Homework: Math 299

Dr. Monks — Spring 2011

The following are the homework assignments for this semester. Check back frequently for updates.

Assignment 0: (Due, Tuesday, February 8, 2011)

- Read Sections 1.1 & 1.2.
- Answer problems #1-8 in Section 1.1. These do not have to by typed and should be handed in on Tuesday. See the syllabus for the correct homework format.
- Read the course syllabus, and download and install the program LyX. Then go through the LyX Tips document on our course website to configure your copy of LyX.

Assignment 1: (Due Thursday, February 10, 2011)

- Read Sections 1.3 & 1.4.
- *Writing:* Hand in a proof for Writing problems #9, 10, or 11 (choose one) in the book. Your proof should explain clearly why the given fact must be true. You should write in standard English word-wrapped paragraphs.
- Download and install Scrambler from the Lurch website and see how many levels you can solve (not by accident, but rather by finding a strategy that you can prove will work in every situation).

Assignment 2: (Due Tuesday, February 15, 2011)

- Read Section 1.5.
- Be ready to do any problems at the end of that section on a pop-quiz.
- Hand in: Prove each of the following Circle-Dot "theorems". You can use the Toy Proof software to ensure that your proof is correct.
  (1) $\bullet\bullet\bigcirc\bigcirc\bullet\bullet$
  (2) $\bullet\bullet\bullet\bullet\bigcirc\bigcirc\bigcirc\bigcirc$
  (3) $\bigcirc\bigcirc\bigcirc\bullet\bullet\bullet$
  (4) $\bullet\bigcirc\bigcirc\bullet\bullet\bullet\bigcirc\bigcirc\bigcirc\bigcirc$
  (5) $\bigcirc\bullet\bigcirc\bullet\bigcirc\bullet\bigcirc$
- Can you prove (with an *expository* proof) that every circle-dot expression can be "proven" in this toy system, or if not, determine (again with proof) exactly those expressions which can?

Assignment 3: (Due Thursday, February 17, 2011)

- Read Sections 2.1 and 2.2 in the textbook.
- Prepare for a quiz on Thursday. The quiz will consist of a few questions selected from problems #29-38, #42 in the book (closed book) and will also ask you to write down the eleven Rules of Inference for Propositional Logic from memory (the ones given in the lecture notes).

Assignment 4: (Due Tuesday, February 22, 2011)

- Read Sections 2.2 and 2.3 in the textbook.
- Be ready to do any problems at the end of the sections in the book up to and including 2.3 on a pop-quiz.
- Prove each one of the following tautologies with a formal proof using only the Rules of Inference for Propositional logic given in the lecture notes. You may also use the *copy* rule if you wish.
  (1) $P \Rightarrow (Q \Rightarrow Q)$
  (2) $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$
  (3) $((P \text{ or } Q) \text{ and } \neg P) \Rightarrow Q$
  (4) $P \text{ or } \neg P$ (hint: don't use or+, try contradition)
  (5) $\neg (P \text{ or } Q) \Leftrightarrow \neg P \text{ and } \neg Q$

- **Writing**: Type (LyX/SWP/LaTeX) an expository style proof to either #67 or #68 (your choice) on Page 22 in the book.

Assignment 5: (Due Thursday, February 24, 2011)

- Read Sections 2.4 and 2.5 in the textbook.
- Quiz: Be able to write down the Rules of Inference for Propositional Logic, the Rules of Inference for Quantifiers, the Rules of Inference for Unique Existence, and the Rules of Inference for Equality.
- Prove each one of the following statements with a formal proof using only the Rules of Inference for Logic and Equality given in the lecture notes. You may also use the *copy* rule if you wish (but you never need to!). You can also use theorems that you were assigned in the previous assignment (whether or not your proved them).

(1) $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$ (contrapositive)
(2) $(P \text{ and } Q) \text{ or } R \Leftrightarrow (P \text{ or } R) \text{ and } (Q \text{ or } R)$ (distributivity of and/or)
(3) $(\neg \forall x, P(x)) \Leftrightarrow \exists y, \neg P(y)$ (DeMorgan)
(4) $(\neg \exists x, P(x)) \Leftrightarrow \forall y, \neg P(y)$ (DeMorgan)

Assignment 6: (Due Tuesday, March 1, 2011)

- Read Sections 3.1 and 3.2 in the textbook.
- Prove each one of the following statements with a formal proof using only the Rules of Inference for Logic and Equality given in the lecture notes. You may also use the *copy* rule if you wish (but you never need to!). You can also use theorems that you were assigned in the previous assignment (whether or not your proved them).

(1) $(P \text{ or } Q) \text{ and } R \Leftrightarrow (P \text{ and } R) \text{ or } (Q \text{ and } R)$ (distributivity of and/or)
(2) $(P \Rightarrow Q) \Leftrightarrow (\neg P \text{ or } Q)$ (alternate definition of implies)
(3) $\rightarrow\leftarrow \Rightarrow Q$ (anything follows from a contradiction)
(4) $(\exists y, \forall x, R(x,y)) \Rightarrow \forall x, \exists y, R(x,y)$
(5) $(\forall x, P(x) \Rightarrow Q(x)) \Rightarrow ((\forall x, P(x)) \Rightarrow \exists x, Q(x))$
(6) $((\forall x, P(x)) \text{ or } (\forall x, Q(x))) \Rightarrow \forall x, P(x) \text{ or } Q(x)$

7.-9. Make up your own unique mathematical or English statements $P(x)$, $Q(x)$, and $R(x,y)$ and use them to translate the formal statements in problems #4-6 into ordinary English sentences that you might write in an essay style proof. Bonus points may be given for sufficiently humorous sentences.

Assignment 7: (Due Thursday, March 3, 2011)

- Read Sections 3.4 and 3.5 in the textbook.
- Be ready to do any problems at the end of those sections or previously assigned sections on a pop-quiz.
- Prove each one of the following statements with a formal proof using only the Rules of Inference for Logic and Equality given in the lecture notes. You may also use the *copy* rule if you wish (but you never need to!). You can also use theorems that you were assigned in the previous assignment (whether or not your proved them).

1. $(P \operatorname{or} Q \Rightarrow R) \Rightarrow (\neg R \Rightarrow \neg P \operatorname{and} \neg Q)$
2. $(\forall a, \forall b, Q(a,b) \Rightarrow Q\left(b,a\right)) \Rightarrow (\forall x, (\exists y, Q(y,x)) \Rightarrow (\exists z, Q\left(x,z\right)))$
3. Fill in the reasons and their input line numbers for the following semi-formal proof.

Theorem. $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$

Proof:

(1) Let x be arbitrary.
(2) Assume $x \in \overline{A \cup B}$
(3) $x \notin A \cup B$
(4) $\neg x \in A \cup B$
(5) Assume $x \in A$
(6) $x \in A \operatorname{or} x \in B$
(7) $x \in A \cup B$
(8) $\rightarrow\leftarrow$
(9) $\leftarrow$
(10) $\neg x \in A$
(11) $x \notin A$
(12) $x \in \overline{A}$
(13) Assume $x \in B$
(14) $x \in A \operatorname{or} x \in B$
(15) $x \in A \cup B$
(16) $\rightarrow\leftarrow$
(17) $\leftarrow$
(18) $\neg x \in B$
(19) $x \notin B$
(20) $x \in \overline{B}$
(21) $x \in \overline{A} \operatorname{and} x \in \overline{B}$
(22) $x \in \overline{A} \cap \overline{B}$
(23) $\leftarrow$
(24) $x \in \overline{A \cup B} \Rightarrow x \in \overline{A} \cap \overline{B}$
(25) $\forall x, x \in \overline{A \cup B} \Rightarrow x \in \overline{A} \cap \overline{B}$
(26) $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$

QED

4. Edit the following essay style proof of the theorem made from the previous problem to add reasons. The final proof with reasons should be in essay style.

Theorem. *The complement of the union of two sets is the intersection of their complement.*

Proof. Let A, B be sets, and $x \in \overline{A \cup B}$. Then x is not an element of $A \cup B$. Thus it is not an element of A or an element of B. So $x \in \overline{A}$ and $x \in \overline{B}$. It follows that $x \in \overline{A} \cap \overline{B}$. Since x was arbitrary, every element of $\overline{A \cup B}$ is also an element of $\overline{A} \cap \overline{B}$ and so $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$. $\square$

Assignment 8: (Due Tuesday, March 8, 2011)

- Read Sections 3.6 and 3.7 in the textbook.
- Be ready to do any problems at the end of those sections or previous sections on a pop-quiz.
- Prove each of the following twice. First with a formal proof. Then prove it again with an essay style proof that translated the formal proof into more readable English. Type all of your work in LyX or LaTeX. Follow the tips I gave you in class and those in Chapter 3.2 when writing your essay proofs.

(1) $A\subseteq B\Rightarrow A\cap B=A$
(2) $A\subseteq B\,\text{and}\,B\subseteq C\Rightarrow A\subseteq C$
(3) $A\subseteq B\Rightarrow A\times C\subseteq B\times C$
(4) $A\cap B\subseteq A\cup B$

Assignment 9: (Due Thursday, March 10, 2011)

- Read Sections 4.1 and 4.2 in the textbook. Be ready to do any problems at the end of those sections or previous sections.
- Review for the midterm exam which will be on Tuesday, March 15 and will cover everything in the course, except for Toy Proofs and Circle Dot proofs. Questions can come from the homework, or from the assigned sections of the textbook or made up by the instructor.
- Prove each of the following twice. First with a formal proof. Then prove it again with an essay style proof that translated the formal proof into more readable English. The capital variables represent sets. Lower case represent elements of sets. For formal definitions about sets you should use the ones given in my lecture notes. Type your work in LyX or LaTeX. Follow the tips I gave you in class and those in Chapter 3.2 when writing your essay proofs.

(1) $\mathcal{P}(A)\cup\mathcal{P}(B)\subseteq\mathcal{P}(A\cup B)$
(2) $\mathcal{P}(B)-\mathcal{P}(A)=\mathcal{P}(B)-\mathcal{P}(A\cap B)$
(3) $A\times C\cap B\times D=A\times D\cap B\times C$
(4) $\overline{\bigcup_{t\in I}A_t}=\bigcap_{t\in I}\overline{A_t}$

Assignment 10: (Due Tuesday, March 29, 2011)

- Read Sections 6.1 and 6.2 in the textbook.
- Be ready to do any problems at the end of those sections or previous sections.
- Write up and hand in semiformal (i.e. using approved shortcuts) proofs of the following statements.
  (1) If $f:A\to B$ and $S\subseteq T\subseteq A$ then $f(S)\subseteq f(T)$.
  (2) If $f:A\to B$ and $S\subseteq B$ and $T\subseteq B$ then $f^{inv}(S\cup T)=f^{inv}(S)\cup f^{inv}(T)$.
  (3) If $A\xrightarrow{f}B$, $B\xrightarrow{g}C$, and $C\xrightarrow{h}D$ then $(h\circ g)\circ f=h\circ(g\circ f)$.

Assignment 11: (Due Thursday, March 31, 2011)

- Read Sections 6.3 in the textbook carefully.
- Be ready to do any problems at the end of that section or previous sections.
- Write up and hand in semiformal (i.e. using approved shortcuts) proofs of the following statements.
  (1) If $f:A\to A$ and f is injective then $f\circ f$ is injective.
  (2) (todo) If $f:A\to A$ and f is surjective then $f\circ f$ is surjective.
  (3) (todo) If $\sim$ is an equivalence relation on A then $\forall x,y,z\in A,\neg x\sim y\,\text{and}\,y\sim z\Rightarrow\neg x\sim z\,\text{and}\,x\neq z$.

Assignment 12: (Due Tuesday, April 5, 2011)

- Read Sections 6.4 and 6.5 in the textbook carefully.
- Be ready to do any problems at the end of those sections or previous sections.
- Write up and hand in semiformal (i.e. using approved shortcuts) proofs of the following statements.
  (1) Let $f : A \to B$ and $g : B \to C$ be bijective functions. Then $g \circ f$ is bijective.
  (2) (low priority todo) Let $f : A \to B$ and $g : B \to C$ be bijective functions. Then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
  (3) Let $f : A \to B$ and define the relation $\sim$ on A such that for all $x, y \in A$
  $$x \sim y \Leftrightarrow f(x) = f(y)$$
  Then $\sim$ is an equivalence relation.

Assignment 13: (Due Thursday, April 7, 2011)

- Read Section 6.6 in the textbook carefully.
- Be ready to take a quiz on the Rules of Inference for Functions.
- Write up and hand in semiformal (i.e. using approved shortcuts) proofs of the following statements.
  (1) Divisibility is transitive.
  (2) (todo) Let $f : A \to B$ be surjective and $S \subseteq B$. Then
  $$S = f(f^{inv}(S))$$

Assignment 14: (Due Tuesday, April 12, 2011)

- Read Sections 5.1 and 5.2 in the textbook carefully.
- Be ready to do any problems at the end of those sections or previous sections.
- Write up and hand in semiformal (i.e. using approved shortcuts) proofs of the following statements.
  (1) Let $f : A \to B$ be injective and $T \subseteq A$. Then
  $$T = f^{inv}(f(T))$$
  (2) If two integers are both divisible by a third number, then so is their sum and their difference.
  (3) For all positive integers n
  $$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$
  (Hint: Induction!)
  (4) For any integer $m > 1$ the relation $\underset{m}{\equiv}$ is an equivalence relation on $\mathbb{Z}$.

Assignment 15: (Due Thursday, April 13, 2011)

- Read Section 5.3 in the textbook carefully.
- Write up and hand in semiformal (i.e. using approved shortcuts) proofs of the following statements.
  (1) Let m be a positive integer and $a, b, c \in \mathbb{Z}$. If $a \underset{m}{\equiv} b$ then $ac \underset{m}{\equiv} bc$.
  (2) Let $n \in \mathbb{N}$. The relation $\subseteq$ on the power set of $\mathbb{O}_n$ is a partial order.

Assignment 16: (Due Tuesday, April 19, 2011)

- Read Section 5.4 in the textbook carefully.
- Study for a quiz on the **Rules of Inference for Induction** and the **Rules of Inference for Divisibility** from the lecture notes.
- Write up and hand in semiformal (i.e. using approved shortcuts) proofs of the following statements. *[New! Improved!! Now with more hints!!]*

  (1) For all $n \in \mathbb{N}$, the number $6^n - 5n - 1$ is divisible by 25. *[Hint: Try induction.]*
  (2) Let B be nonempty and define $\pi_A : A \times B \to A$ by
  $$\forall z \in A \times B, \forall x \in A, \forall y \in B, z = (x, y) \Rightarrow \pi_A(z) = x$$
  The function π_A is surjective. *[Hint: this is easier than it looks.]*

(3) Let $\sim$ be an equivalence relation on a set A and $a, b \in A$. Then

$$[a] = [b] \Leftrightarrow a \sim b.$$

[Hint: You can't use the Theorem in the notes, because that is what I'm asking you to prove!]

(4) Let $a_1, a_2, a_3, \cdots$ be a sequence of positive integers such that $a_1 = 1$, $a_2 = 3$, and for all $n \geq 3$,

$$a_n = 2a_{n-1} - a_{n-2}$$

(Note: this sort of definition is called a **recursive definition** *because each term in the sequence after the first few are defined in terms of previous terms of the sequence.)*

(a) Compute a_n for all $n \leq 10$. *[Hint: You do not need to prove anything... just compute the values.]*

(b) Prove that for all $n \geq 1$,

$$a_n = 2n - 1$$

[Hint: Use strong induction!]

(Note: this sort of definition of a_n is called a **closed formula** *because a_n is given only in terms of n, not in terms of previous terms in the sequence.)*

Assignment 18: (Due Tuesday, April 26, 2011)

- Read Section 5.5 in the textbook carefully.
- Write up and hand in semiformal (i.e. using approved shortcuts) proofs of the following statements.

(1) The number 91 is composite.

(2) For any positive integers a, b we have

$$\gcd(a, b) = \gcd(a, a + b).$$

(3) Let $f : \mathbb{R} \to (0 \ldots 1]$ be the function such that $f(x) = \frac{1}{1+x^2}$ for all real numbers x.

(a) f is surjective.

(b) f is not injective.

Comments:

- $(0 \ldots 1]$ *is interval notation for the set of all real numbers greater than or equal to 1. So for example, you can use facts like, e.g.* $x \in (0 \ldots 1] \Leftrightarrow 0 < x \leq 1 \text{ and } x \in \mathbb{R}$ *"by definition of interval notation" or simply "by algebra". You can also use recipies derived from this definition. For example, since you know that* $\frac{\pi}{4} \in \mathbb{R}$ *and also that* $0 < \frac{\pi}{4} \leq 1$ *(both by arithmetic) you can conclude that* $\pi \in (0 \ldots 1]$. *In other words, we can assume you know what interval notation means from algebra. But if you want an official definition to use for this particular notation itself it could be:*

$$(a \ldots b] = \{x \in \mathbb{R} : a < x \leq b\}$$

- *In part (b) it asks you to prove that the function is NOT injective. So formally that statement says "$\neg(f \text{ is injective})$". Thus you don't prove the last statement in your proof with injective+, but rather with $\neg$+, i.e. proof by contradiction.*

(4) For each of the following mathematical statements, rewrite the statement in formal symbolic form, say what kind of statement it is, and what rule of inference you might try to use to prove it. (Note: the symbols names for famous sets, like $\mathbb{Q}$ for the set of rational numbers, can be found in the lecture notes pdf.)

Example: Consider the statement:

There is no largest negative rational number.

We could rewrite this formally as this:

$$\neg \exists x, x \in \mathbb{Q} \text{ and x<0 and } \forall y, y \in \mathbb{Q} \text{ and } y < 0 \Rightarrow y \leq x$$

This statement is a *negation* (i.e. its $\neg$ something). So we might try to prove it with a proof by contradiction ($\neg$+)

(a) The product of an irrational number and a nonzero rational number is irrational.
(b) The sum of the squares of two odd integers cannot be a perfect square.
(c) Every nonzero rational number has a multiplicative inverse.
(d) There are two distinct irrational numbers whose sum is rational.

Assignment 19: (Due Thursday, April 28, 2011)

- Read Sections 4.3 and 4.4 in the textbook carefully.
- Write up and hand in the following.
  (1) Look at the definition of the set of rational numbers, $\mathbb{Q}$, in the Lecture Notes.
    (a) Use it to make your own rules of inference for *Rational+* and *Rational-*. [Hint: For Rational+ you might *Conclude* $x \in \mathbb{Q}$ (or equivalently in English: *Conclude x is rational*), while for Rational- you might want *Show* $x \in \mathbb{Q}$ (or equivalently in English: *Show x is rational* as an input.
    (b) Here's a new definition:

      Definition: Let r be a real number. We say that the real number s is the multiplicative inverse of r if and only if $rs = 1$.

      Use this definition to make your own rules of inference for *Multiplicative Inverse+* and *Multiplicative Inverse-*.
    (c) Use your rules of inference above to prove that every nonzero rational number has a rational multiplicative inverse. Note: you may not prove this simply "by arithmetic", i.e. you can't just say "Every nonzero rational number has a multiplicative inverse by arithmetic.". The same goes for facts about integers. Any concept that we have a definition for, you have to use the definition. Other facts (like how to add fractions, etc.) you can justify "by arithmetic".
  (2) For each of the following mathematical statements, rewrite the statement in formal symbolic form, say what kind of statement it is, and what rule of inference you might try to use to prove it.
    (a) No power set is empty.
    (b) A positive integer has an odd number of divisors if and only if it is a perfect square.
    (c) Not every number that is divisible by only 1 and itself is prime.
    (d) If a number is composite, then it has a prime divisor less than or equal to its square root.
    (e) Every integer exceeding 1 has a prime divisor.

Assignment 20: (Due Tuesday, May 3, 2011)

- Read Sections 4.5 and 4.6 in the textbook carefully.
- Study for a quiz on the **Rules of Inference for Relations** from the lecture notes.
- Write up and hand in each of the following.
  (1) Give a combinatorial argument for the following statement
    (a) *For any natural number $n \geq 1$,*

$$n! = n \cdot (n-1)!$$

      Hint: $k!$ counts the number of ways you can place k distinct things in order, i.e. the number of permutations of k distinct things.
  (2) Consider the statement:

    For any natural numbers n and and k with $k \leq n$,

$$\binom{n}{k} = \binom{n}{n-k}$$

    (a) Give a semi-formal proof for this statement using algebra. (Hint: Use the definition of choose: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.)

(b) Give a purely combinatorial proof for this statement. (Hint: Use the fact that $\binom{a}{b}$ counts the number of ways you can choose b things from a things if order doesn't matter and duplicates are not allowed, i.e. the number of b-element subsets of a set with a elements.)

(3) Let $f : A \to B$ and $g \subseteq B \times A$ such that for all $b \in B$ and for all $a \in A$,

$$(b, a) \in g \Leftrightarrow (a, b) \in f$$

Prove that g is a function if and only if f is bijective.
Hint: Remember the Function Application Rules too!

(4) Read 4.3 on page 106 of the textbook. Then rewrite it as a Theorem followed by a semi-formal proof using only our shortcuts. Use line numbers for your lines and only one statement per line. Be sure to give reasons. Is the author skipping steps? Is he omitting reasons? Using other shortcuts? Comment briefly.

Assignment 21: (Due Thursday, May 5, 2011)

- **Writing**

(1) Type up and hand in Type up two essay style word-wrapped expository proofs for the following theorem. In one proof give a purely combinatorial argument. In the other give a proof by induction.

Theorem. *For any positive integer n,*

$$1 + 3 + 5 + \cdots (2n - 1) = n^2$$

(2) Here's a definition from precalculus.

Definition. Let $f : \mathbb{R} \to \mathbb{R}$. We say f is *periodic* if and only if

$$\exists \lambda > 0, \forall x \in \mathbb{R}, f(x) = f(x + \lambda)$$

(a) Make a Periodic+ and Periodic- rule from this definition.
(b) Use your rules to prove in an expository style word wrapped proof that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $\forall x \in \mathbb{R}, f(x) = \sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{3}\right)$ is periodic.

(3) Translate the following English sentences into formal mathematical symbolic statements.
(a) Nonempty sets have a nonempty union.
(b) Union is distributive over intersection.

(4) Read 6.5 on page 169 of the textbook. Then rewrite it as a Theorem followed by a semi-formal proof using only our shortcuts. Use line numbers for your lines and only one statement per line. Be sure to give reasons. Is the author skipping steps? Is he omitting reasons? Using other shortcuts? Can you improve it with chain-of-equality notation? Comment briefly.

Assignment 22: (Due Tuesday, May 10, 2011)

- **Writing**

(1) Type up two essay style proofs for each of the following theorem. In one proof give a purely combinatorial argument. In the other give a direct proof using the binomial theorem. *Note: You are not allowed to just compute the sums!*

$$\binom{2012}{0} + \binom{2012}{2} + \cdots + \binom{2012}{2012} = \binom{2012}{1} + \binom{2012}{3} + \cdots + \binom{2012}{2011}$$

(2) Type a *formal* proof of the following tautology. Don't use any shortcuts and give line numbers for lines and reason inputs.

$$P \Rightarrow (Q \Rightarrow P)$$

(3) Type a *semi-formal* proof of the following theorem. Use numbered lines and approved shortcuts.

Theorem: Let A, B be sets.

$$A - B \subseteq \overline{B - A}$$

(4) Type an *essay style* proof for the following theorem.

Theorem: If a, b, c, d and m are positive integers such that $a \underset{m}{\equiv} b$ and $c \underset{m}{\equiv} d$ then

$$ac \underset{m}{\equiv} bd$$

(5) Read 5.2 on page 134 of the textbook. Is his proof a combinatorial argument? Explain.

Assignment 23: (This assignment does not need to be handed in.)

- **Writing:** Use essay style proofs for the following problems where appropriate.
  (1) Let $\sim$ be an equivalence relation on a set A. For each $x \in A$ define the equivalence class of x to be the set $[x] = \{y : y \sim x\}$.
    (a) Show that any two distinct equivalence classes are disjoint, i.e.
    $$\forall x, \forall y, [x] \cap [y] = \emptyset \,\mathrm{or} [x] = [y]$$
    (b) Prove that $A = \bigcup_{x \in A} [x]$.
    (*This proves that any equivalence relation on a set partitions it into disjoint equivalence classes).*
  (2) Let $a_1, a_2, a_3, \cdots$ be a sequence of positive integers such that $a_1 = 1$, $a_2 = 4$, and for all $n \geq 3$,
    $$a_n = 2a_{n-1} - a_{n-2} + 2$$
    (a) Compute a_n for all $n \leq 10$. (You do not need to prove anything.)
    (b) Conjecture a closed formula for the value of a_n.
    (c) Prove your conjecture from part b using strong induction.
  (3) Read 3.6 on page 80 of the textbook. The author calls this a "disproof". What does he mean by that? Rewrite it as a Theorem followed by a semi-formal proof using only our shortcuts. Use line numbers for your lines and only one statement per line. Be sure to give reasons. Is the author skipping steps? Is he omitting reasons? Using other shortcuts? Comment briefly.