Math 104 - Mathematics for Elementary Teachers
Department of Mathematics
University of Scranton
Prepared by: Dr. Ken Monks

The following is a detailed list of the mathematical topics covered in the Mathematics for Elementary Teachers course at the University of Scranton. Each of the twenty eight major sections below correspond to roughly fifty minutes worth of lecture material. The remainder of the class time in the course will be used for reviews, exams, and additional topics chosen by the instructor. In some cases more than one lecture may be used to cover a particular subject.

These are the notes I made for myself in order to prepare my lectures in the course, and therefore they are written at a level which is more sophisticated than what I would expect from the students. I will explain all of these concepts in detail and present many examples of their use in class. Thus, students should use this as a "checklist" for the mathematical concepts they should strive to learn, but should not be overly concerned if they don't understand something below in the exact form that it is written here. All will be made clear in class. However, this is a precise description of exactly what we will cover in the course, and therefore can be a valuable reference.

1. Patterns and Sequences
   a. Sequences

      Definition: A **number sequence** of numbers is just an ordered list of numbers. Each number in the list is called a **term** of the sequence. We usually write such a sequence by writing the numbers from left to right, separated by commas:

      $$a_1, a_2, a_3, \ldots$$

      In this case a_1 is called the **first term,** a_2 the **second term**, and so on.

      i. finite sequences

         Definition: A sequence is **finite** if it has only finitely many terms, i.e. if the sequence is

         $$a_1, a_2, \ldots, a_k$$

         for some whole number k.

      ii. infinite sequences

         Definition: A sequence is **infinite** if it has one term for each positive whole number, i.e. the sequence is of the form

         $$a_1, a_2, a_3, \ldots$$

         where for each positive whole number k there is a term a_k in the sequence.

   b. Periodic sequences

Informally, a *periodic* sequence is an infinite sequence whose terms repeat.

Definition: An infinite sequence of numbers

$$a_1, a_2, a_3, \ldots$$

is **periodic** if there is a number $k > 0$ such that $a_i = a_{i+k}$ for every positive whole number i. In this situations we say the sequence has **period** $k.$ A sequence of period one is called a **constant sequence**.

Overbar Notation: If $a_1, a_2, a_3, \ldots$ is periodic with period $k,$ we can abbreviate this by writing:

$$\overline{a_1, a_2, a_3, \ldots, a_k}$$

n^{th} **term**: The n^{th} term of the periodic sequence

$$\overline{a_1, a_2, a_3, \ldots, a_k}$$

is a_r where r is the remainder when n is divided by k (and $a_0 = a_k$).

c. Arithmetic sequence

Definition: The infinite sequence $a_1, a_2, a_3, \ldots$ is an **arithmetic sequence** if the difference between any two consecutive terms is a fixed number d called the **common difference** of the arithmetic sequence.

n^{th} **term**: If $a_1, a_2, a_3, \ldots$ is an arithmetic sequence with common difference $d,$ then the n^{th} term is

$$a_1 + (n-1)d$$

Fact: If $a_1, a_2, a_3, \ldots$ is an arithmetic sequence with common difference $d,$ then

$$a_1 + a_2 + \cdots + a_n = n\left(a_1 + \frac{(n-1)}{2}d\right)$$

in particular, if $a_1 = 1$ and $d = 1$ we have

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

d. Geometric sequence

Definition: The infinite sequence $a_1, a_2, a_3, \ldots$ is a **geometric sequence** if the ratio of any two consecutive terms is a fixed number r called the **common ratio** of the geometric sequence.

n^{th} **term**: If $a_1, a_2, a_3, \ldots$ is a geometric sequence with common ratio r then the n^{th} term is

$$a_1 r^{n-1}$$

Fact: If $a_1, a_2, a_3, \ldots$ is a geometric sequence with common ratio r then

$$a_1 + a_2 + a_3 + \cdots + a_n = a_1 \frac{1 - r^n}{1 - r}$$

e. Method of Finite Differences

Method: Given the first k terms of a sequence $a_1, a_2, a_3, \ldots, a_k$

1. Compute the sequence of consecutive differences

$$a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1}.$$

(Note: If this new sequence is a constant sequence, then the original sequence was arithmetic.)

2. Iterate this process until you obtain the first few terms of a constant sequence. (Note: You might never obtain a constant sequence. In this case the method fails. The method also fails if you have only one term left after iterating.)

3. Under the assumption that the final sequence you obtained by this process is actually a constant sequence, back calculate to obtain additional terms of the original sequence.

f. Other sequences

Definition: The **Fibonacci numbers** are the numbers in the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

where each term is the sum of the two terms immediately preceding it (except the first two terms, which are both one).

Definition: The **triangular numbers** are the numbers in the sequence

$$1, 3, 6, 10, 15, 21, 28, \ldots$$

where the n^{th} term is the sum $1 + 2 + 3 + \cdots + n$.

2. Sets and set operations

a. elements and membership

A set is a collection of objects called the **elements** or **members** of the set. The following statements are all equivalent and can be used interchangeably: "x is an element of the set S", "$x \in S$", "x is a member of S", "x is in S", "S contains x".

b. finite sets and notation

Definition: A set is **finite**, if it contains finitely many elements. (This means there is a nonnegative whole number n so that the elements of the set can be put in one-to-one correspondence with the set of numbers $\{1, 2, \ldots, n\}$.) If S is a finite set and $a_1, \ldots, a_k$ are its elements, then we can denote this set S by

$$\{a_1, \ldots, a_k\}.$$

There is a set having no elements called the **empty set.** The empty set is denoted $\{\}$ or $\emptyset$. The number of elements in a finite set is called the **cardinality** of the set. The cardinality of a finite set S is denoted $n(S)$ (or $\#(S)$ or $|S|$).

c. infinite sets and set builder notation

Definition: A set is **infinite**, if it is not finite. If S is an infinite set whose elements consist of all things x satisfying some statement P then we can write S in set-builder notation:

$$\{x : P\}.$$

d. equality and subsets

Definition: Two sets are **equal** if they contain the same elements.

Definition: Let A and B be sets. A is a **subset** of B if every element in A is also an element of B. A is a **proper subset** of B if A is a subset of B and A is not equal to B.

Notation: $A \subseteq B$ means "A is a subset of B". $A \subset B$ means "A is a proper subset of B".

e. set operations

i. union

Definition: Let A and B be sets. The **union** of A and B is the set of all things which are either in A or in B (or in both) We denote this set by $A \cup B$, i.e.

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

ii. intersection

Definition: Let A and B be sets. The **intersection** of A and B is the set of all things which are both in A and in B. We denote this set by $A \cap B$, i.e.

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

iii. difference

Definition: Let A and B be sets. The **difference** of A and B is the set of all elements of A which are not elements of B. We denote this set by $A - B$, i.e.

$$A - B = \{x : x \in A \text{ and } x \notin B\}$$

iv. complement

Definition: Suppose all sets in a particular discussion are subsets of a set U (called a **universal set**). If A is a subset of U then the **complement** of A is the set of all elements (in U) which are not in A. We denote this set by $\overline{A}$. Thus $\overline{A} = U - A$ or equivalently

$$\overline{A} = \{x : x \notin A\}.$$

v. Cartesian product

Definition: An **ordered pair** is a pair of objects in a particular order, denoted (x,y). x is called the **first coordinate** and y is called the **second coordinate** of the ordered pair (x,y)

Definition: Let A and B be sets. The **Cartesian product** of A and B is the set of all ordered pairs whose first coordinate is in A and second coordinate is in B. We denote this set by $A \times B$, i.e.

$$A \times B = \{(x,y) : x \in A \text{ and } y \in B\}$$

f. Venn diagrams

3. Functions
   a. definition

Definition: Let A and B be sets. f is a **function** from A to B if f is a rule that assigns exactly one element of B to each element of A. We denote this by $f : A \to B$. A is called the **domain** of the function and B is called the **codomain**. If $x \in A$ then $f(x)$ is the element of B that f assigns to A.

   b. representation
      i. as a rule

Notation: We often specify a function by a statement of the form $f(x) = P$ where P is some expression involving x. If the domain for a function given by a rule is not specified, it is assumed to be the largest set of real numbers for which the rule is defined.

      ii. as a table

Notation: If $f : \{a_1, a_{2,} \ldots, a_n\} \to B$ we can represent f in a table as follows:

x	$f(x)$
a_1	$f(a_1)$
a_2	$f(a_2)$
$\vdots$	$\vdots$
a_n	$f(a_n)$

      iii. as a graph or set of ordered pairs

Definition: Given $f : A \to B$, the **graph** of f, is the set of all ordered pairs $(x,f(x))$ for which x in A, i.e. it is

$$\{(x,f(x)) : x \in A\}.$$

If A and B are sets of real numbers, we can draw the graph of f by coloring all the points in the plane (x,y) for which $y = f(x)$.

      iv. as Venn-like diagrams

We sometimes represent a function, f, by drawing circles to represent the domain and range, points in the circles to represent the elements of the sets, and then draw arrows from each x in the domain to $f(x)$ in the codomain.

4. Logic
   a. statements

Definition: A **statement** is an expression which is either true or false (but not both).

   b. deduction and proof

Definition: An **axiom** (or **premise** or **given**) is a statement which is assumed to be true. A **proof** is a sequence of statements, each of which either: a) is an axiom, or b) follows from the previous statements by rules of logic (called **rules of inference**). Using the rules of logic to determine the truth of a statement is called **deductive reasoning**.

   c. conditional statements
      i. definition

Definition: A **conditional statement** is a statement of the form "If P then Q" where P and Q are statements. P is called the **hypothesis** of the conditional statement and Q is called the **conclusion.**

Notation: There are many other ways we can write the conditional statement "If P then Q" in the English language and in mathematics. For example: "P implies Q", "Q, if P", "Q follows from P", and "$P \Rightarrow Q$".

      ii. converse

Definition: The **converse** of the conditional statement "If P then Q" is the statement

"If Q then P".

      iii. inverse

Definition: The **inverse** of the conditional statement "If P then Q" is the statement

"If not P then not Q".

      iv. contrapositive

Definition: The **contrapositive** of the conditional statement "If P then Q" is the statement

"If not Q then not P".

Fact: The contrapositive of a conditional statement is logically equivalent to the conditional statement itself.

d. negation

Definition: The **negation** of a statement, P, is the statement “not P”.

e. rules of inference

i. modus ponens

Rule of Inference: (modus ponens) Given that the statements P and “If P then Q” are both true, we can conclude that Q is also true.

ii. contraposition

Rule of Inference: (contraposition) Given that the statements “not Q” and “If P then Q” are both true, we can conclude that “not P” is also true.

f. logical operators: and, or, not, implies, if and only if

Definition: If P and Q are statements then so are: “not P”,“P and Q”, “P or Q”,“P implies Q”, and “P if and only if Q”. The truth value of these statements is determined by the truth values of P and Q as given in the following **truth table.**

P	Q	not P	P and Q	P or Q	P implies Q	P if and only if Q
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

g. Venn diagram approach to statements with quantifiers

5. Natural Numbers

a. definition

Definition: A **natural number** is a non-negative whole number. The set of all natural numbers is an infinite set denoted $\mathbb{N}$, i.e.

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}$$

b. base b representation of natural numbers

Definition: If n is a natural number, and b is a natural number greater than 1, then we define the **base b representation of n** to be the unique sequence of digits

$$d_k d_{k-1} \cdots d_2 d_1 d_{0(b)}$$

such that $d_0 + d_1 b + d_2 b^2 + \cdots + d_{k-1} b^{k-1} + d_k b^k = n$ and each digit, d_i, is between 0 and

$b - 1$ inclusive. (Note: if $b > 10$ we use the letters of the alphabet in order for the extra digits, e.g. $A = 11$, $B = 12$, etc.)

c. conversion between bases

Algorithm: To convert $d_k d_{k-1} \cdots d_2 d_1 d_{0(b)}$ to base ten, compute the sum $d_0 + d_1 b + d_2 b^2 + \cdots + d_{k-1} b^{k-1} + d_k b^k$.

Definition: For any non-negative integers a, b with $b \neq 0$, let $a \bmod b$ denote the remainder when a is divided by b. Also let $\lfloor \frac{a}{b} \rfloor$ denote the greatest integer which is less than or equal to $\frac{a}{b}$ (i.e. the quotient).

Algorithm: To convert a positive integer n from base ten to base $b > 1$:

Method A: (compute the digits from right to left)

1. Let $n_0 = n$ and $d_0 = n \bmod b$, and for $i > 0$ compute $n_i = \frac{n_{i-1} - d_{i-1}}{b}$ and $d_i = n_i \bmod b$.
2. Then

$$n = d_k d_{k-1} \cdots d_2 d_1 d_{0(b)}$$

where k is the largest integer for which $n_k \neq 0$.

Method B: (compute the digits from left to right)

1. Find the whole number k so that $b^k \leq n < b^{k+1}$.
2. Let $n_k = n$, $d_k = \lfloor \frac{n_k}{b^k} \rfloor$ and for $k > i \geq 0$ compute $n_i = n_{i+1} - d_{i+1} b^{i+1}$ and $d_i = \lfloor \frac{n_i}{b^i} \rfloor$.
3. Then $n = d_k d_{k-1} \cdots d_2 d_1 d_{0(b)}$.

d. other representations: Roman numerals, Mayan numerals

Roman Numerals: The value of the digits in Roman Numerals is:

Roman Digit	Value
I	1
V	5
X	10
L	50
C	100
D	500
M	1000

1. A **Roman numeral** is a string of Roman Digits satisfying the following rules.
2. The digits are always written in decreasing order of value from left to right (except for rule #4 below).
3. No more than three of a particular digit can appear consecutively in the string.
4. The only exception to rule #1, are the pairs shown in the following table:

Exceptional Pair	Value
IV	4
IX	9
XL	40
XC	90
CD	400
CM	900

5. The integer that the Roman numeral represents is equal to the sum of the values of the digits (with the exceptional pairs counting as a single digit).

6. Addition and Subtraction of Natural Numbers

a. properties

i. closure

Fact: If a and b are natural numbers then so is $a + b$.

Notation: The numbers a and b in the expression $a + b$ are called **summands**.

ii. commutativity

Fact: If a and b are natural numbers then $a + b = b + a$.

iii. associativity

Fact: If a, b and c are natural numbers then $(a + b) + c = a + (b + c)$.

Notation: If a, b and c are natural numbers then $a + b + c = (a + b) + c$ (which is equal to $a + (b + c)$ by associativity). We can make similar definitions for sums of more than three numbers also to eliminate extraneous parentheses.

iv. identity

Fact: There is a natural number called 0 which has the property that for any natural number a,

$$a + 0 = a$$

and

$$0 + a = a.$$

The number 0 is called the **additive identity**.

b. definition of inequality

Definition: A natural number n is **positive** if it is not equal to zero.

Definition: Let a and b be natural numbers. We say $a < b$ if $a + n = b$ for some positive natural number n.
We also write

Statement	Meaning
$a \leq b$	$a < b$ or $a = b$
$a > b$	$b < a$
$a \geq b$	$b \leq a$
$a \nless b$	$b \leq a$
$a \nleq b$	$b < a$
$a \ngtr b$	$a \leq b$
$a \ngeq b$	$a < b$

.

c. subtraction

i. definition

Definition: If a and b are natural numbers with $b < a$ then $a - b$ is defined to be the unique natural number n so that

$$b + n = a.$$

ii. properties

Subtraction is **not** commutative or associative.

d. algorithms for addition and subtraction in base b

Algorithm: Let $\ldots d_k d_{k-1} \cdots d_1 d_{0(b)}$ and $\ldots e_k e_{k-1} \cdots e_1 e_{0(b)}$ be the base b representation of two natural numbers. Let $f_0 = d_0 + e_0 \bmod b.$ For each integer $i > 0,$ compute $c_i = (d_{i-1} + e_{i-1} - f_{i-1})/b$ and $f_i = d_i + e_i + c_i \bmod b.$ Then

$$\ldots f_k f_{k-1} \cdots f_1 f_{0(b)}$$

is the base b representation for the number $\ldots d_k d_{k-1} \cdots d_1 d_{0(b)} + \ldots e_k e_{k-1} \cdots e_1 e_{0(b)}.$

e. other algorithms for subtraction

i. Regrouping using $d_k 0 \cdots 0_{(b)} = (d_k - 1)(b-1)(b-1) \cdots (b-1)0 \cdots 0_{(b)} + 10 \cdots 0_{(b)}$ where each summand has the same number of zeros, i.e. we carry a $b - 1$ and also a 1 to the next column to the right and write both vertically above that column.

f. estimation

7. Multiplication of Natural Numbers

a. definition

Definition: Let a and b be natural numbers then $0 \times b = 0,$ $1 \times b = b$ and for $a > 1,$ $a \times b$

is the natural number $\underbrace{b + b + \cdots + b}_{a \text{ summands}}$.

Notation: The product $a \times b$ is also written ab or $(a)(b)$ or $a(b)$ or $(a)b$. The numbers a and b in the expression $a \times b$ are called **factors**.

b. properties

i. closure

Fact: If a and b are natural numbers then so is ab.

ii. commutativity

Fact: If a and b are natural numbers then $ab = ba$.

iii. associativity

Fact: If a,b and c are natural numbers then $(ab)c = a(bc)$.

iv. identity

Fact: There is a natural number called 1 which has the property that for any natural number a,

$$a \times 1 = a$$

and

$$1 \times a = a.$$

The number 1 is called the **multiplicative identity**.

v. distributive

Fact: If a,b and c are natural numbers then $a(b + c) = (ab) + (ac)$.

c. algorithms for multiplication in base b

Algorithm: (for multiplication a natural number by a single digit number in base b) Let $\ldots d_k d_{k-1} \cdots d_1 d_{0(b)}$ be the base b representation of a natural number and let $0 \leq n < b$. Let $f_0 = d_0 n \bmod b$. For each integer $i > 0$, compute $c_i = (d_{i-1} n - f_{i-1})/b$ and $f_i = d_i n + c_i \bmod b$. Then

$$\ldots f_k f_{k-1} \cdots f_1 f_{0(b)}$$

is the base b representation for the number $\ldots d_k d_{k-1} \cdots d_1 d_{0(b)} \times n$.

Algorithm: (for multiplication a natural number by a power of b in base b) Let $\ldots d_k d_{k-1} \cdots d_1 d_{0(b)}$ be the base b representation of a natural number and let n be a natural number. Then

$$\ldots d_k d_{k-1} \cdots d_1 d_0 \underbrace{00\cdots0}_{n \text{ zeros}} {}_{(b)}$$

is the base b representation for the number $\ldots d_k d_{k-1} \cdots d_1 d_{0(b)} \times b^n$ (see below for a definition of b^n)

.

Algorithm: (for multiplication two natural numbers in base b) Let $d = \ldots d_k d_{k-1} \cdots d_1 d_{0(b)}$ and $\ldots e_k e_{k-1} \cdots e_1 e_{0(b)}$ be the base b representation of two natural numbers. Then the digits of the base b representation of the product $\ldots d_k d_{k-1} \cdots d_1 d_{0(b)} \times \ldots e_k e_{k-1} \cdots e_1 e_{0(b)}$ can be found by computing the base b representation of $(e_0 d) + (e_1 b d) + (e_2 b^2 d) + (e_3 b^3 d) + \cdots$ using the previous two algorithms and the algorithm for computing the base b representation of a sum.

- **d.** other algorithms
  - **i.** Horizontal algorithm (expanded form using the distributive law)
  - **ii.** Vertical algorithm
  - **iii.** Traditional algorithm
  - **iv.** Lattice method
  - **v.** Russian peasant method
- **e.** estimation

8. Division and Exponents for Natural Numbers

- **a.** The Division Theorem

Fact: Let a and b be any natural numbers with $b \neq 0$. Then there exist unique natural numbers q and r such that

$$a = bq + r \text{ and } 0 \leq r < b.$$

In this situation q is called the **integer quotient** and r is called the **remainder** of a divided by b.

- **b.** definition and notation for division

Definition: If a and b are natural numbers with $b \neq 0$ such that the remainder when a is divided by b is 0. Then $a \div b$ is defined to be the unique natural number q so that

$$bq = a.$$

- **c.** algorithms for division
  - **i.** repeated subtraction
  - **ii.** intermediate algorithm: repeated subtraction with non-optimal multiples allowed to be subtracted
  - **iii.** traditional algorithm
- **d.** exponents
  - **i.** definition

Definition: For any number b and any natural number n which are not both zero, $b^0 = 1$, $b^1 = b$, and for $n > 1$

$$b^n = \underbrace{b \times b \times \cdots \times b}_{n \text{ factors}}.$$

ii. laws of exponents

Fact: (Laws of Exponents) For any number $a \neq 0$ and any natural numbers n and m with $m < n$,

$$a^n a^m = a^{n+m}$$

and

$$a^n \div a^m = a^{n-m}.$$

iii. precedence of operations

Notation: In order to minimize the need for parentheses, operators are ranked or given a precedence relative to each other. Operators in parentheses have the highest precedence. Then the arithmetic operators are ranked as follows:

exponentiation

multiplication, division

addition, subtraction

Negation is treated as multiplication by -1 in determining precedence.

9. Divisibility

a. definition of $b \mid a$

Definition: Let a and b be natural numbers with $b \neq 0.$ Then b **divides** a if $a = bn$ for some natural number $n.$ In this case we write $b \mid a$ and say b is a **divisor** of a (or a is a multiple of b). If b does not divide a we write $b \nmid a.$

Note: $b \mid a$ is a statement, not a number.

b. definition of even and odd

Definition: Let a be a natural number. We say a is **even** if $2 \mid a.$ If a is not even then we say it is **odd**.

c. properties of $\mid$

Facts: Let $a,$ b and c be natural numbers.
1. If a number divides two other numbers then it also divides their sum, i.e.

$$\text{if } a \mid b \text{ and } a \mid c \text{ then } a \mid (b+c).$$

2. If a number divides one of two numbers but not the other then it does not divide their sum, i.e.

$$\text{if } a \mid b \text{ and } a \nmid c \text{ then } a \nmid (b+c).$$

3. If one number divides another number, then it will divide the product of that number with any other natural number, i.e.

$$\text{if } a \mid b \text{ then } a \mid bc.$$

d. definition of prime number

Definition: A natural number a is a **prime number** if $a > 1$ and the only divisors of a are 1 and a.

e. Sieve of Eratosthenes

Algorithm: To find all of the primes less than n.
1. Write down all of the numbers $2,3,4,\ldots,n-1$.
2. Circle the left most number which is not circled or crossed out.
3. Cross out all of the multiples of the number you just circled in step #2 (except the number itself).
4. If there are any numbers left which are not circled or crossed out, go to step #2.
5. The numbers that are circled are the primes less than n.

f. tests for divisibility by 2,3,4,5,6,7,8,9,10,11

Divisibility Tests: Let $d_k \cdots d_{0(\text{ten})}$ be the base ten representation of a natural number n. Then

a. $2 \mid n$ if and only if $2 \mid d_0$ (i.e. if n ends with $0, 2, 4, 6,$ or 8)
b. $3 \mid n$ if and only if $3 \mid d_0 + d_1 + \cdots + d_k$ (i.e. if 3 divides the sum of the digits)
c. $4 \mid n$ if and only if $4 \mid d_1 d_{0(\text{ten})}$ (i.e. if 4 divides the number formed by the last two digits)
d. $5 \mid n$ if and only if $5 \mid d_0$ (i.e. if n ends with 0 or 5)
e. $6 \mid n$ if and only if $3 \mid n$ and $2 \mid n$
f. $7 \mid n$ if and only if $7 \mid d_k \cdots d_{1(\text{ten})} - 2d_0$
g. $8 \mid n$ if and only if $8 \mid d_2 d_1 d_{0(\text{ten})}$ (i.e. if 8 divides the number formed by the last three digits)
h. $9 \mid n$ if and only if $9 \mid d_0 + d_1 + \cdots + d_k$ (i.e. if 9 divides the sum of the digits)
i. $10 \mid n$ if and only if $10 \mid d_0$ (i.e. if n ends with 0)
j. $11 \mid n$ if and only if $11 \mid (d_0 + d_2 + d_4 + \cdots) - (d_1 + d_3 + d_5 + \cdots)$ or $11 \mid (d_1 + d_3 + d_5 + \cdots) - (d_0 + d_2 + d_4 + \cdots)$

g. testing a number to see if it is prime

Fact: A natural number $n > 1$ is prime if and only if n is not divisible by any prime number which is less than or equal to $\sqrt{n}$.

10. Prime factorization, GCD, LCM

a. Fundamental Theorem of Arithmetic

Theorem: Every natural number greater than 1 can be written uniquely as a product of prime numbers (where the factors are written in increasing order).

i. finding the prime factorization
    A. by factor tree

b. GCD
    i. definition

Definition: Let a and b be natural numbers which are not both zero. Then $\gcd(a,b)$ is the largest natural number d such that $d \mid a$ and $d \mid b$.

ii. computing GCD(a,b)
    A. by listing divisors

Algorithm: Let a and b be natural numbers. To compute $\gcd(a,b)$,
1. List all the divisors of a.
2. List all the divisors of b
3. Compare the two lists to find the biggest number they have in common.

B. by prime factorization

Fact: Let $a > 1$ and $b > 1$. If the prime factorization of a is $p_1^{r_1}p_2^{r_2}\cdots$ and the prime factorization of b is $p_1^{s_1}p_2^{s_2}\cdots$ (where p_i is the i^{th} prime number) then the prime factorization of $\gcd(a,b)$ is
$p_1^{t_1}p_2^{t_2}\cdots$ where $t_i = \min(r_i, s_i)$.

C. by the Euclidean algorithm

Fact: Let a and b be natural numbers. Let r be the remainder when a is divided by b. Then $\gcd(a,b) = \gcd(b,r)$.

Algorithm: To find the $\gcd(a,b)$ continually replace the larger of the two numbers with the remainder obtained by dividing the smaller into the larger, until a zero remainder is obtained. The last nonzero remainder obtained in this fashion is the $\gcd(a,b)$.

c. LCM
    i. definition

Definition: Let a and b be nonzero natural numbers. Then $\operatorname{lcm}(a,b)$ is the smallest positive natural number d such that $a \mid d$ and $b \mid d$.

ii. computing LCM(a,b)
    A. by listing multiples

Algorithm: Let a and b be natural numbers. To compute $\operatorname{lcm}(a,b)$,
1. List all the nonzero multiples of a which are less than or equal to ab
2. List all the nonzero multiples of b which are less than or equal to ab
3. Compare the two lists to find the smallest number they have in common.

B. by prime factorization

Fact: Let $a > 1$ and $b > 1.$ If the prime factorization of a is $p_1^{r_1} p_2^{r_2} \cdots$ and the prime factorization of b is $p_1^{s_1} p_2^{s_2} \cdots$ (where p_i is the i^{th} prime number) then the prime factorization of $\operatorname{lcm}(a,b)$ is $p_1^{t_1} p_2^{t_2} \cdots$ where $t_i = \max(r_i, s_i).$

C. by the Euclidean algorithm with LCM(a,b)=ab/GCD(a,b)

Fact: For any nonzero natural numbers a, b, $\operatorname{lcm}(a,b)\gcd(a,b) = ab$

11. Integers

a. definition of additive inverse or opposite

Definition: Let a be a natural number. The **additive inverse** (or **opposite**) of a is a unique number n such that

$$a + n = 0.$$

The opposite of a is denoted ${}^{-}a.$

Definition: An **integer** is either a natural number or the additive inverse of a natural number. The set of all integers is denoted $\mathbb{Z}$, i.e.

$$\mathbb{Z} = \{\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3, \ldots\}.$$

b. addition of integers

Definition: Let a and b be nonzero natural numbers. Then

1. ${}^{-}a + {}^{-}b = {}^{-}(a+b)$
2. if $a > b$ then ${}^{-}a + b = {}^{-}(a-b)$
3. if $a < b$ then ${}^{-}a + b = b - a.$

c. subtraction of integers

Definition: Let a and b be natural numbers. Then $a - b = a + {}^{-}b.$

d. multiplication of integers

Definition: Let a and b be nonzero natural numbers. Then

1. ${}^{-}a \times {}^{-}b = ab$
2. $a \times {}^{-}b = {}^{-}(ab)$
3. ${}^{-}a \times b = {}^{-}(ab)$

e. properties of integer arithmetic

Facts: For all integers $a, b, c.$

1. $a + b$ is an integer (closure of $+$)
2. $a + b = b + a$ (commutativity of $+$)

3. $(a+b)+c = a+(b+c)$ (associativity of $+$)
4. $a+0 = a$ and $0+a = a$ (identity of $+$)
5. there is an integer ${}^{-}a$ such that $a + {}^{-}a = 0$ (additive inverses exist)
6. ab is an integer (closure of $\times$)
7. $ab = ba$ (commutativity of $\times$)
8. $(ab)c = a(bc)$ (associativity of $\times$)
9. $1 \times a = a$ and $a \times 1 = a$ (identity of $+$)
10. $a(b+c) = (ab)+(ac)$ (distributive law)

f. division of integers

Definition: If a and b are natural numbers with $b \neq 0$ such that

$$bq = a.$$

for some integer q, then $a \div b$ is defined to be the integer q.

g. definition of inequality of integers

Definition: Let a and b be integers. We say $a < b$ if $a+n = b$ for some positive natural number n. The statements $a \leq b,\ a > b,\ a \geq b,\ a \nleq b,$ etc. are defined in the same manner as they are for natural numbers.

12. Rational Numbers

a. definition of fraction

Definition: For any numbers a and b with $b \neq 0$ the symbol $\frac{a}{b}$ is called a **fraction.** a is called the **numerator** of the fraction and b is called the **denominator** of the fraction. If a and b are integers with $b > 0$ and $\gcd(a,b) = 1$ then $\frac{a}{b}$ is called a **reduced fraction**. Any number a can be represented as a fraction by writing it as $\frac{a}{1}$.

Definition: Every reduced fraction represents a number called a **rational number**. The set of all rational numbers is denoted $\mathbb{Q},$ i.e. $\mathbb{Q}$ is the set of all reduced fractions. Every integer n is also a rational number, since $n = \frac{n}{1}$.

b. equality of fractions

i. reducing fractions

Definition: Let $a,\ b$, and k be numbers with $b \neq 0$ and $k \neq 0.$ Then

$$\frac{a}{b} = \frac{ka}{kb}.$$

i.e. the fractions $\frac{a}{b}$ and $\frac{ka}{kb}$ represent the same number.

GCD Method: If a and b are integers with $b \neq 0$ then $\frac{a}{b}$ represents the rational number whose reduced fraction representation is: $\frac{\left(\frac{a}{\gcd(a,b)}\right)}{\left(\frac{b}{\gcd(a,b)}\right)}$. i.e. to **reduce** the fraction $\frac{a}{b}$, divide the numerator and denominator by $\gcd(a,b)$.

Prime Factorization Method: Factor the numerator and denominator and cancel like factors.

Non-optimal Cancellation: Repeatedly divide the numerator and denominator by a common nontrivial divisor until a reduced fraction is obtained.

ii. cross multiplication

Fact: Let $a,\ b,\ c,$ and d be numbers with $b \neq 0$ and $d \neq 0.$ Then

$$\frac{a}{b} = \frac{c}{d} \text{ if and only if } ad = bc.$$

c. rules for signs in fractions

Definition: For integers a, b with $b \neq 0$

$$\frac{a}{{}^{-}b} = \frac{{}^{-}a}{b} = {}^{-}\left(\frac{a}{b}\right)$$

and

$$\frac{{}^{-}a}{{}^{-}b} = \frac{a}{b}.$$

d. inequality of fractions

i. cross multiplication

Definition: Let $a,\ b,\ c,$ and d be numbers with $b > 0$ and $d > 0.$ Then

$$\frac{a}{b} < \frac{c}{d} \text{ if and only if } ad < bc.$$

Similar definitions for $>, \leq, \geq, \nleq,$ etc. can be made as usual.

e. mixed numbers

Definition: A symbol of the form

$$n\frac{a}{b}$$

where n, $a,$ and b are integers with $a \geq 0$ and $b > 0$ is called a **mixed number**. The mixed number expression $n\frac{a}{b}$ represents the number $n + \frac{a}{b}$. (see below for definition of $+$ for rational numbers)

f. improper fractions

Definition: A fraction of the form $\frac{a}{b}$ where $a > b$ is called an **improper fraction**.

13. Rational Arithmetic

a. addition, subtraction, multiplication, and division of fractions

Definition: For any fractions $\frac{a}{b}, \frac{c}{d},$

$$
\begin{aligned}
\frac{a}{b} + \frac{c}{d} &= \frac{ad+bc}{bd} \\
\frac{a}{b} - \frac{c}{d} &= \frac{a}{b} + \frac{^{-}c}{d} \\
\frac{a}{b} \times \frac{c}{d} &= \frac{ac}{bd} \\
\frac{a}{b} \div \frac{c}{d} &= \frac{a}{b} \times \frac{d}{c}
\end{aligned}
$$

Note: For any fraction $\frac{a}{b}$

$$\frac{a}{b} = \frac{a1}{1b} = \frac{a}{1} \times \frac{1}{b} = \frac{a}{1} \div \frac{b}{1} = a \div b.$$

Thus any fraction can be interpreted as division, i.e. $\frac{a}{b} = a \div b.$

b. arithmetic with natural numbers and fractions

Note: We can do arithmetic involving both natural numbers and fractions by writing the natural numbers in their reduced fraction form, e.g. if n is a natural number and $\frac{a}{b}$ is a fraction then

$$
\begin{aligned}
n + \frac{a}{b} &= \frac{n}{1} + \frac{a}{b} \\
n - \frac{a}{b} &= \frac{n}{1} - \frac{a}{b} \\
n \times \frac{a}{b} &= \frac{n}{1} \times \frac{a}{b} \\
n \div \frac{a}{b} &= \frac{n}{1} \div \frac{a}{b} \\
\frac{a}{b} \div n &= \frac{a}{b} \div \frac{n}{1}
\end{aligned}
$$

and so on.

c. negative exponents

Definition: If n is a positive integer and a is a number other than zero, then

$$a^{-n} = \frac{1}{a^n}$$

Comment: The Laws of Exponents (section 8.c.ii) for natural number exponents hold for integer exponenets as well.

d. properties of rational arithmetic

Facts: For all rational numbers $a, b, c.$

1. $a + b$ is a rational number (closure of $+$)
2. $a + b = b + a$ (commutativity of $+$)
3. $(a + b) + c = a + (b + c)$ (associativity of $+$)
4. $a + 0 = a$ and $0 + a = a$ (identity of $+$)
5. there is a rational number ^{-}a such that $a + {}^{-}a = 0$ (additive inverses exist)
6. ab is a rational number (closure of $\times$)
7. $ab = ba$ (commutativity of $\times$)

8. $(ab)c = a(bc)$ (associativity of $\times$)
9. $1 \times a = a$ and $a \times 1 = a$ (identity of $+$)
10. $a(b+c) = (ab)+(ac)$ (distributive law)
11. if $a \neq 0$ then there exists a rational number $\frac{1}{a}$ such that $a \times \frac{1}{a} = 1$ and $\frac{1}{a} \times a = 1$ (multiplicative inverses exist)

14. Decimal Notation

a. definition: place value in base b

Definition: Let b is a natural number greater than 1, then we define a **base b expansion of a real number** to be a sequence of digits

$$d_k \cdots d_0 . d_{-1} d_{-2} \ldots_{(b)}.$$

where each digit, d_i, is between 0 and $b-1$ inclusive. (Note: if $b > 10$ we use the letters of the alphabet in order for the extra digits, e.g. $A = 11$, $B = 12$, etc.) This expression represents the number $d_k b^k + \cdots + d_1 b + d_0 + d_{-1}\frac{1}{b} + d_{-2}\frac{1}{b^2} + d_{-3}\frac{1}{b^3} + \cdots$. Such a number is called a **real number**. The set of all real numbers is denoted $\mathbb{R}$. The base ten expansion of a real number is called a **decimal expansion**.

Note: More than one base b expansion can represent the same real number, for example $1.0000\ldots = 0.9999\ldots$ in base ten.

b. finite decimals and repeating decimals

Notation: Just as with periodic sequences we also use the overbar notation to denote periodic sequences of digits in a base b expansion. For example $1.2\overline{34} = 1.23434343434\ldots$.

Definition: A base b expansion which ends repeating 0's is called a **finite expansion** and can be abbreviated by truncating the zeros. If the base is ten then a finite expansion is called a **finite decimal**.

Fact: A base b expansion represents a rational number if and only if its digits are eventually repeating.

c. criteria for a fraction to have a finite decimal

Fact: A reduced fraction $\frac{a}{b}$ can be represented by a finite decimal if and only if no prime number other than 2 or 5 divides b.

d. converting a fraction to a repeating decimal

Algorithm: To convert a reduced fraction $\frac{a}{b}$ to a repeating decimal, use long division.

e. converting a repeating decimal to a fraction

Algorithm: To convert a repeating decimal, x, to a fraction.
1. Count the number of digits that repeat (i.e. under the overbar) and call this n.

2. Compute $y = 10^n x - x$ by subtraction (the repeated parts can be cancelled)
3. Let k be the number of digits to the right of the decimal point in the finite decimal expansion of y. Then $10^k y$ is an integer.
4. Then $x = \frac{10^k y}{10^k(10^n - 1)}$. (In most cases you should reduce this fraction.)

15. Ratio, Percent, Scientific Notation

a. ratio notation (a:b)

Definition: The **ratio** of any two positive numbers a and b is the fraction $\frac{a}{b}$. This ratio is also written $a : b$.

b. definition of proportion

Definition: An equation of the form

$$\frac{a}{b} = \frac{c}{d}$$

is called a **proportion**. (Here $a, b, c,$ and d are numbers with $b \neq 0$ and $d \neq 0$.)

c. definition of %

Definition: The symbol $\%$ simply means $\frac{1}{100}$. In particular, if n is a number then $n\% = n\frac{1}{100}$.

d. scientific notation

Definition: An expression of the form $a \times 10^n$ where n is an integer and a is a number such that $1 \leq a < 10$ is called **scientific notation** for the number $a(10^n)$. The number a is called the **mantissa** and the number n is called the **characteristic**.

e. word problems

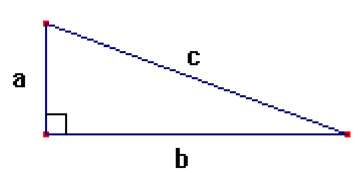
16. Real Numbers and Irrational Numbers

a. definition of irrational

Definition: A real number which is not rational is called an **irrational number**.

b. Pythagorean theorem

Theorem: In the right triangle:



the lengths of the sides shown satisfy

$$a^2 + b^2 = c^2.$$

c. irrationality of $\sqrt{2}$

Fact: $\sqrt{2}$ is not equal to any rational number.

d. definition of n^{th}-roots

Definition: Let a and b be real numbers and n a positive integer. If $b^n = a$ then b is called a n^{th} root of a. If a has an n^{th} root, the principle n^{th} root of a is the n^{th} root that has the same sign as a denoted $\sqrt[n]{a}$.

Facts:
1. if a is positive and n is even then a has two n^{th} roots, $\sqrt[n]{a}$ and $-\sqrt[n]{a}$.
2. if a is positive and n is odd then a has one positive n^{th} root, $\sqrt[n]{a}$.
3. if a is negative and n is even then a has no n^{th} roots.
4. if a is negative and n is odd then a has one negative n^{th} root, $\sqrt[n]{a}$.

e. product of two square roots

Fact: For any positive real numbers a and b,

$$\sqrt{a}\,\sqrt{b} = \sqrt{ab}.$$

f. properties of real arithmetic

Facts: For all real numbers a, b, c.
1. $a + b$ is a real number (closure of $+$)
2. $a + b = b + a$ (commutativity of $+$)
3. $(a + b) + c = a + (b + c)$ (associativity of $+$)
4. $a + 0 = a$ and $0 + a = a$ (identity of $+$)
5. there is a real number ^{-}a such that $a + {}^{-}a = 0$ (additive inverses exist)
6. ab is a real number (closure of $\times$)
7. $ab = ba$ (commutativity of $\times$)
8. $(ab)c = a(bc)$ (associativity of $\times$)
9. $1 \times a = a$ and $a \times 1 = a$ (identity of $+$)
10. $a(b + c) = (ab) + (ac)$ (distributive law)
11. if $a \neq 0$ then there exists a real number $\frac{1}{a}$ such that $a \times \frac{1}{a} = 1$ and $\frac{1}{a} \times a = 1$ (multiplicative inverses exist)

17. Algebraic Expressions, Equations, Inequalities

a. simplifying algebraic expressions

Definition: A real variable is a single symbol which represents a single unspecified real number. A (real) expression is a collection of symbols which represents a real number. To simplify an expression means to write a shorter or simpler expression which represents the same real number as the original expression.

Techniques: To simplify an algebraic expression, we make use of the following facts. Let x be any expression and a and b numbers.
1. $1x$ can be replaced with x.
2. $0 + x$ can be replaced with x.
3. $ax + bx$ can be replaced by $(a + b)x$.
4. $-(-x)$ can be replaced by x.

b. solving linear equations in one variable

Definition: An **equation** is a statement of the form $A = B$ (where A and B are usually expressions). To **solve** an equation means to find all the values of the variables in A and B which make the statement true.

Facts: For any real expressions a, b, and c,

$$a = b \text{ if and only if } a + c = b + c$$

and if $c \neq 0$ then

$$a = b \text{ if and only if } ac = bc.$$

c. solving linear inequalities in one variable

Definition: An **inequality** is a statement of the form $A < B$ (where A and B are usually expressions). To **solve** an inequality means to find all the values of the variables in A and B which make the statement true.

Note: statements of the form $A \leq B$, $A > B$, and $A \geq B$ are also inequalities.

Facts: For any real expressions a, b, and c,

$$a < b \text{ if and only if } a + c < b + c.$$

If $c > 0$ then

$$a < b \text{ if and only if } ac < bc.$$

If $c < 0$ then

$$a < b \text{ if and only if } ac > bc.$$

(Analogous facts hold for $\leq, \geq,$ and $>$)

i. plotting the solution on a number line

Definition: To plot the solution of an equation or inequality in one variable on a number line means to color or indicate all of those points on the line whose coordinate is a solution.

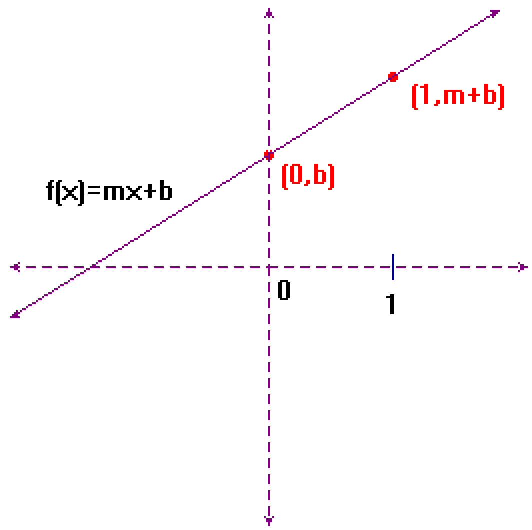
d. substitution

Fact: When solving inequalities or equations, you may substitute an equivalent expression for any expression in the statement without changing the set of solutions. (See **simplifying** above)

18. Functions and their Graphs

a. linear functions: $f(x) = mx + b$

Fact: Let m and b be real numbers. The graph of $f(x) = mx + b$ is the straight line through the points $(0, b)$
and $(1, m + b)$.



i. slope

Definition: The **slope** of the graph of $f(x) = mx + b$ is m.

ii. intercept

Definition: The **intercept** of the graph of $f(x) = mx + b$ is b.

iii. point-point formula for slope

Fact: If $x_1 \neq x_2$ then the line which passes through the points (x_1, y_1) and (x_2, y_2) has slope $\frac{y_1 - y_2}{x_1 - x_2}$.

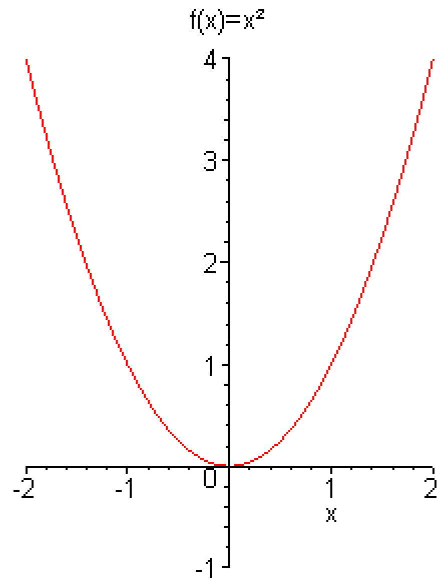
b. step functions

Definition: Let x be a real number. The **floor** of x is the largest integer which is less than or equal to x and is denoted $\lfloor x \rfloor$. The **ceiling** of x is the smallest integer which is greater than or equal to x and is denoted $\lfloor x \rfloor$.

c. quadratic function

Definition: A **quadratic function** is a function of the form $f(x) = ax^2 + bx + c$ where $a, b,$ and c are real numbers and $a \neq 0$.

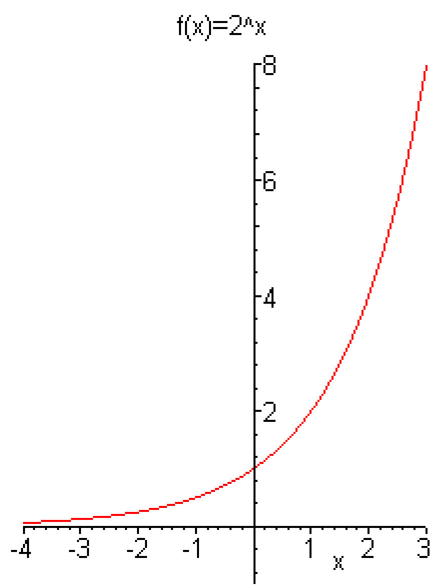
Fact: The graph of a quadratic function is a parabola with max/min value at $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.



d. exponential functions

i. doubling: 2^x

Definition: The function $f(x) = 2^x$ is called the **doubling function.**



ii. compound interest and e^x

Definition: e is an irrational number whose value to seven decimal places is $2.7182818.$

Fact: If you deposit and amount P in an interest bearing account which has continuously compounded interest at an interest rate of $r,$ then the amount A that you will have in that account after t years is given by

$$A = Pe^{rt}.$$

19. Probability - Single Stage Experiments

a. probability of an event in a finite sample space

Definition: The set of all possible outcomes of an experiment is called the **sample space**. A subset of the sample space is called an **event**. A sample space is **finite** if it has finitely many outcomes.

Fact: If all of the outcomes of a finite sample space, $S,$ are equally likely, then the probability of an event $E \subseteq S$ is

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{\#(E)}{\#(S)}.$$

Notice that this implies the following facts:

- $0 \leq P(E) \leq 1$ (every probability is between 0 and 1 inclusive)
- $P(S) = 1$ (the probability the event S is 1)
- $P(\emptyset) = 0$ (the probability of the empty set is zero)
- $P(\overline{E}) = 1 - P(E)$ (the probability of the complement of a set plus the probability of the set add to 1)

b. probability of mutually exclusive events (addition property)

Fact: The probability of an event $E,$ which has outcomes $e_1, e_2, \ldots, e_n$ is the sum of the probabilities of the outcomes,

$$P(E) = P(e_1) + P(e_2) + \ldots + P(e_n).$$

c. probability of a non-mutually exclusive events (two events only)

Fact: For any two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Note: If A and B are disjoint, then $A \cap B = \emptyset,$ so $P(A \cap B) = 0$ and so in this case

$$P(A \cup B) = P(A) + P(B).$$

d. definition of odds

Definition: If E is an event with probability $P(E)$ then the **odds in favor** of E are $P(E)$ to $1 - P(E)$. Conversely if the odds in favor of E are n to m then $P(E) = \frac{n}{n+m}$. Similarly, if E

is an event with probability $P(E)$ then the **odds against** E are $1 - P(E)$ to $P(E)$. If the odds against of E are m to n then $P(E) = \frac{n}{n+m}$.

e. experimental probabilities and simulations

20. Probability - Multistage Experiments

a. probability if independent events (multiplication property)

Definition: If two events are such that the occurrence of either one does not affect the probability of the occurrence of the other we say the events are **independent.**

Fact: If A and B are independent events and C is the multistage event consisting of first obtaining an outcome in event A followed by an outcome in event B then

$$P(C) = P(A)P(B).$$

b. tree diagram for multistage experiments

c. probability of complementary events

Fact: If the probability of and event E is $P(E)$ then the probability of the even not happening is $1 - P(E)$.

d. expected value

Definition: If each of the outcomes $e_1, \ldots, e_n$ in an event, E, has a value of $v_1, \ldots, v_n$ respectively, then the expected value of the event E is

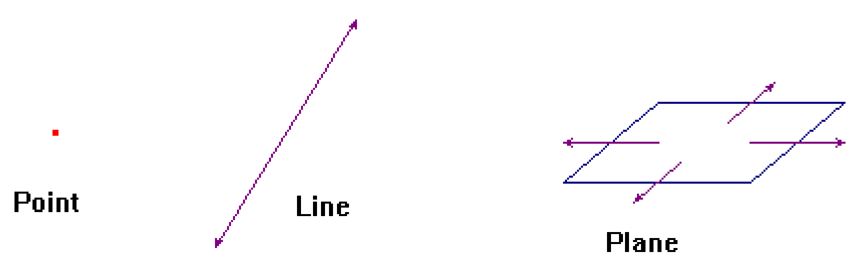
$$P(e_1)v_1 + P(e_2)v_2 + \cdots + P(e_n)v_n.$$

21. Plane Geometry

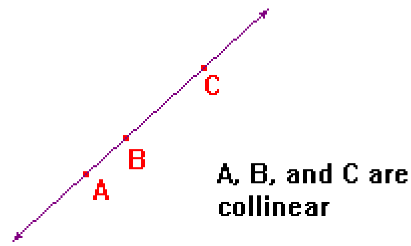
a. basic terms: point, line, plane, ray, segment, half-plane, angle, collinear, coplanar

Definitions and facts:

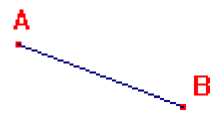
1. **Point**, **Line**, and **Plane** are undefined terms. Lines and Planes can be thought of as certain sets of points.



2. A set of points is **collinear** if all of the points in the set are contained in a single line. A set of lines and/or points is **coplanar** if they are contained in a single plane.

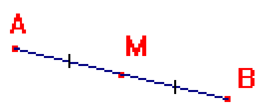


3. A **line segment** consists of two distinct points called **endpoints** and all of the points between the endpoints. If the endpoints are *A* and *B* we denote the line segment by $\overline{AB}$.



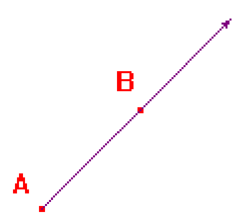
Line Segment $\overline{AB}$

4. The **midpoint** of a line segment $\overline{AB}$ is the point *M* on *AB* such that the lengths of $\overline{AM}$ and $\overline{MB}$ are equal.



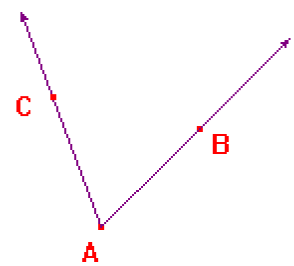
Midpoint M

5. A line in a plane partitions the plane into three disjoint sets, the points on the line and two **half-planes**. All of the points in a given half-plane are said to be on the same side of the line.
6. A point on a line partitions the line into three disjoint sets, the set containing the point itself and two half-lines. A **ray** consists of point on a line (called the **vertex** of the ray) and all of the points in one of the half-lines determined by the point.



Ray $\overrightarrow{AB}$

7. An **angle** consists of two rays with a common vertex. If the two rays together form a line, then the angle is called a **straight angle**.



Angle CAB

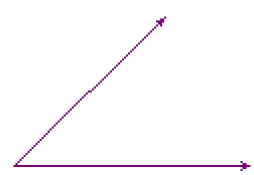
b. angle measurement (degrees, minutes, seconds)

Fact: Every angle can be assigned a measurement. A straight angle has measurement 180°. A **minute** is $\frac{1}{60}$ of a degree and a **second** is $\frac{1}{60}$ of a minute.

c. types of angles

i. acute

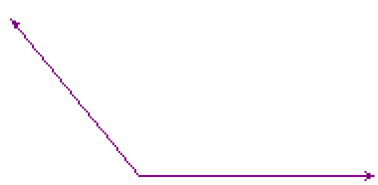
Definition: An angle is **acute** if its measurement is greater than 0° but less than 90°.



An Acute Angle

ii. obtuse

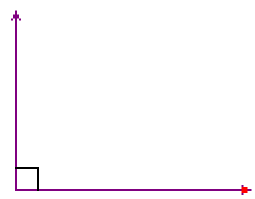
Definition: An angle is **obtuse** if its measurement is greater than 90° but less than 180°.



An Obtuse Angle

iii. right

Definition: An angle is **right** if its measurement is 90°.

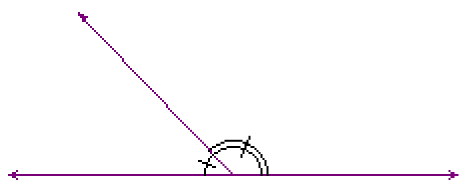


A Right Angle

d. relationships among angles

i. supplementary

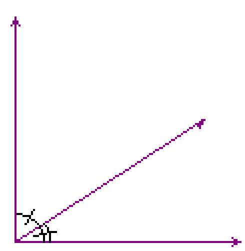
Definition: Two angles are **supplementary** if the sum of their measurements is 180°.



Supplementary Angles

ii. complementary

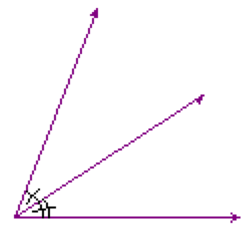
Definition: Two angles are **complementary** if the sum of their measurements is 90°.



Complementary Angles

iii. adjacent

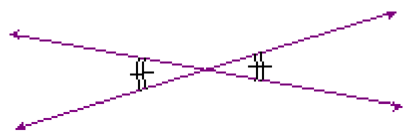
Definition: Two angles are **adjacent** if they share a common ray.



Adjacent Angles

iv. vertical

Definition: Nonadjacent angles formed by two intersecting lines are called **vertical angles**.

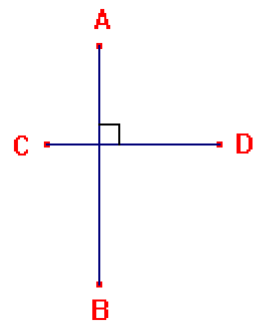


Vertical Angles

e. relationships among lines

i. perpendicular

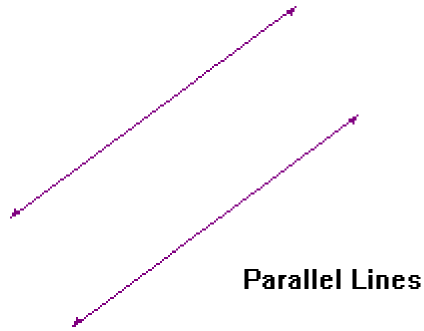
Definition: Two lines or segments which intersect at right angles are said to be **perpendicular**.



Perpendicular Segments

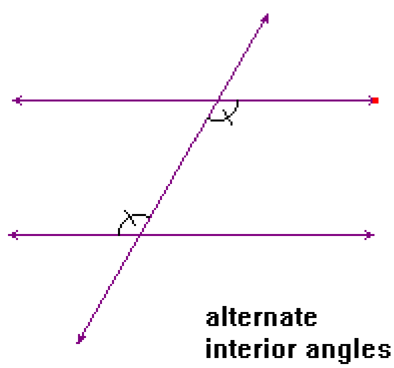
ii. parallel

Definition: Two coplanar lines which do not intersect are said to be **parallel**.



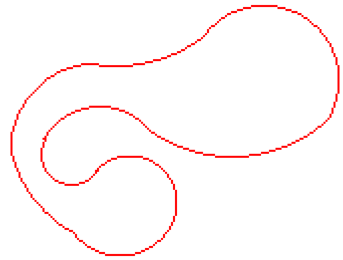
A. alternate interior angle theorem

Fact: Two lines in a plane intersected by a third line (called a **transversal**) are parallel if and only if the alternate interior angles formed have equal measure.



f. simple closed curves and convex sets

Definition and facts: A curve drawn in a plane with a continuous motion that does not intersect itself and starts and stops at the same point is called a **simple closed curve**. A simple closed curve divides the plane into three disjoint regions, the curve itself and an interior and an exterior. We say the interior region is **bounded** by the simple closed curve. A region in the plane is **convex** if every line segment whose endpoints are in the region, is itself contained in the region.



A simple closed curve

g. circles

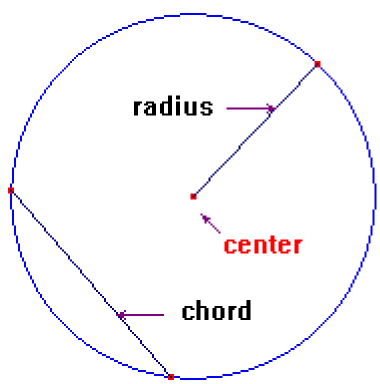
i. definition

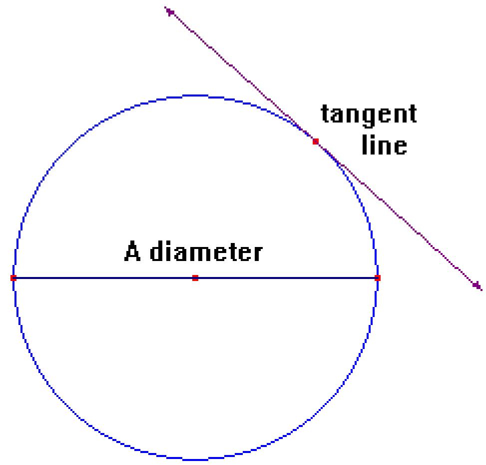
Definition: The set of all points in the plane that are a fixed distance from a given point is called a **circle**. The given point that is equidistant from all of the points on the circle is called the **center** of the circle.

ii. basic terms: radius, diameter, chord, tangent

Definitions: Let S be a circle with center C and let P and Q be any points on the circle.
1. $\overline{CP}$ is called a radius of the circle.
2. $\overline{PQ}$ is called a chord of the circle.
3. If C is on $\overline{PQ}$ then $\overline{PQ}$ is called a diameter of the circle.
4. Any line which intersects the circle in only one point is called a **tangent line** to the circle

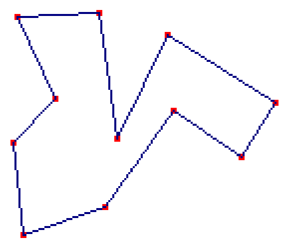
A circle





iii. polygons

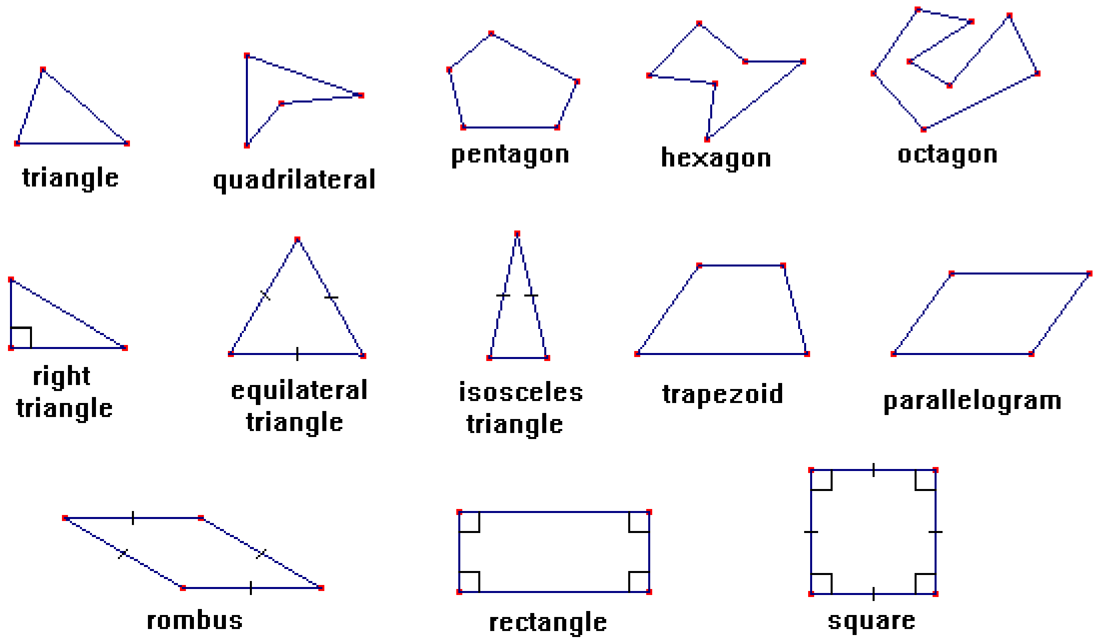
Definitions: A simple closed curve that is the union of finitely many line segments is called a **polygon**. Its interior is called a **polygonal region**. The line segments which make up a polygon are called its **sides**.



a polygon

iv. types of polygons

Definitions:
1. A polygon with three sides is called a **triangle**.
2. A polygon with four sides is called a **quadrilateral**.
3. A polygon with five sides is called a **pentagon**.
4. A polygon with three sides is called a **hexagon**.
5. A polygon with eight sides is called an **octagon**.
6. A triangle with one right angle is called a **right triangle**.
7. A triangle with all three sides of equal length is called an **equilateral triangle**.
8. A triangle with at least two sides of equal length is called an **isosceles triangle**.
9. A quadrilateral with exactly one pair of parallel sides is called a **trapezoid**.
10. A quadrilateral with two pairs of parallel sides is called a **parallelogram**.
11. A quadrilateral with all four sides of equal length is called a **rhombus**.
12. A quadrilateral with four right angles is called a **rectangle**.
13. A rhombus with four right angles is called a **square**.



22. Polygons and Tessellations

a. sum of the angle measures in convex polygons

Fact: The sum of the angles of a convex polygon with n sides is

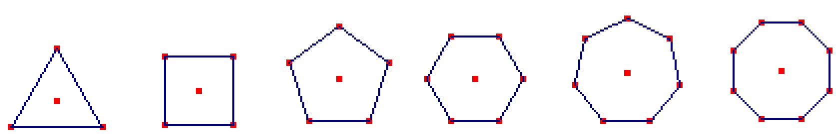
$$(n-2)180°.$$

b. congruence of line segments and angles

Definition: Two line segments are **congruent** if they are the same length. Two angles are **congruent** if they have the same measure.

c. regular polygons

Definition: A **regular polygon** is a polygon whose sides are all congruent to one another and whose angles are all congruent to one another.



Regular Polygons

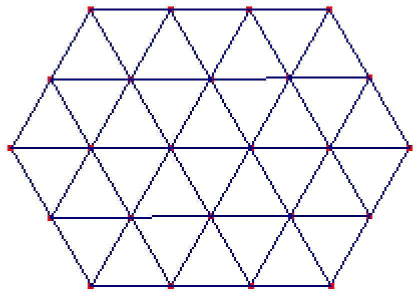
d. tessellations

Definition: An arrangement of figures which do no overlap (except perhaps along their boundaries) which covers a region is called a **tessellation** of that region.

i. regular

Definition: A **regular tessellation** of the plane is a tessellation of the plane by a single regular polygon.

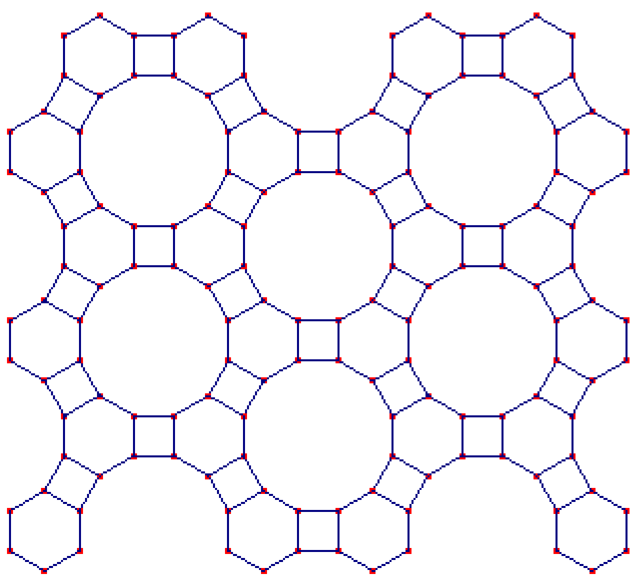
Fact: The only regular polygons which can tessellate the plane by themselves are the equilateral triangle, the square, and the regular hexagon.



tessellating by regular triangles

ii. semi-regular

Definition: A **semiregular tessellation** of the plane is a tessellation of the plane by more than one type of regular polygon such that the each vertex is surrounded by the same arrangement of polygons.



a semi-regular tessellation

iii. with congruent tiles which are not regular

23. Three Dimensional Geometry

a. polyhedra

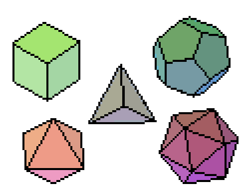
Definition: A surface that encloses a connected region of space and that consists of a finite union of polygonal regions meeting only at their boundaries is called a **polyhedron**. A polyhedron is **convex** if the line segment connecting any two points in the region the polyhedron encloses is itself contained in the region that the polyhedron encloses.

i. regular polyhedra

Definition: A convex polyhedron whose faces are congruent regular polygons, the same number of which meet at each vertex, is called a **regular polyhedron**.

Fact: There are only five types of regular polyhedra:
1. tetrahedron - 4 triangular faces
2. cube - 6 square faces
3. octahedron - 8 triangular faces
4. dodecahedron - 12 pentagonal faces
5. icosahedron - 20 triangular faces

The Five Regular Polyhedra



ii. semi-regular polyhedra

Definition: A convex polyhedron whose faces consist of two or more types of congruent regular polygons, in the same arrangement at each vertex, is called a **semiregular polyhedron**.

b. pyramids and right pyramids

Definition: A **pyramid** is a convex polyhedron having one face (called the **base**) that can be any polygon and having triangles for all of its other faces (called the **sides**). A **right**

pyramid is a pyramid whose sides are isoscles triangles.

c. prisms and right prisms

Definition: A **prism** is a convex polyhedron having two parallel congruent faces (called the **bases**) that can be any polygon and having quadrilaterals for all of its other faces (called the **sides**). A **right pyramid** is a pyramid whose sides are all rectangles.

d. cones and cylinders

Definition: A **cone** consists of a circle (called the **base**), a point not in the same plane as the circle (called the **vertex**) and all line segments connecting the vertex to a point on the circle. A **cylinder** is the surface of the solid figure obtained by taking two circles in distinct, parallel planes, and all line segments connecting any point in the interior or on one circle to a point on or in the other.

e. spheres

Definition: A **sphere** is the set of all points in space that are a fixed distance from a given point called the **center** of the sphere. Any line segment joining a point on the sphere to its center is called a **radius** of the sphere.

i. maps, latitude, longitude

24. Symmetry, Rotation, Reflection

a. symmetry in the plane

i. reflection: lines of symmetry

Definition: A line in a plane is a **line of symmetry** for a given figure, if the figure maps to itself when reflected across that line.

ii. rotational symmetry about a point

Definition: A figure in a plane has **rotational symmetry** if the figure maps to itself when rotated about that point through some angle.

b. symmetry in 3-space

i. reflection: planes of symmetry

Definition: A plane in space is a **plane of symmetry** for a given solid figure, if the figure maps to itself when reflected across that plane.

ii. rotational symmetry about a line

Definition: A figure in a plane has **rotational symmetry** about a given line if the figure maps to itself when rotated about that line through some angle.

25. Measurements and Units

a. English units: length, area, volume, mass, temperature

Conversions:

Length

Name	Unit	Conversion
Inch	in	$\frac{1}{12}$ ft
Foot	ft	12 in
Yard	yd	3 ft
Mile	mi	5280 ft

Volume

Name	Unit	Conversion
Ounce	oz	$\frac{1}{8}$ c
Cup	c	8 oz
Pint	pt	2 c
Quart	qt	2 pt
Gallon	gal	4 qt

Mass

Name	Unit	Conversion
Ounce	oz	$\frac{1}{16}$ lb
Pound	lb	16 oz
Ton	tn	2000 lb

Temperature: to convert a temperature from $°C$ to $°F$,

$$F = \frac{9}{5}C + 32$$

b. metric units: length, area, volume, mass, temperature

Conversions:

Length

Name	Unit	Conversion
millimeter	mm	$\frac{1}{1000}$ m
centimeter	cm	$\frac{1}{100}$ m
meter	m	1 m
kilometer	km	1000 m

Volume

Name	Unit	Conversion
milliliter	ml	$\frac{1}{1000}$ l
Liter	l	1 l

Mass

Name	Unit	Conversion
milligram	mg	$\frac{1}{1000}$ g
gram	g	1 g
kilogram	kg	1000 g

Temperature: to convert a temperature from $°F$ to $°C$,

$$C = \frac{5}{9}(F - 32)$$

or from $°C$ to $°F$,

$$F = \frac{9}{5}C + 32$$

c. conversions between metric and English units

Length

Name	Unit	Conversion
inch	in	2.54 cm

Volume

Name	Unit	Conversion
Liter	l	~1.06 qt

Mass

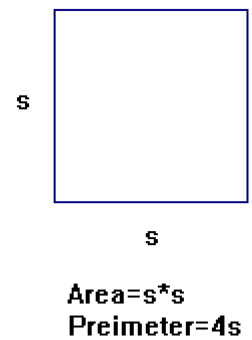
Name	Unit	Conversion
kilogram	kg	~2.2 lb

d. converting between units (factor label method)

26. Area and Perimeter

a. squares

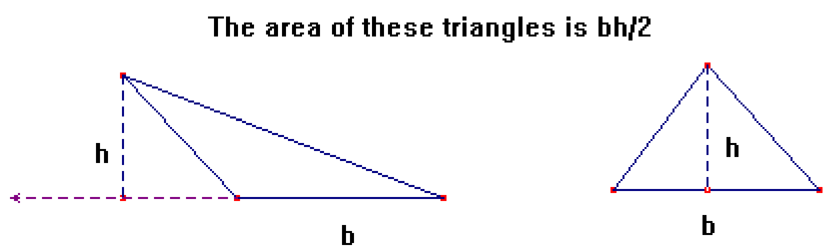
Fact: If a square has side length s then it has area s^2 and perimeter $4s$.



b. triangles

Definition: Given a triangle $\triangle ABC,$ let P be the point on $\overleftrightarrow{AB}$ such that $\overleftrightarrow{CP}$ is perpendicular to $\overleftrightarrow{AB}.$ The **height** of the triangle with respect to base $\overline{AB}$ is defined to be the length of the line segment $\overline{CP}$.

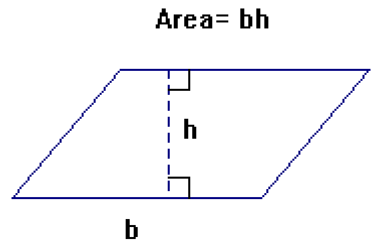
Fact: If a triangle has a side of length b and height h then it has area $\frac{1}{2}bh.$



c. parallelograms and rectangles

Definition: The **height** of a parallelogram with respect to a pair of sides is defined to be the length of the shortest line segment connecting those sides.

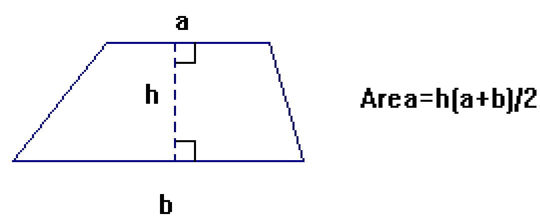
Fact: If a parallelogram has a side of length b and height h with respect to that side then it has area $bh.$



d. trapezoids

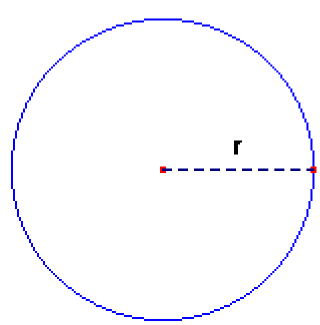
Definition: The **height** of a trapezoid is defined to be the length of the shortest line segment connecting the parallel sides.

Fact: If a trapezoid has a parallel sides of length a and b and height h then it has area $\frac{a+b}{2}h$.



e. circles

Fact: The area of a circle whose radius is length r is πr^2. The circumference (=perimeter) is $2\pi r$.



$$\text{Area} = \pi r^2$$
$$\text{Circumference} = 2\pi r$$

27. Volume and Surface Area

a. right prisms

Fact: The volume of a right prism is its height times the area of its base. The surface area is the sum of the areas of its sides and the areas of its two bases.

b. cylinders

Fact: The volume of a cylinder of is also its height times the area of its base. The surface area is twice the area of the base plus the product of the circumference of the base times the height.

c. pyramids

Definition: The height of a pyramid is the shortest distance from the point which is common to all of its sides to its base.

Fact: The volume of a pyramid is one third of its height times the area of its base.

d. cone

Definition: The height of a cone is the shortest distance from its vertex to its base.

Fact: The volume of a pyramid is one third of its height times the area of its base.

e. sphere

Fact: The volume of a sphere whose radius has length r is $\frac{4}{3}\pi r^3$. The surface area of a sphere whose radius has length r is $4\pi r^2$.

28. Congruence and Constructions

a. definition of congruence for polygons

Definition: Two polygons are congruent if there is a distance preserving transformation which maps one onto the other.

b. ruler and compass constructions

(I will illustrate these constructions for you using Cabri)

- **i.** congruent segments
- **ii.** congruent angles
- **iii.** congruent triangle (by SSS theorem)
- **iv.** perpendicular bisector
- **v.** angle bisector
- **vi.** perpendicular to a line through a given point
- **vii.** circumscribed circle about a triangle
- **viii.** a line parallel to a given line through a point not on the given line

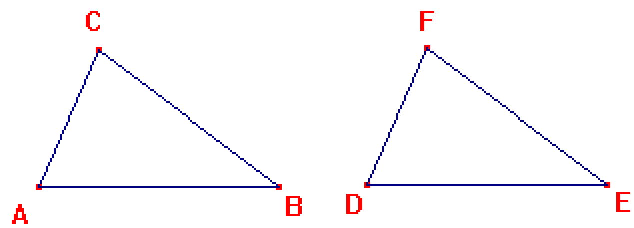
c. the triangle inequality

Fact: Given a triangle $\triangle ABC,$ let c be the length of $\overline{AB},$ a be the length of $\overline{BC}$, and b be the length of $\overline{AC}.$ Then

$$a + b \leq c$$

d. ASA, SAS, SSS, AAS theorems

Fact: Given two triangles $\triangle ABC$ and $\triangle DEF,$ if



a. (ASA) $\angle A \cong \angle D$, $\overline{AB} \cong \overline{DE},$ and $\angle B \cong \angle E$ or

b. (SAS) $\overline{AB} \cong \overline{DE},$ $\angle B \cong \angle E$ and $\overline{BC} \cong \overline{EF}$ or

c. (SSS) $\overline{AB} \cong \overline{DE},$ $\overline{AC} \cong \overline{DF}$ and $\overline{BC} \cong \overline{EF}$ or

d. (AAS) $\angle A \cong \angle D$, $\angle B \cong \angle E$ and $\overline{BC} \cong \overline{EF}$

then $\triangle ABC \cong \triangle DEF.$