Introduction to Formal Proofs - A Toy Proof System

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The following is an exercise designed to introduce you to some of the main concepts used in doing formal mathematical proofs. Read the following and try the exercises. We will discuss how this applies to actual mathematical proofs in class.

We begin by defining our language.

Definition: A ***TOY FORMULA*** is either

a. •
b. o
c. *wv* where *w* and *v* are toy formulas
d. all the formulas are obtained by a-c.

In other words, toy formulas are just strings of circles (o's) and dots (•'s). For example o, •, o•, oooo, and o•oo•• are all examples of toy formulas.

These toy formulas correspond to the mathematical statements we can make. Unlike normal mathematical statements, these formulas don't MEAN anything.

We now define some **axioms**, which are toy formulas that are, by definition, automatically provable.

Axiom 1: o•

Axiom 2: •o

We next specify some rules of inference which allow us to deduce new provable toy formulas, called **theorems**, from the ones we already have proven.

The notation we will use to describe these rules of inference is as follows:

```
INPUTLINE1
INPUTLINE2
```

```
:
INPUTLINEn
------------
OUTPUTLINE1
OUTPUTLINE2
:
OUTPUTLINEm
```

This describes a rule of inference in the following manner... if all of the `INPUTLINE`s are proven, then we can conclude that all of the `OUTPUTLINE`s are proven as well. Usually we will specify whole families of such rules of inference by using meta-variables which stand for an arbitrary toy formula (such as *w* and *v* in the Rules below).

In other words, if we interpret this notation as a "recipe" for deducing lines in our proofs, then we can interpret the above notation as:

In order to show `OUTPUTLINE1,...,OUTPUTLINEm`

1. First show `INPUTLINE1`,
2. Then show `INPUTLINE2`,

:

n. Then show `INPUTLINEn`.

Here are the ***Rules of Inference*** for our toy system.

Rule of inference 1: For any formulas *w* and *v*,

wv
vw
---
w

Rule of inference 2: For any formulas *w* and *v*,

w
v
-----
w•*v*

Rule of inference 3: For any formulas *w* and *v*,

wv•
-----

*w*o

That's it! You might want to stop reading here and try some proofs on your own before reading the discussion below.

Here is an example:

Thm A: •

Proof.

1. o• Axiom 1
2. •o Axiom 2
3. • Rule 1; 2.,1.

QED

Comments: note that when giving a rule as a reason I give the line numbers used as inputs in the rule in the order that they correspond to the lines in the rule. Thus in line 3 above I could not say it is because of Rule 1; 1,2. If I had used Rule 1 with lines 1 and 2 then *w* would be o and *v* would be • and
so we would have concluded o instead of •

Let's try something with rule 2...

Thm B: •••

Proof.

1. • Thm A
2. ••• Rule 2; 1.,1.

QED

Fun fun fun!

How about something with Rule 3?

Thm C: ••o

Proof.

1. ••• Thm B
2. •••••• Rule 2; 1.,1.
3. ••o Rule 3; 2.

QED

EXERCISES:

1. Prove the following:
   - **Thm D:** o
   - **Thm E:** •oo
   - **Thm F:** o••o
   - **Thm G:** oooo
   - **Thm H:** •o•
   - **Thm I:** •ooo

2. Now the question that immediately comes to mind is:

Is EVERY formula provable in this toy proof system?

If so, then this would be analogous to having an inconsistent system in normal logic. I know the answer to this, but I will let you think about it. See if you can figure out whether or not every formula is provable in this system. If not every formula is provable, can you determine exactly which formulas are provable and which are not?