

NONLECTURE NOTES - THE ART OF PROBLEM SOLVING

MATH 101 - DR. MONKS - UNIVERSITY OF SCRANTON

1 The Way of Problem Solving

Contemplation within activity is a million times better than contemplation within stillness.

- Hakuin

Problem Solving is the art of deciding what to do when faced with a problem *that you have not been shown how to solve before*. It exemplifies the way that mathematicians approach mathematical research or solve problems that typically appear on high quality mathematical competitions. .

- **Art:** Problem solving is an art. Like any art it requires proper attitude, practice, creativity, and passion to master. Like any artist the problem solver creates works of wonder and surprise and sublime aesthetic value.
- **Beauty:** A correct solution is better than no solution. A massive straightforward slog (a.k.a. *dumb-assing*) that gets the correct answer is better than no solution at all. But a *clever* correct solution is better than a straightforward or obvious solution. All else being equal, the shorter the solution, the better. A solution that does not require a calculator or computer is better than one that does. A solution that does not require algebra is better than one that does.
- **Fellowship:** As with any art form, we can benefit from interacting with other artists. By aspiring to learn from those who are more experienced, by cooperating with our peers, and by assisting those who are less experienced, everyone benefits. Several minds can produce several perspectives on the same problem. As with any group of artisans, problem solvers naturally bond together into a community of people who share a common interest.

The life of Zen attainment is not like standing on a riverbank watching the current and appreciating the water or landscape as a witness; it is jumping into the current and becoming one with it.

- Leggett

1.1 Some Characteristics of a Good Problem

- Your first impression is “This is impossible!”.
- You are surprised, or delighted by the question itself.
- It is simple to state, but hard to answer.
- You’ve never seen a question just like it before.
- You don’t know immediately how to solve it.
- It is addictive. You want to know the answer.
- It has an obvious, straightforward, ugly, messy, lengthy solution, but also a clever, ingenious, short, elegant solution.
- It only requires only very elementary mathematics, but is quite challenging nonetheless.
- It has some symmetry, or a story, or a pattern or picture that is aesthetically pleasing.

1.2 The Problem Solving Mindset

There are several attitudes or psychological perspectives that are needed to be a successful problem solver.

- **concentration:** it is easy to get distracted or frustrated by a difficult problem. Problem solving requires sometimes lengthy, intense, focused concentration on a single topic.
- **confidence:** it is important to believe that you will eventually be able to solve a problem, even if you have no idea how to do it at first. Even if you are a beginner at problem solving, you should approach a problem with a confident attitude. Don’t worry that you might not remember a key theorem or an important fact. Every problem has to be solved with what you already know.
- **creativity:** a problem solver must always remain open to all and any ideas that may come to mind and always on the lookout for new ways to approach a problem. A change of perspective, a reinterpretation of the question, a nonstandard approach to a otherwise familiar situation can have tremendous benefits. It can also be a dead end. But if even one idea in ten is fruitful, that may be the only one you need to solve the problem.
- **peripheral vision:** when looking at the night sky we can see fainter objects by not looking directly at them. The receptors on the sides of our eyes are more sensitive to faint light than those in the center. Similarly, when solving a problem, we should not always think about solving the problem itself directly, but rather allow ourselves to

ponder things that are perhaps only vaguely related to the problem. This is similar to a smell or gut instinct or intuition that leads you in a certain direction without being 100% certain why you think you ought to go that way. The more you practice, the more reliable your instincts will become.

- **thinking on your feet:** problem solvers strive to develop the ability to think on their feet with the minimal amount of assistance possible. A solution that does not require a calculator or computer is better than one that does. A solution that does not require a pencil and paper is better than one that does. A short elementary solution that does not require any advanced theorems or previously proven results is better than one that does. The problem solver solves problems in the shower, while lying in bed before going to sleep or right after waking up, while running or biking or hiking or driving in the car. The problem solver may actually look forward to time in the waiting room at the doctor's office or dentist as it provides uninterrupted time to work on their problems.
- **stay loose:** The mind is a more flexible and fluid canvass than pencil and blank paper. We can manipulate ideas freely in our mind. Putting something down on paper tends to make it more concrete and cast in stone. The more you practice, the better you will become at not needing paper and pencil to do mathematics. As you do you will sometimes find that you have more success solving difficult problems if you don't use paper than if you do! Especially at the beginning, when you first approach a problem, it is important to stay loose and flexible. Working mostly in your head is often the best way to do that. Once you have an epiphany and see the crux move, it may then be time to break out the paper or calculators.
- **be careful:** without accuracy and care, stupid mistakes can easily turn an otherwise correct solution into an incorrect one. Also some problems may be easy to solve if you do them correctly but a hideous nightmare if you make a small mistake. The problem solver must also strive to be sure that every case has been considered and that there is no omission in the solution that could catastrophic.

2 Problem Solving Strategies

If you would be a real seeker after truth, it is necessary that at least once in your life you doubt, as far as possible, all things.

- Descartes

A *Problem Solving Strategy* is mathematical or psychological ideas for starting and approaching problems. They are a checklist of sorts that the problem solver goes through when faced with a problem they do not know how to solve.

- **Diligence and Care** - sometimes there is an obvious brute force way to attack a problem, and all it requires is effort, diligence, and care to see it through. Sometimes just making an exhaustive list of all possibilities is enough to solve the problem. For example, in the Census Taker problem the solution was easy to find if you made a complete careful list of every possible three ages whose product is 36.
- **Impose or Preserve Symmetry** - whenever there are patterns or symmetries in a problem, try to preserve them and use them to your advantage. If there aren't symmetries or patterns, try to impose or construct them.
- **Consider a Simpler Problem** - when faced with a problem that is too difficult to figure out immediately, consider a simpler but similar problems that you can try to answer first. Sometimes finding a solution to the easier problem will give the key insights needed to solve the more difficult one. If the problem asks you to prove something for 20 things, first try it for one thing, then two things, then three things, and so on. Doing so can lead you to notice a pattern or develop an insight for what is going on that is too difficult to notice in the original problem.
- **Extreme Principle** - when faced with a problem that has a lot of different values or shapes, focus on the extremes – the largest ones or the smallest ones – first. They are special since you have an extra piece of information about them that you don't have about the other cases – namely nothing is larger than whatever the largest thing is (and similarly for smaller/smallest).