CHAPTER 1

Apprentice Level

Can I use my fingers?

When faced with doing arithmetic by hand or in your head, under time pressure, without a calculator, in a mathematics contest or on an exam, wouldn't it be nice to have some easy way to do things that is both accurate and efficient?

It turns out that the way we are taught to do arithmetic is not necessarily the fastest or best or even easiest way for the purpose of actual problem solving. Let's explore some alternate methods of doing arithmetic that will soon have your calculator collecting dust in a closet somewhere.

1.1 Subtracting the (Left to) Right Way

I don't borrow. My mother told me "Neither a borrower nor a lender be."

The traditional way of subtracting two numbers is to write one on top of the other with their unit's digits aligned and then to subtract one column at a time, working from right to left by "borrowing" from other columns as necessary. But this method has three serious disadvantages in practice. The first is that when subtracting infinite repeated decimals such as $1.\overline{123} - 0.\overline{25}$, i.e.

$$\begin{array}{r} 1.123123123123\ldots \\ -0.252525252525\ldots \\ \hline \end{array}$$

there is no "rightmost" digit to start from. The second is that the bookkeeping necessary to "borrow" can, in some cases, be quite messy and complicated. The third is that by subtracting from right to left we do not get a rough estimate of what our answer is going to be until the very end of the computation.

It is possible to eliminate all of these problems by subtracting in a completely different way, using what we will call the *Left-to-Right* method. To use this method we begin the same way as the traditional method, writing the numbers on top of each other with their unit's digits aligned.

Whenever we do this there can be three flavors of digit columns:

- *Good columns*: the digit on top is larger than the digit on the bottom
- *Bad columns*: the digit on top is smaller than the digit on the bottom
- *Invisible columns*: the two digits are equal

For example, $\substack{5\\2}$ is a Good column, $\substack{3\\7}$ is a Bad column, and $\substack{4\\4}$ is an Invisible column. Since both Good and Bad columns are not Invisible, we will say they are *Visible*.

Each column is then subtracted individually, using *mod ten arithmetic*. In other words, whenever the result is negative we add ten to it (or equivalently, we add ten to the top digit before subtracting the bottom digit whenever the top digit is smaller than the bottom digit). However, there is one catch. Regardless of whether the column we are subtracting is Good, Bad, or Invisible, we incur a *penalty point* and must decrease our answer for that column by one (still working mod ten) *if the first Visible column to its right is Bad*. This penalty is not incurred if there are no Visible columns to the right of the column being subtracted.

Left-to-Right Subtraction
Any digit of $a-b$ can be computed by subtracting the corresponding digits in a and b mod ten and decreasing by one mod ten whenever there is a Bad column that is the first Visible column to the right of the column being subtracted.

For example, if we want to compute

$$154399223764075 - 24891803595078$$

we begin as usual by placing the first number on top of the second, with their unit's digits lined up.

$$\begin{array}{r} 1\,5\,4\,3\,9\,9\,2\,2\,3\,7\,6\,4\,0\,7\,5 \\ -\;\;2\,4\,8\,9\,1\,8\,0\,3\,5\,9\,5\,0\,7\,8 \\ \hline \end{array}$$

Notice that in this example, some columns are Good, some are Bad, and some are Invisible.

$$
\begin{array}{ccccccccccccccccc}
 & \text{Good}\downarrow & \text{Good}\downarrow & & \text{Bad}\downarrow & & \text{Good}\downarrow & \text{Bad}\downarrow & \text{Good}\downarrow & & \text{Good}\downarrow & \text{Bad}\downarrow & \text{Bad}\downarrow & & & \text{Bad}\downarrow \\
 & 1 & 5 & 4 & 3 & 9 & 9 & 2 & 2 & 3 & 7 & 6 & 4 & 0 & 7 & 5 \\
- & & 2 & 4 & 8 & 9 & 1 & 8 & 0 & 3 & 5 & 9 & 5 & 0 & 7 & 8 \\
\hline
\end{array}
$$

So why is it called the Left-to-Right method?

We can now compute any digit of the answer directly. This can be useful if we only require certain digits of the answer as opposed to the entire answer. But in most situations where we want to compute the entire answer, or just a good estimate of the answer, it makes more sense to start with the leftmost column and work our way from left to right.

Oh, that's why!

So in this example we could begin with the first column on the left. Since a missing digit can be considered to be zero, we think of this column as ${1 \atop 0}$. The first Visible column to its right is ${5 \atop 2}$, which is Good. So the first (leftmost) digit in the answer is $1-0$, or 1.

Writing this down,

$$
\begin{array}{r}
154399223764075 \\
-\ \ 24891803595078 \\
\hline
1
\end{array}
$$

we can move to the next column, ${5 \atop 2}$. Since the ${4 \atop 4}$ column is invisible, the first Visible column to its right is ${3 \atop 8}$, which is Bad. So the penalty point rule applies. We subtract $5-2$ to get 3, and then decrease this by one because of the penalty point. So the second digit from the left in our answer is 2.

$$
\begin{array}{r}
154399223764075 \\
-\ \ 24891803595078 \\
\hline
12
\end{array}
$$

The next column we encounter is ${4 \atop 4}$. The first Visible column to its right is still the Bad column ${3 \atop 8}$. So the penalty point rule applies again. We subtract $4-4$ to get 0, and then decrease this by one to get -1 which we add ten to to get 9 for the next digit (alternatively we can add ten to the 0 before subtracting the 1).

Continuing to the right we find that the first Visible column to the right of ${3 \atop 8}$ is ${9 \atop 1}$, which is Good, so no penalty point is applied and the next digit of the answer is obtained by subtracting $3-8$ which is -5. But as this result is negative we add ten to it to get 5.

So after computing the first four digits we have

$$\begin{array}{r} 1\,5\,4\,3\,9\,9\,2\,2\,3\,7\,6\,4\,0\,7\,5 \\ -\quad 2\,4\,8\,9\,1\,8\,0\,3\,5\,9\,5\,0\,7\,8 \\ \hline 1\,2\,9\,5 \end{array}$$

Let's pause at this point to notice that by computing this messy subtraction from left to right, we have already obtained an estimate of the answer with four significant digits: 129500000000000. So if a good approximation is all we need, then we are already done.

If we require the entire answer, we can continue in this manner to obtain the rest of the digits. The only remaining columns that will not incur a penalty point are $\substack{9\\9}$, $\substack{2\\0}$, $\substack{3\\3}$, and $\substack{5\\8}$. Note that $\substack{5\\8}$ does not incur the penalty point because there is no Bad column which is the first Visible column to its right (because there are no columns to its right!)

Right on!

We can continue in this manner to get the remaining digits in the answer. See if you can finish the calculation yourself (Problem 1).

The fact that we can compute the digits of the answer in any order can be quite useful, as illustrated by the following example.

Example 1 *Let x and y be the digits indicated in the following subtraction problem. What is $x+y$?*

$$\begin{array}{r} 5\,0\,6\,3\,1\,0\,1\,9 \\ -4\,2\,6\,3\,3\,0\,1\,5 \\ \hline x\quad y \end{array}$$

SOLUTION. Since the first Visible column to the left of $\substack{6\\6}$ is the Bad column $\substack{1\\3}$ the penalty applies, so instead of $6-6=0$ the value of x is one less than that mod ten. So $x=9$. For y we notice that there is no Bad column to the right of $\substack{1\\3}$ so that y is $1-3$ mod ten. So $y=8$. Thus $x+y=9+8=17$.

Answer: $\boxed{17}$ ■

Notice that we did not have to compute the rest of the digits in order to answer this question.

The Left to Right method of subtraction does explicitly what many problems solvers often do implicitly in their heads when faced with a subtraction problem. For example, when asked to compute the difference 1000 –

234 we might do this mentally as follows: "200 *less than* 1000 *is* 800*, and* 30 *more below that is* 770*, and* 4 *below that is* 766". In this method, the solver is working from left to right, obtaining the most significant information first and filling in the details as they proceed.

like $1234 - 0 = ?$

or the left to right tool for the left to right situation?

In situations like this, where the particular numbers involved in a subtraction problem are small or nice in some way, and we can see some clever fast efficient trick to compute the difference in your head, the full left to right subtraction method might be overkill. We should always use the right tool for the right situation.

Your Turn!

Use the Left-to-Right subtraction method to answer the following questions. Resist the urge to use your more familiar method of right to left subtraction! Don't compute any more digits than are necessary to answer the question being asked.

Problem 1 *Complete the calculation of*

$$154399223764075 - 24891803595078$$

that we started above by computing the remaining digits from left to right. Then check your answer by adding your answer to the smaller number to verify that you get the larger. Finally write a farewell note to subtraction by the traditional right-to-left method and say goodbye to it forever!

Problem 2 *Compute*

$$\begin{array}{r} 1\,2\,3\,4\,5\,6\,7\,8\,9\,0 \\ -\;\;9\,8\,7\,6\,5\,4\,3\,2\,1 \\ \hline \end{array}$$

Problem 3 *Compute the exact value of* $0.\overline{123} - 0.\overline{98}$*. Use the overbar to indicate repeating digits.*

Problem 4 *Compute the following (a) to five significant digits and (b) exactly.*

$$\begin{array}{r} 5\,0\,8\,7\,9\,0\,0\,5 \\ -\;\;\;8\,8\,2\,0\,1\,3 \\ \hline \end{array}$$

Problem 5 *Dwind of the Forest is responsible for maintaining the daily forest leaf-count. One windy fall day exactly* 145872370 *leaves fell from the trees. If Dwind counted* 980402412 *leaves on all trees at the start of the day, how many leaves were left on the trees at the end of the day?*

Problem 6 *Let x, y, and z be the digits indicated in the following subtraction. What is the product xyz?*

$$\begin{array}{r} 3\,0\,0\,0\,0\,0\,1\,6 \\ -\quad 2\,6\,0\,0\,0\,1\,8 \\ \hline \mathrm{x}\;\,\mathrm{y}\;\;\;\;\;\;\mathrm{z}\;\;\;\;\;\; \end{array}$$

Problem 7 *Why does the Left-to-Right method work? In particular, why do we decrease the result of subtracting a digit column by one when there is a bad column that is the first visible column to the right of the column we are subtracting?*