Homework: Math 299

Dr. Monks Spring 2010

The following are the homework assignments for this semester. Check back frequently for updates.

Assignment 0: (Due Thu, Feb 4)

- Read Sections 1.1 & 1.2.
- Do all problems, but no need to hand them in. You may be quizzed on them in class.
- Read the course syllabus, and downloading and install the program LyX.

Assignment 1: (Due Tue, Feb 9)

- Read Sections 1.3 & 1.4.
- Be ready to do any problems at the end of those sections on a pop-quiz.
- *Writing:* Hand in your proofs for Writing problems #9, 10, 11, and 27 in the book.
- Download and install Scrambler from the Lurch website and try to beat some levels (not by accident, but rather by finding a strategy that you can prove will work in every situation.

Assignment 2: (Due Thu, Feb 11)

- Read Section 1.5.
- Be ready to do any problems at the end of that section on a pop-quiz.
- Hand in: Prove each of the following Circle-Dot "theorems". You can use the Toy Proof software to ensure that your proof is correct.
  (1) •○•○•○•
  (2) ••○○••
  (3) •••••○○○○○
  (4) ○○○○○•••••
  (5) ○•○○••○○○•••
- Can you prove (with a *convincing explanation* proof) that every circle-dot expression can be "proven" in this toy system, or if not, determine (again with proof) exactly those expressions which can?

Assignment 3: (Due Tue, Feb 16)

- Download and install LyX on your computer (or use it in the Math Lab).
- Rewrite your proofs from Assignment #1 in LaTeX, using either LyX, LaTeX, or Scientific Word/Workplace. Each of your proofs should be in essay format.
- Read Sections 2.1 and 2.2 in the textbook.
- Be ready to do any problems at the end of those sections on a pop-quiz.
- Determine exactly which circle-dot strings can be proven and which cannot, and defend your answer with an essay-style convincing explanation type proof.

Assignment 4: (Due Thu, Feb 18)

- Read Sections 2.2 and 2.3 in the textbook.
- Be ready to do any problems at the end of the sections in the book up to and including 2.3 on a pop-quiz.
- **Writing**: Type (LyX/SWP/LaTeX) a solution to either #67 or #68 (your choice) on Page 22 in the book.

Assignment 5: (Due Tue, Feb 23)

- Read Sections 2.4 and 2.5 in the textbook.
- Be ready to do any problems at the end of those sections on a pop-quiz.

- Verify each of the following is a tautology with a truth table. Then prove each one with a formal proof using only the Natural Deduction rules given in lecture (the five introduction $(+)$ rules and five elimination $(-)$ rules for the logical operators *and, or* $,\neg,\Rightarrow,\Leftrightarrow$). You may also use the $\rightarrow\leftarrow +$ rule.

(1) $P \Rightarrow P$
(2) $P \,\text{and}\, \neg P \Rightarrow Q$
(3) $P \Rightarrow (Q \Rightarrow Q)$
(4) $((P \,\text{or}\, Q) \,\text{and}\, \neg P) \Rightarrow Q$
(5) $P \,\text{or}\, \neg P$
(6) $\neg (P \,\text{or}\, Q) \Leftrightarrow \neg P \,\text{and}\, \neg Q$

Assignment 6: (Due Tue, Mar 2)

- Read Sections 3.1 and 3.2 in the textbook.
- Be ready to do any problems at the end of those sections or previous sections on a pop-quiz.
- Prove each of the following tautologies with a formal proof using only the Natural Deduction rules given in lecture (the five introduction $(+)$ rules and five elimination $(-)$ rules for the logical operators *and, or* $,\neg,\Rightarrow,\Leftrightarrow$). You may also use the $\rightarrow\leftarrow +$ rule and the `Copy` rule. You can also use theorems that you were assigned in the previous assignment (whether or not your proved them).

(1) $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$ (contrapositive)
(2) $(P \,\text{and}\, Q) \,\text{or}\, R \Leftrightarrow (P \,\text{or}\, R) \,\text{and}\, (Q \,\text{or}\, R)$ (distributivity of and/or)
(3) $(P \,\text{or}\, Q) \,\text{and}\, R \Leftrightarrow (P \,\text{and}\, R) \,\text{or}\, (Q \,\text{and}\, R)$ (distributivity of and/or)
(4) $(P \Rightarrow Q) \Leftrightarrow (\neg P \,\text{or}\, Q)$ (definition of implies)
(5) $\rightarrow\leftarrow \Rightarrow Q$ (anything follows from a contradiction)

Assignment 7: (Due Tue, Mar 2)

- Read Sections 3.4 and 3.5 in the textbook.
- Be ready to do any problems at the end of those sections or previous sections on a pop-quiz.
- **Writing**: Type (LyX/SWP/LaTeX) a solution to #15, #26, #48, #70, and #89 in Chapter 2 of the book.

Assignment 8: (Due Thu, Mar 4)

- Read Sections 3.6 and 3.7 in the textbook.
- Be ready to do any problems at the end of those sections or previous sections on a pop-quiz.
- Prove each of the following tautologies with a formal proof using only the Natural Deduction rules given in lecture (the five introduction $(+)$ rules and five elimination $(-)$ rules for the logical operators and, or,$\neg,\Rightarrow,\Leftrightarrow$, and quantifiers $\forall$ and $\exists$, the $\rightarrow\leftarrow +$ rule, and `Copy` rule). You can also use theorems that you were assigned in the previous assignment (whether or not your proved them).

(1) $(\neg\forall x, P(x)) \Leftrightarrow \exists y, \neg P(y)$ (DeMorgan)
(2) $(\neg\exists x, P(x)) \Leftrightarrow \forall y, \neg P(y)$ (DeMorgan)
(3) $(\exists y, \forall x, R(x,y)) \Rightarrow \forall x, \exists y, R(x,y)$
(4) $(\forall x, P(x) \Rightarrow Q(x)) \Rightarrow ((\forall x, P(x)) \Rightarrow \exists x, Q(x))$
(5) $((\forall x, P(x)) \,\text{or}\, (\forall x, Q(x))) \Rightarrow \forall x, P(x) \,\text{or}\, Q(x)$

6.-10. Make up your own unique mathematical or English statements $P(x)$, $Q(x)$, and $R(x,y)$ and use them to translate the formal statements in problems #1-5 into ordinary English sentences that you might write in an essay style proof. Bonus points may be given for sufficiently humorous sentences.

Assignment 9: (Due Tue, Mar 9)

- Read Sections 4.1 and 4.2 in the textbook.
- Be ready to do any problems at the end of those sections or previous sections on a pop-quiz.

- Prove each of the following twice. First with a formal proof. Then prove it again with an essay style proof that translated the formal proof into more readable English. The capital variables represent sets. Lower case represent elements of sets. For formal definitions about sets you should use the ones given in my lecture notes for the Geometry Course on my web site. For definitions not given there you may use a definition given into the book (translate it into formal notation for the formal proof). Type your work in LyX or LaTeX. Follow the tips I gave you in class and those in Chapter 3.2 when writing your essay proofs.

(1) $A \subseteq B \Rightarrow A \cap B = A$
(2) $A \subseteq B \,\text{and}\, B \subseteq C \Rightarrow A \subseteq C$
(3) $A \subseteq B \Rightarrow A \times C \subseteq B \times C$
(4) $A \cap B \subseteq A \cup B$
(5) Problem #50 in Chapter 2
(6) Problem #52 in Chapter 2
(7) Problem #67 in Chapter 2
(8) Problem #89 in Chapter 2

Assignment 10: (Due Thu, Mar 25)

- Read Sections 6.1 and 6.2 in the textbook.
- Be ready to do any problems at the end of those sections or previous sections on a pop-quiz.
- Write up and hand in solutions to the following questions.

#1. Fill in the reasons and their input line numbers for the following semi-formal proof.

Theorem. $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$

Proof:

(1) Let x be arbitrary.
(2) Assume $x \in \overline{A \cup B}$
(3) $x \notin A \cup B$
(4) $\neg x \in A \cup B$
(5) Assume $x \in A$
(6) $x \in A \,\text{or}\, x \in B$
(7) $x \in A \cup B$
(8) $\rightarrow\leftarrow$
(9) $\leftarrow$
(10) $\neg x \in A$
(11) $x \notin A$
(12) $x \in \overline{A}$
(13) Assume $x \in B$
(14) $x \in A \,\text{or}\, x \in B$
(15) $x \in A \cup B$
(16) $\rightarrow\leftarrow$
(17) $\leftarrow$
(18) $\neg x \in B$
(19) $x \notin B$
(20) $x \in \overline{B}$
(21) $x \in \overline{A} \,\text{and}\, x \in \overline{B}$
(22) $x \in \overline{A} \cap \overline{B}$
(23) $\leftarrow$
(24) $x \in \overline{A \cup B} \Rightarrow x \in \overline{A} \cap \overline{B}$
(25) $\forall x, x \in \overline{A \cup B} \Rightarrow x \in \overline{A} \cap \overline{B}$
(26) $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$

QED

#**2.** Edit the following essay style proof of the theorem from the previous problem to add reasons. The final proof with reasons should be in essay style.

Theorem. *The complement of the union of two sets is the intersection of their complement.*

Proof. Let A, B be sets, and $x \in \overline{A \cup B}$. Then x is not an element of $A \cup B$. Thus it is not an element of A or an element of B. So $x \in \overline{A}$ and $x \in \overline{B}$. It follows that $x \in \overline{A} \cap \overline{B}$. Since x was arbitrary, every element of $\overline{A \cup B}$ is also an element of $\overline{A} \cap \overline{B}$ and so $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$. □

#**3.** For each of the following mathematical statements, rewrite the statement in formal symbolic form, say what kind of statement it is, and what rule of inference you might try to use to prove it. (Note: the symbols names for famous sets, like $\mathbb{Q}$ for the set of rational numbers, can be found in the Geometry lecture notes pdf.)

Example: Consider the statement:

There is no largest negative rational number.

We could rewrite this formally as this:

$\neg\exists x, x \in \mathbb{Q}\,\text{and}\,\text{x<0}\,\text{and}\,\forall y, y \in \mathbb{Q}\,\text{and}\,y < 0 \Rightarrow y \leq x$

This statement is a *negation* (i.e. its ¬ something). So we might try to prove it with a proof by contradiction (¬+)

(a) The product of an irrational number and a nonzero rational number is irrational.

(b) The sum of the squares of two odd integers cannot be a perfect square.

(c) Every nonzero rational number has a multiplicative inverse.

(d) There are two distinct irrational numbers whose sum is rational.

(e) No power set is empty.

(f) A positive integer has an odd number of divisors if and only if it is a perfect square.

(g) Not every number that is divisible by only 1 and itself is prime.

(h) If a number is composite, then it has a prime divisor less than or equal to its square root.

(i) Every integer exceeding 1 has a prime divisor.

#**4.** Write a formal version of the definition of *divisible* given on page 79 in the book.

#**5. (a-c)** Choose any three formal definitions about sets from the Geometry Lecture notes pdf, and write a Plus and a Minus rule for the symbol that is derived from those definitions.

WRITING

For each of the following claims, first translate it into a formal statement. Then give a semi-formal proof of the claim using shortcuts discussed in class (but giving complete reasons and input line numbers for everything). Finally, write a essay style proof based on your semi-formal proof.

#**6.** Nonempty sets have a nonempty union.

#**7.** Union is distributive over intersection.

#**8.** Divisibility is transitive.

#**9.** If two numbers are both divisible by a third number, then so is their sum and their difference.

Assignment 11: (Due Tue, Mar 30)

- Read Sections 6.3 in the textbook carefully.
- Be ready to do any problems at the end of those sections or previous sections on a pop-quiz.

- **Writing:** Type up essay style proofs for the following questions in Chapter 6: #29, 31, 32, 33, 49, 50

Assignment 12: (Due Thu, Apr 1 [no fooling!])

- Read Sections 6.4 and 6.5 in the textbook carefully.
- Be ready to do any problems at the end of those sections or previous sections on a pop-quiz.
- Hand in your answers to the following questions in Chapter 6 (do not use the answers given in the back of the book): #57, 59, 60, 61, 63, 64, 65, 67, 68, 69, 70, 71, 78, 79, 81, 82, 84, 85, 88, 91, 92

Assignment 13: (Due Thu, Apr 8)

- Read Sections 6.6 in the textbook carefully.
- Be ready to do any problems at the end of that section or previous sections on a pop-quiz.
- **Writing:** There are only four problems this time, so do a good and careful job! Look up or derive proof recipes for an *injective+, injective-, surjective+* and *surjective-* proof recipes. (I mentioned these in class so they should be in your notes. Some of them are also in the proof recipes handout at the website.) You do not have to type up or hand in these recipes, but you **should use them** for the next two problems. Wow me with your care and cleverness!
  (1) Prove that the composition of injective functions is injective. [*Type up both a semi-formal proof (two columns with statements and reasons but using shortcuts) and an essay style proof based on your semi-formal one.*]
  (2) Prove that the composition of surjective functions is surjective. [*Type up both a semi-formal proof and an essay style proof based on your semi-formal one.*]

Assignment 14: (Due Tue, Apr 14)

- Read Sections 5.1 and 5.2 in the textbook carefully.
- Be ready to do any problems at the end of those sections or previous sections on a pop-quiz.
- **Writing:** There are only four problems this time. In these questions you should use the following definition of *inverse function.*

  Definition: Let $f : A \to B$. Then g is a inverse function of f if and only if (1) $g : B \to A$, (2) $g \circ f = \mathrm{id}_A$ and (3) $f \circ g = \mathrm{id}_B$.

  Try to make both an *inverse function+* and *inverse function-* recipe to use in your proofs (you don't have to type them up). Type up both a semi-formal proof and an essay style proof based on your semi-formal proof for each of the following theorems. You may use the theorem that a function has an inverse if and only if it is bijective.
  (1) A bijective function has a unique inverse function.
  (2) Let $f : A \to B$ and $g : B \to C$ be bijective fuctions. Then $g \circ f$ is bijective and
  $$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

Assignment 15: (Due Thu, Apr 16)

- Read Section 5.3 in the textbook carefully.
- Be ready to do any problems at the end of those sections or previous sections on a pop-quiz.
- **Writing:** There are six problems this time. In these questions you should use the following definition of *inverse image.*

  Definition: Let $f : A \to B$ and $S \subseteq B$. Then the *inverse image of S by f* is the set $f^{\mathrm{inv}}(S)$ given by
  $$f^{inv}(S) = \{x \in A : f(x) \in S\}.$$

  Try to make both an *inverse image+* and *inverse image-* recipe to use in your proofs (you don't have to type them up). Type up both a semi-formal proof and an essay style proof based on your semi-formal proof for each of the following theorems.

(1) Let $f : A \to B$ be surjective and $S \subseteq B$. Then
$$S = f(f^{inv}(S))$$
(2) Let $f : A \to B$ and $T \subseteq A$. Then
$$T \subseteq f^{inv}(f(T))$$
(3) Let $f : A \to B$ be injective and $T \subseteq A$. Then
$$T = f^{inv}(f(T))$$

Assignment 16: (Due Tue, Apr 20)

- Read Section 5.4 in the textbook carefully.
- Be ready to do any problems at the end of those sections or previous sections on a pop-quiz.
- **Writing:** Type up both a semi-formal proof and an essay style proof based on your semi-formal proof for each of the following theorems.
  (1) If $f : A \to B$ and $S \subseteq T \subseteq A$ then $f(S) \subseteq f(T)$.
  (2) If $f : A \to B$ and $S \subseteq B$ and $T \subseteq B$ then $f^{inv}(S \cup T) = f^{inv}(S) \cup f^{inv}(T)$.
  (3) Let $\sim$ be an equivalence relation on a set A. For each $x \in A$ define the equivalence class of x to be the set $[x] = \{y : y \sim x\}$.
    (a) Show that any two distinct equivalence classes are disjoint, i.e.
    $$\forall x, \forall y, [x] \cap [y] = \emptyset \,\text{or} [x] = [y]$$
    (b) Prove that $A = \bigcup_{x \in A} [x]$.
    (*This proves that any equivalence relation on a set partitions it into disjoint equivalence classes).*
  (4) Let $f : A \to B$. Define a relation $\sim$ on A by
  $$\forall x \in A, \forall y \in A, x \sim y \Leftrightarrow f(x) = f(y).$$
    (a) Prove $\sim$ is an equivalence relation on A.
    (b) Let $\widetilde{A}$ be the set of equivalence classes for $\sim$. Define $g : \widetilde{A} \to B$ by
    $$\forall \text{z} \in \widetilde{A}, \forall x \in A, z = [x] \Rightarrow g(z) = f(x).$$
    Prove that g is injective. [Note: you can use the fact that g is actually a function without proving it.]
  (5) Prove problem #51 in Chapter 5 by induction.
  (6) Prove problem #56 in Chapter 5 by induction.

Assignment 17: (Due Thu, Apr 22)

- Read Section 5.5 in the textbook carefully.
- Be ready to do any problems at the end of those sections or previous sections on a pop-quiz.
- **Writing:** Type up a semi-formal proof and an essay style proof based on your semi-formal proof for each of the following theorems.
  (1) Prove problem #55 in Chapter 5 by induction.
  (2) Let $f : B \to C$. Prove that f is injective if and only if for all sets A and for all functions $g, h : A \to B$
  $$f \circ g = f \circ h \Rightarrow g = h$$

Assignment 18: (Due Tue, Apr 27)

- Read Sections 4.3 and 4.4 in the textbook carefully.
- Be ready to do any problems at the end of those sections or previous sections on a pop-quiz.
- **Writing:** Type up a semi-formal proof and an essay style proof based on your semi-formal proof for each of the following theorems.

(1) Let $f : A \to B$. Prove that f is surjective if and only if for all sets C and for all functions $g, h : B \to C$

$$g \circ f = h \circ f \Rightarrow g = h$$

(2) Let $a_1, a_2, a_3, \cdots$ be a sequence of positive integers such that $a_1 = 1$, $a_2 = 3$, and for all $n \geq 3$,

$$a_n = 2a_{n-1} - a_{n-2}$$

(Note: this sort of definition is called a *recursive definition* because each term in the sequence after the first few are defined in terms of previous terms of the sequence.)

  (a) Compute a_n for all $n \leq 10$. (You do not need to prove anything.)
  (b) Use strong induction to prove that for all $n \geq 1$,

$$a_n = 2n - 1$$

(Note: this sort of definition of a_n is called a *closed formula* because a_n is given only in terms of n, not in terms of previous terms in the sequence.)

(3) Let's change the sequence in the previous question just a tiny bit. Let $a_1, a_2, a_3, \cdots$ be a sequence of positive integers such that $a_1 = 1$, $a_2 = 4$, and for all $n \geq 3$,

$$a_n = 2a_{n-1} - a_{n-2} + 2$$

  (a) Compute a_n for all $n \leq 10$. (You do not need to prove anything.)
  (b) Conjecture a closed formula for the value of a_n.
  (c) Prove your conjecture from part b using strong induction.

(4) Prove that if a, b, c, d and m are positive integers, with $a \underset{m}{\equiv} b$ and $c \underset{m}{\equiv} d$ then

$$a + c \underset{m}{\equiv} b + d$$

(5) Let A, B, C be sets. Prove that

$$(A - B) \cap C \subseteq (A \cap C) \cup (B \cap C)$$

Assignment 19: (Due Thu, Apr 29)

- Read Sections 4.5 and 4.6 in the textbook carefully.
- Be ready to do any problems at the end of those sections or previous sections on a pop-quiz.
- **Writing:** Type up two essay style proofs for each of the following theorems. In one proof give a purely combinatorial argument. In the other give a proof by induction or algebra.

  (1) For any natural numbers n and and k with $k \leq n$,

$$\binom{n}{k} = \binom{n}{n-k}$$

  (2) For any positive integer n,

$$1 + 3 + 5 + \cdots (2n - 1) = n^2$$

Assignment 20: (Due Tue, May 4)

- In the book, do problems #1,3,5,7,9,11,17,19,21,25 in Chapter 5. You do not have to type these and you do not have to prove anything. Just explain briefly or show your work.
- **Writing:** It's time to play *Convince the Proof Prof!* Show me you are learning how to write proofs!

(1) Type up two essay style proofs for each of the following theorem. In one proof give a purely combinatorial argument. In the other give a direct proof using the binomial theorem. *Note: You are not allowed to just compute the sums!*

$$\binom{2010}{0} + \binom{2010}{2} + \cdots + \binom{2010}{2010} = \binom{2010}{1} + \binom{2010}{3} + \cdots + \binom{2010}{2009}$$

(2) Type a *formal* proof of the following tautology. You do not have to type an essay style proof.

$$P \Rightarrow (Q \Rightarrow P)$$

(3) Type a *semi-formal* proof of the following theorem. You do not have to type an essay style proof.

Theorem: Let A, B be sets.

$$A - B \subseteq \overline{B - A}$$

(4) Type both an essay style proof and a semi-formal style proof for the following theorem.

Theorem: If a, b, c, d and m are positive integers such that $a \underset{m}{\equiv} b$ and $c \underset{m}{\equiv} d$ then

$$ac \underset{m}{\equiv} bd$$

(5) Prove the following by strong induction. Type both an essay style prof and a semi-formal style proof.

Theorem: For every integer n larger than 11 there are nonnegative integers s, t such that $n = 3s + 7t$.

(6) Give a semi-formal and essay style proof for the following.

Theorem: Define $\pi_A : A \times B \to A$ by

$$\forall z \in A \times B, \forall x \in A, \forall y \in B, z = (x, y) \Rightarrow \pi_A(z) = x$$

The function π_A is surjective.

Assignment 21: (Due Thu, May 6)

- **Writing:** It's the next round of *Convince the Proof Prof!* Show me you are learning how to write proofs by writing a semiformal and essay style proof for each of the following theorems. The domain of all variables in the following theorems is the set of real numbers. You may continue to use facts about the arithmetic, ordering, and the definition of real numbers in your proofs by giving *by arithmetic* as the justification. You will need the following definition.

Definition: Let $f : \mathbb{R} \to \mathbb{R}$. We say f is *periodic* if and only if

$$\exists \lambda > 0, \forall x \in \mathbb{R}, f(x) = f(x + \lambda)$$

(1) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $\forall x \in \mathbb{R}, f(x) = \sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{3}\right)$ is periodic.
(2) Prove that the composition of periodic functions is periodic.
(3) *Blast from the past*! Give a formal proof (no shortcuts) of the following theorem. You do not need to give an essay style proof.

$$(\exists x, \forall y, P(x, y)) \text{ and } (\forall x, P(x, x) \Rightarrow \exists y, Q(x, y)) \Rightarrow \exists y, \exists x, Q(x, y)$$