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Math 299
Classroom Example Extravaganza!

1. Theorem: $P \text{ and } Q \Rightarrow Q$

Proof:

(1)	Assume $P \text{ and } Q$	
(2)	Q	by and $-$;1
(3)	$\leftarrow$	
(4)	$P \text{ and } Q \Rightarrow Q$	by $\Rightarrow +$;1,2,3

You Win!!

Formal proofs give us *Objectivity* to know that our proof is correct. They can also be written in *Expository* style by deleting the formatting and numbering, and adding a few English words and punctuation to make the grammar correct. Note that instead of distinguising between "+" and "−" rules, we just refer to both as "definition of". If we do this to the above proof, here is what we might obtain as an *Expository* style proof.

Proof: Assume $P \text{ and } Q$. Then Q is true by the definition of and. Therefore, $P \text{ and } Q \Rightarrow Q$ by the definition of implies.
QED

2. Theorem: $(P \text{ or } Q) \text{ and } \neg P \Rightarrow Q$

Proof:

(1)	Assume $(P \text{ or } Q) \text{ and } \neg P$	
(2)	$P \text{ or } Q$	by and-;1
(3)	$\neg P$	by and-;1
(4)	Assume P	
(5)	Assume $\neg Q$	
(6)	$\rightarrow\leftarrow$	by $\rightarrow\leftarrow$ +;4,3
(7)	$\leftarrow$	
(8)	Q	by $\neg -$;5,6
(9)	$\leftarrow$	
(10)	$P \Rightarrow Q$	by $\Rightarrow$ +;4,8
(11)	Assume Q	
(12)	Q	by copy;11
(13)	$\leftarrow$	
(14)	$Q \Rightarrow Q$	by $\Rightarrow$ +;11,12
(15)	Q	by or-;2,10,14
(16)	$\leftarrow$	
(17)	$(P \text{ or } Q) \text{ and } \neg P \Rightarrow Q$	by $\Rightarrow$ +;1,15

You Win!

3. Theorem: $\neg\neg P \Leftrightarrow P$

Proof:

(1)	Assume $\neg(\neg P)$	
(2)	Assume $(\neg P)$	
(3)	$\rightarrow\leftarrow$	by $\rightarrow\leftarrow +$;2,1
(4)	$\leftarrow$	
(5)	P	by $\neg-$;2,3
(6)	$\leftarrow$	
(7)	$\neg\neg P \Rightarrow P$	by $\Rightarrow +$;1,5
(8)	Assume P	
(9)	Assume $\neg P$	
(10)	$\rightarrow\leftarrow$	by $\rightarrow\leftarrow +$;8,9
(11)	$\leftarrow$	
(12)	$\neg\neg P$	by $\neg+$;9,10
(13)	$\leftarrow$	
(14)	$P \Rightarrow \neg\neg P$	by $\Rightarrow +$;8,12
(15)	$\neg\neg P \Leftrightarrow P$	by $\Leftrightarrow +$;7,14

You Win!!!

4. Theorem: $(\exists x, P(x)) \Rightarrow (\neg\forall y, \neg P(y))$

Proof:

(1)	Assume $\exists x, P(x)$	
(2)	$P(c)$ for some c	by $\exists-$;1
(3)	Assume $\forall y, \neg P(y)$	
(4)	$\neg P(c)$	by $\forall-$;3
(5)	$\rightarrow\leftarrow$	by $\rightarrow\leftarrow +$;2,4
(6)	$\leftarrow$	
(7)	$\neg\forall y, \neg P(y)$	by $\neg+$;3,5
(8)	$\leftarrow$	
(9)	$(\exists x, P(x)) \Rightarrow (\neg\forall y, \neg P(y))$	by $\Rightarrow +$;1,7

We Are Victorious!

Theorem: P or $\neg P$

Proof:

(1)	Assume $\neg(P$ or $\neg P)$	
(2)	Assume P	
(3)	P or $\neg P$	by or $+$;2
(4)	$\rightarrow\leftarrow$	by $\rightarrow\leftarrow$ $+$;3,1
(5)	$\leftarrow$	
(6)	$\neg P$	by $\neg +$;2,4
(7)	Assume $\neg P$	
(8)	P or $\neg P$	by or $+$;7
(9)	$\rightarrow\leftarrow$	by $\rightarrow\leftarrow$ $+$;8,1
(10)	$\leftarrow$	
(11)	P	by $\neg -$;7,9
(12)	$\rightarrow\leftarrow$	by $\rightarrow\leftarrow$ $+$;11,6
(13)	$\leftarrow$	
(14)	P or $\neg P$	by $\neg -$;1,12

Awesome Uber Coolness!!!

Theorem: $x = y$ and $y = z \Rightarrow x = z$

Proof:

(1)	Assume $x = y$ and $y = z$	
(2)	$x = y$	by and $-$;1
(3)	$y = z$	by and $-$;1
(4)	$x = z$	by substitution;3,2
(5)	$\leftarrow$	
(6)	$x = y$ and $y = z \Rightarrow x = z$	by$\Rightarrow$ $+$;1,4

We Win!!

Theorem: $(\exists! x, x \leq a) \Rightarrow \neg\forall y, \neg(y \leq a)$

Proof:

(1)	Assume $\exists! x, x \leq a$	
(2)	$\exists x, x \leq a \text{ and } \forall z, z \leq a \Rightarrow z = x$	by $\exists!-$;1
(3)	$c \leq a \text{ and } \forall z, z \leq a \Rightarrow z = c$ for some c	by $\exists-$;2
(4)	$c \leq a$	by and $-$;3
(5)	Assume $\forall y, \neg(y \leq a)$	
(6)	$\neg(c \leq a)$	by $\forall-$;5
(7)	$\rightarrow\leftarrow$	by $\rightarrow\leftarrow +$;4,6
(8)	$\leftarrow$	
(9)	$\neg\forall y, \neg(y \leq a)$	by $\neg+$;5,7
(10)	$\leftarrow$	
(11)	$(\exists! x, x \leq a) \Rightarrow \neg\forall y, \neg(y \leq a)$	by $\Rightarrow +$;1,9

QED!

Theorem: $A \subseteq A$

Proof:

(1)	Let $x \in A$	
(2)	$x \in A$	by copy;1
(3)	$A \subseteq A$	by $\subseteq +$;1,2

QED

Theorem $A - B \subseteq A$

Proof:

(1)	Let $x \in A - B$	
(2)	$x \in A$	by $--$;1
(3)	$A - B \subseteq A$	by $\subseteq +$;1,2

QED!

Theorem: $\overline{B} \subseteq \overline{A} \Rightarrow A \subseteq B$

Proof:

(1)	Assume $\overline{B} \subseteq \overline{A}$	
(2)	Let $x \in A$	
(3)	Assume $\neg x \in B$	
(4)	$x \notin B$	by $\notin +$;3
(5)	$x \in \overline{B}$	by complement $+$;4
(6)	$x \in \overline{A}$	by $\subseteq -$;5,1
(7)	$x \notin A$	by complement $-$;6
(8)	$\neg x \in A$	by $\notin -$;7
(9)	$\rightarrow\leftarrow$	by $\rightarrow\leftarrow +$;2,8
(10)	$\leftarrow$	
(11)	$x \in B$	by $\neg -$;3,9
(12)	$A \subseteq B$	by $\subseteq +$;2,11
(13)	$\leftarrow$	
(14)	$\overline{B} \subseteq \overline{A} \Rightarrow A \subseteq B$	by $\Rightarrow +$;1,12

QED!!

Theorem: $(A \cap B) \times B \subseteq A \times B$

Proof:

(1)	Let $x \in (A \cap B) \times B$	
(2)	$x = (s, t)$ for some $s \in A \cap B$ and $t \in B$	by $\times -$;1
(3)	$s \in A$ and $s \in B$	by $\cap -$;2
(4)	$s \in A$	by and $-$;3
(5)	$(s, t) \in A \times B$	by $\times +$;4,2
(6)	$x \in A \times B$	by substitution;2,5
(7)	$(A \cap B) \times B \subseteq A \times B$	by $\subseteq +$;1,6

QED!

Theorem: $A \cap \bigcup_{i \in I} B_i \subseteq \bigcup_{i \in I} (A \cap B_i)$

Proof:

(1) Let $x \in A \cap \bigcup_{i \in I} B_i$

(2) $x \in A$ by $\cap-$;1

(3) $x \in \bigcup_{i \in I} B_i$ by $\cap-$;1

(4) $x \in B_k$ for some $k \in I$ by indexed union -;3

(5) $x \in A \cap B_k$ by $\cap+$;2,4

(6) $x \in \bigcup_{i \in I} (A \cap B_i)$ by indexed union +; 4,5

(7) $A \cap \bigcup_{i \in I} B_i \subseteq \bigcup_{i \in I} (A \cap B_i)$ by $\subseteq +$;1,6

QED!

Here is an example of induction.

Theorem: For all positive integers n we have

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Proof:

(1) $1 = \frac{1(1+1)}{2}$ by arithmetic.

(2) Let $k \in \mathbb{N}$ and $1 \leq k$.

(3) Assume $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$

(4) So by substitution and arithmetic,

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= (1 + 2 + \cdots + k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)((k+1)+1)}{2} \end{aligned}$$

(5) $\leftarrow$

(6) $\forall n \geq 1, 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ by induction;1,2,3,4,5