2) - T(p_1)) + \frac{u_1}{u_1 v_2 - u_2 v_1}(T(p_3) - T(p_1))$
55. $M\binom{1}{0}$ is the first column of M and $M\binom{0}{1}$ is the second column of M
56. M is completely determined by $p_1, p_2, p_3, T(p_1), T(p_2), T(p_3)$
57. $T(p_1) = Mp_1 + B$

58. $\quad B = T(p_1) - Mp_1$
59. $\quad B$ is completely determined by $p_1, p_2, p_3, T(p_1), T(p_2), T(p_3)$
60. $\quad T$ is completely determined by $p_1, p_2, p_3, T(p_1), T(p_2), T(p_3)$
61. T is completely determined by where it sends any three noncollinear points

□

13.9 Contraction Factor for Affine Maps

Theorem 13.11. *Let $\alpha, \beta, \gamma \in \mathbb{C}$ and $c = |\alpha| + |\beta|$. Then the map $T = \text{affineC}(\alpha, \beta, \gamma)$ is a contraction mapping if and only if $c < 1$. Further, if T is a contraction mapping then c is a contraction factor for T.*

Proof.

1.	Let $\alpha, \beta, \gamma \in \mathbb{C},\ c = |\alpha| + |\beta|$, and $T = \text{affineC}(\alpha, \beta, \gamma)$	
2.	Define $d = d_{\text{Euc}}$.	
3.	Let $z, w \in \mathbb{C}$	-
4.	Define $q = z - w$	-
5.	$|q| = |z - w|$	substitution
6.	$= d(z, w)$	def of d_{Euc}
7.	Define $r = |q|$, $r_1 = |\alpha|$, $r_2 = |\beta|$	-
8.	Define $\theta = \text{Arg}(q)$, $\theta_1 = \text{Arg}(\alpha)$, $\theta_2 = \text{Arg}(\beta)$	-
9.	$q = re^{i\theta}$, $\alpha = r_1 e^{i\theta_1}$, and $\beta = r_2 e^{i\theta_2}$	def polar form
10.	$d(T(z), T(w)) = d(\alpha z + \beta\overline{z} + \gamma, \alpha w + \beta\overline{w} + \gamma)$	def of affineC
11.	$= \left|(\alpha z + \beta\overline{z} + \gamma) - (\alpha w + \beta\overline{w} + \gamma)\right|$	definition of d_{Euc}
12.	$= \left|\alpha(z - w) + \beta(\overline{z} - \overline{w})\right|$	arithmetic
13.	$= \left|\alpha(z - w) + \beta(\overline{z - w})\right|$	property of conjugates
14.	$= \left|\alpha q + \beta\overline{q}\right|$	definition of q
15.	$\leq |\alpha q| + |\beta\overline{q}|$	by the triangle inequality
16.	$= |\alpha||q| + |\beta||\overline{q}|$	property of absolute value
17.	$= |\alpha||q| + |\beta||q|$	property of conjugates
18.	$= |q|(|\alpha| + |\beta|)$	arithmetic
19.	$= cd(z, w)$	definitions of c, q
20.	$d(T(z), T(w)) \leq cd(z, w)$	transitivity
($\Leftarrow$)		
21.	Assume $c < 1$	-
22.	$c > 0$	def of c and property of $\| \|$

23.	$0 < c < 1$	by the two last two lines
24.	T is a contraction mapping	definition of contraction mapping
25.	$c < 1 \Rightarrow T$ is a contraction mapping	

($\Rightarrow$)

26.	Assume T is a contraction mapping	-
27.	T has contraction factor s for some $s \in (0..1)$	def of contraction mapping
28.	Define $u = 0$, $v = e^{i(\frac{\theta_2-\theta_1}{2})}$	-
29.	$d(T(v), T(u)) = \left\| \alpha v + \beta\overline{v} + \gamma - (\alpha u + \beta\overline{u} + \gamma) \right\|$	complex notation
30.	$= \left\| \alpha v + \beta\overline{v} \right\|$	subst $w = 0$ and arithmetic
31.	$= \left\| \alpha e^{i(\frac{\theta_2-\theta_1}{2})} + \beta e^{-i(\frac{\theta_2-\theta_1}{2})} \right\|$	substitution
32.	$= \left\| r_1 e^{i\theta_1} e^{i(\frac{\theta_2-\theta_1}{2})} + r_2 e^{i\theta_2} e^{-i(\frac{\theta_2-\theta_1}{2})} \right\|$	substitution
33.	$= \left\| r_1 e^{i(\theta_1 + \frac{\theta_2-\theta_1}{2})} + r_2 e^{i(\theta_2 - \frac{\theta_2-\theta_1}{2})} \right\|$	property of exponentials
34.	$= \left\| r_1 e^{i(\frac{\theta_1+\theta_2}{2})} + r_2 e^{i(\frac{\theta_1+\theta_2}{2})} \right\|$	arithmetic
35.	$= \left\| (r_1 + r_2) e^{i(\frac{\theta_1+\theta_2}{2})} \right\|$	distributive law
36.	$= \left\| r_1 + r_2 \right\| \left\| e^{i(\frac{\theta_1+\theta_2}{2})} \right\|$	property of $\| \|$
37.	$= \left\| r_1 + r_2 \right\|$	property of exponentials
38.	$= r_1 + r_2$	def of $\| \|$ (since $r_1, r_2 \geq 0$)
39.	$= \left\| \alpha \right\| + \left\| \beta \right\|$	substitution
40.	$= c$	substitution
41.	$d(T(v), T(u)) = c$	transitivity
42.	$c = d(T(v), T(u))$	substitution
43.	$\leq sd(v, u)$	def of contraction map
44.	$= sd(e^{i(\frac{\theta_2-\theta_1}{2})}, 0)$	substitution
45.	$= s \left\| e^{i(\frac{\theta_2-\theta_1}{2})} - 0 \right\|$	def of d_{Euc}
46.	$= s \left\| e^{i(\frac{\theta_2-\theta_1}{2})} \right\|$	arithmetic
47.	$= s$	property of exponentials
48.	c is a contraction factor for T	substitution
49.	$c < 1$	def of contraction factor
50.	T is a contraction mapping $\Rightarrow c < 1$	-

□

13.10 Attractor Size Theorem

Theorem 13.12. *Let $W = [w_0, \ldots, w_n]$ be an IFS, $c_0, \ldots, c_n$ the contraction factors of $w_0, \ldots, w_n$ respectively, and $q_0, \ldots, q_n$ the fixed points of $w_0, \ldots, w_n$ respectively. Define $c = \max\{c_0, \ldots, c_n\}$ and $r = \max\left\{d(q_i, q_j) : i, j \in \mathbb{O}_n\right\}$. Then for any $a \in F_W$ and any $i \in \mathbb{O}_n$,*

$$d_{\mathrm{Euc}}(a, q_i) \leq \frac{1}{1-c} r$$

Proof.

1. Let $W = [w_0, \ldots, w_n]$ be an IFS on $\mathbb{R}^k$, -
2. Let $c_0, \ldots, c_n$ the contraction factors of $w_0, \ldots, w_n$ respectively, and -
3. Let $q_0, \ldots, q_n$ the fixed points of $w_0, \ldots, w_n$, respectively. -
4. Define $c = \max\{c_0, \ldots, c_n\}$, $d = d_{\mathrm{Euc}}$, and $r = \max\left\{d(q_i, q_j) : i, j \in \mathbb{O}_n\right\}$.
5. Let $i, j \in \mathbb{O}_n$ -
6. Let $z \in \mathbb{R}^k$ -
7. $d(w_j(z), q_i) \leq d(w_j(z), w_j(q_j)) + d(w_j(q_j), q_i)$
8. $\leq c_j d(z, q_j) + d(q_j, q_i)$
9. $\leq c d(z, q_j) + r$
10. $\forall i, j \in \mathbb{O}_n, \forall z \in \mathbb{R}^k, d(w_j(z), q_i) \leq c d(z, q_j) + r$
11. Let $a \in F_w$ and $i \in \mathbb{O}_n$ -
12. $a = \Phi(t_1 t_2 \ldots)$ for some $t_1 t_2 \ldots \in \Sigma_{n+1}$
13. $= \lim_{n \to \infty} w_{t_1} w_{t_2} \ldots w_{t_n}(q_i)$
14. Let $n \in \mathbb{N}$ -
15. $d(w_{t_n}(q_i), q_i) \leq c d(q_i, q_{t_n}) + r \leq cr + r = (1+c)r$
16. $d(w_{t_{n-1}} w_{t_n}(q_i), q_i) \leq c d(w_{t_n}(q_i), q_{t_{n-1}}) + r \leq c(1+c)r + r = (1+c+c^2)r$
17. $\vdots$
18. $d(w_{t_1} w_{t_2} \circ \cdots \circ w_{t_{n-1}} w_{t_n}(q_i), q_i) \leq (1 + c + c^2 + \cdots + c^n) r$
19. $\leq (1 + c + c^2 + \cdots) r$
20. $\leq \frac{1}{1-c} r$
21. $\forall n, d(w_{t_1} w_{t_2} \circ \cdots \circ w_{t_{n-1}} w_{t_n}(q_i), q_i) \leq \frac{1}{1-c} r$
22. Let $\varepsilon > 0$ -
23. For some $N > 0, \forall n \geq N, d(a, w_{t_1} \circ \cdots \circ w_{t_n}(q_i)) < \varepsilon$
24. For some n, $n > N$
25. $d(a, q_i) \leq d(a, w_{t_1} \circ \cdots \circ w_{t_n}(q_i)) + d(w_{t_1} \circ \cdots \circ w_{t_n}(q_i), q_i)$
26. $\leq d(a, w_{t_1} \circ \cdots \circ w_{t_n}(q_i)) + \frac{1}{1-c} r$

27. $< \varepsilon + \frac{1}{1-c} r$
28. $\forall \varepsilon > 0, d(a, q_i) < \frac{1}{1-c} r + \varepsilon$
29. $d(a, q_i) \leq \frac{1}{1-c} r$

□

Corollary 13.13. $F_W \subseteq \bigcap_{i=0}^{n} \overline{B}(q_i; \frac{r}{1-c})$

Proof.

1. Let $W = [w_0, \ldots, w_n]$ be an IFS on $\mathbb{R}^k$, $c_0, \ldots, c_n$ the contraction factors of
2. $w_0, \ldots, w_n$ respectively, and $q_0, \ldots, q_n$ the fixed points of $w_0, \ldots, w_n$ respectively.
3. Let $a \in F_W$ -
4. Let $i \in \mathbb{O}_n$ -
5. $d(a, q_i) \leq \frac{1}{1-c} r$
6. $a \in \overline{B}(q_i; \frac{r}{1-c})$
7. $\forall i \in \mathbb{O}_n, a \in \overline{B}(q_i; \frac{r}{1-c})$
8. $a \in \bigcap_{i=0}^{n} \overline{B}(q_i; \frac{r}{1-c})$
9. $F_W \subseteq \bigcap_{i=0}^{n} \overline{B}(q_i; \frac{r}{1-c})$

□