

Fun Facts: Chaos and Fractals

© 2002 - 2006 Ken Monks

by Ken Monks

Department of Mathematics

University of Scranton

Course: Math 320 - Chaos and Fractals

Revised: Spring 2004

This is **not** a complete set of lecture notes for Math 320, Chaos and Fractals. Additional material will be covered in class and discussed in the textbook.

Logic

In this section we give an informal overview of logic and proofs. For a more formal introduction see any logic textbook.

Variables, Expressions, and Statements

Definition A *set* is a collection of items called the **members** (or **elements**) of the set.

Remark An element is either in a set or it is not in a set, it cannot be in a set more than once.

Definition An **expression** is an arrangement of symbols which represents an element of a set called the **domain** (or **type**) of the expression.

Remark It is not necessary that we know specifically which element of the domain an expression represents, only that it represents some unspecified element in that set.

Definition The element of the domain that the expression represents is called a **value** of that expression.

Definition A **variable** is an expression consisting of a single symbol.

Definition A **constant** is an expression whose domain contains a single element.

Definition A **statement** (or **Boolean expression**) is an expression whose domain is $\{true, false\}$.

Remark We do not have to know if a statement is true or false, just that it is either true or false.

Definition The value of a statement is called its **truth value**.

Definition To **solve** a statement is to determine the set of all elements for which the statement is true.

Remark More precisely, if a statement contains n variables, x_1, \dots, x_n , then to solve the statement is to find the set of all n -tuples (a_1, \dots, a_n) such that each a_i is an element of the domain of x_i and the statement becomes true when x_1, \dots, x_n are replaced by a_1, \dots, a_n respectively. Any such n -tuple is called a **solution** and the set

of all solutions is called the **solution set**.

Definition An **equation** is a statement of the form $A = B$ where A and B are expressions.

Definition An **inequality** is a statement of the form $A \star B$ where A and B are expressions and \star is one of $\leq, \geq, >, <$, or \neq .

Propositional Logic

The Five Logical Operators

Definition Let P, Q be statements. Then the expressions

1. $\sim P$
2. P and Q
3. P or Q
4. $P \Rightarrow Q$
5. $P \Leftrightarrow Q$

are also statements whose truth values are completely determined by the truth values of P and Q as shown in the following table

P	Q	$\sim P$	P and Q	P or Q	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Rules of Inference and Proof

Definition A **rule of inference** is a rule which takes zero or more statements (or other items) as input and returns one or more statements as output.

Notation An expression of the form

$$\begin{array}{c}
 P_1 \\
 \vdots \\
 P_k \\
 \hline
 Q_1 \\
 \vdots \\
 Q_n
 \end{array}$$

represents a rule of inference whose inputs are P_1, \dots, P_k and outputs are Q_1, \dots, Q_n .

Notation The rule of inference shown above can also be expressed in **recipe notation**

as

Show P_1

⋮

Show P_k

Conclude Q_1

⋮

Conclude Q_n

or equivalently,

To show Q_1, \dots, Q_n

Show P_1

⋮

Show P_k

Definition A **formal logic system** consists of a set of statements and a set of rules of inference.

Definition A **proof** in a formal logic system consists of a finite sequence of statements (and other inputs to the rules of inference) such that each statement follows from the previous statements in the sequence by one or more of the rules of inference.

Natural Deduction

Definition The symbol \leftarrow is an abbreviation for “end assumption”.

Definition The rules of inference for propositional logic are shown in Table 1.

Table 1: Rules of inference for Propositional Logic		
<div style="border: 1px solid black; padding: 2px; margin-bottom: 5px;">$\text{and } +$</div> To show W and V 1. Show W 2. Show V	<div style="border: 1px solid black; padding: 2px; margin-bottom: 5px;">$\text{and } -$</div> To show W 1. Show W and V	<div style="border: 1px solid black; padding: 2px; margin-bottom: 5px;">$\text{and } -$</div> To show V 1. Show W and V
<div style="border: 1px solid black; padding: 2px; margin-bottom: 5px;">$\Rightarrow +$</div> To show $W \Rightarrow V$ 1. Assume W 2. Show V 3. \leftarrow	<div style="border: 1px solid black; padding: 2px; margin-bottom: 5px;">$\Rightarrow -$ (modus ponens)</div> To show V 1. Show W 2. Show $W \Rightarrow V$	
<div style="border: 1px solid black; padding: 2px; margin-bottom: 5px;">$\Leftrightarrow +$</div> To show $W \Leftrightarrow V$ 1. Show $W \Rightarrow V$ 2. Show $V \Rightarrow W$	<div style="border: 1px solid black; padding: 2px; margin-bottom: 5px;">$\Leftrightarrow -$</div> To show $W \Rightarrow V$ 1. Show $W \Leftrightarrow V$	<div style="border: 1px solid black; padding: 2px; margin-bottom: 5px;">$\Leftrightarrow -$</div> To show $V \Rightarrow W$ 1. Show $W \Leftrightarrow V$
<div style="border: 1px solid black; padding: 2px; margin-bottom: 5px;">$\text{or } +$</div> To show W or V 1. Show W	<div style="border: 1px solid black; padding: 2px; margin-bottom: 5px;">$\text{or } +$</div> To show W or V 1. Show V	<div style="border: 1px solid black; padding: 2px; margin-bottom: 5px;">$\text{or } -$ (proof by cases)</div> To show U 1. Show W or V 2. Show $W \Rightarrow U$ 3. Show $V \Rightarrow U$
<div style="border: 1px solid black; padding: 2px; margin-bottom: 5px;">$\sim +$ (proof by contradiction)</div> To show $\sim W$ 1. Assume W 2. Show $\rightarrow\leftarrow$ 3. \leftarrow		<div style="border: 1px solid black; padding: 2px; margin-bottom: 5px;">$\sim -$ (proof by contradiction)</div> To show W 1. Assume $\sim W$ 2. Show $\rightarrow\leftarrow$ 3. \leftarrow
<div style="border: 1px solid black; padding: 2px; margin-bottom: 5px;">$\rightarrow\leftarrow +$</div> To show $\rightarrow\leftarrow$ 1. Show W 2. Show $\sim W$		<div style="border: 1px solid black; padding: 2px; margin-bottom: 5px;">$\rightarrow\leftarrow -$</div> To show W 1. Show $\rightarrow\leftarrow$

Remark Note that the inputs “Assume -” and “ \leftarrow ” are not themselves statements but rather inputs to rules of inference that may be inserted into a proof at any time. There is no reason however, to insert such statements unless you intend to use one

of the rules of inference that requires them as inputs.

Example Prove $P \Rightarrow (P \text{ or } Q)$ and verify it with a truth table

Example Prove $(P \text{ or } Q) \Rightarrow \sim(\sim P \text{ and } \sim Q)$ and verify it with a truth table

Equality

Definition The equality symbol, $=$, is defined by two rules of inference:

<p>Reflexive =</p> <p>To show $x = x$</p>	<p>Substitution</p> <p>To show W with the n^{th} free occurrence of x replaced by y</p> <ol style="list-style-type: none"> 1. Show W 2. Show $x = y$
---	--

Remark Note that in the Reflexive rule there are no inputs, so you can insert a statement of the form $x = x$ into your proof at any time.

Example Given $x = y$ and $y = z$, prove $x = z$.

Predicate Logic

Quantifiers

Definition The symbols \forall and \exists are **quantifiers**. The symbol \forall is called “for all”, “for every”, or “for each”. The symbol \exists is called “for some” or “there exists”.

Definition If W is a statement and x is any variable then $\forall x, W$ and $\exists x, W$ are both statements. The rules of inference for these quantifiers are given in Table 2.

Notation If x is a variable, t an expression, and $W(x)$ a statement then $W(t)$ is the statement obtained by replacing every free occurrence of x in $W(x)$ with (t) .

<p>$\forall +$</p> <p>To show $\forall x, W(x)$</p> <ol style="list-style-type: none"> 1. Let s be arbitrary 2. Show $W(s)$ 	<p>$\forall -$</p> <p>To show $W(t)$</p> <ol style="list-style-type: none"> 1. Show $\forall x, W(x)$
<p>$\exists +$</p> <p>To show $\exists x, W(x)$</p> <ol style="list-style-type: none"> 1. Show $W(t)$ 	<p>$\exists -$</p> <p>To show $W(t)$ for some t</p> <ol style="list-style-type: none"> 1. Show $\exists x, W(x)$

Remark Note that there are restrictions on the rules of inference for quantifiers which are not listed in Table 2 (see the Proof Recipes sheet for details). In most situations they are not a concern.

Remark *Precedence: Quantifiers have a lower precedence than \Leftrightarrow . Thus they quantify the largest statement to their right possible unless specifically limited by parentheses.*

Example Prove $(\exists x, P(x)) \Rightarrow (\sim \forall y, \sim P(y))$

Sets, Functions, Numbers

Some Definitions from Set theory

The symbol \in is formally undefined, but it means “is an element of”. Many of the definitions below are informal definitions that are sufficient for our purposes.

Set notation and operations

Finite set notation:	$x \in \{x_1, \dots, x_n\} \Leftrightarrow x = x_1 \text{ or } \dots \text{ or } x = x_n$
Set builder notation:	$x \in \{y : P(y)\} \Leftrightarrow P(x)$
Cardinality:	$\#S =$ the number of elements in a finite set S
Subset:	$A \subseteq B \Leftrightarrow \forall x, x \in A \Rightarrow x \in B$
Set equality:	$A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A$
Def. of \notin :	$x \notin A \Leftrightarrow \sim(x \in A)$
Empty set:	$\exists \emptyset, \forall x, x \notin \emptyset$
Relative Complement:	$x \in B - A \Leftrightarrow x \in B \text{ and } x \notin A$
Intersection:	$x \in A \cap B \Leftrightarrow x \in A \text{ and } x \in B$
Union:	$x \in A \cup B \Leftrightarrow x \in A \text{ or } x \in B$
Indexed Intersection:	$x \in \bigcap_{i \in I} A_i \Leftrightarrow \forall i, i \in I \Rightarrow x \in A_i$
Indexed Union:	$x \in \bigcup_{i \in I} A_i \Leftrightarrow \exists i, i \in I \text{ and } x \in A_i$
Two convenient abbreviations:	$(\forall x \in A, P(x)) \Leftrightarrow \forall x, x \in A \Rightarrow P(x)$ $(\exists x \in A, P(x)) \Leftrightarrow \exists x, x \in A \text{ and } P(x)$

Some Famous Sets

The Natural Numbers	$\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$
The Integers	$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
The Rational Numbers	$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}, b \neq 0, \text{ and } \gcd(a, b) = 1 \right\}$
The Real Numbers	$\mathbb{R} = \{x : x \text{ can be expressed as a decimal number}\}$
The Complex Numbers	$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}$ where $i^2 = -1$
The positive real numbers	$\mathbb{R}^+ = \{x : x \in \mathbb{R} \text{ and } x > 0\}$
The negative real numbers	$\mathbb{R}^- = \{x : x \in \mathbb{R} \text{ and } x < 0\}$
The positive reals in a set A	$A^+ = A \cap \mathbb{R}^+$
The negative reals in a set A	$A^- = A \cap \mathbb{R}^-$
The first n positive integers	$\mathbb{I}_n = \{1, 2, \dots, n\}$
The first $n + 1$ natural numbers	$\mathbb{O}_n = \{0, 1, 2, \dots, n\}$

Cartesian products

Ordered Pairs:	$(x, y) = (u, v) \Leftrightarrow x = u \text{ and } y = v$
Ordered n -tuple:	$(x_1, \dots, x_n) = (y_1, \dots, y_n) \Leftrightarrow x_1 = y_1 \text{ and } \dots \text{ and } x_n = y_n$
Cartesian Product:	$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$
Cartesian Product:	$A_1 \times \dots \times A_n = \{(x_1, \dots, x_n) : x_1 \in A_1 \text{ and } \dots \text{ and } x_n \in A_n\}$
Power of a Set	$A^n = A \times A \times \dots \times A$ where there are n “ A ’s” in the Cartesian product

Functions and Relations

Def of relation:	R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$
Def of function:	$f : A \rightarrow B \Leftrightarrow f \subseteq A \times B$ and $(\forall x, \exists y, (x, y) \in f)$ and $(\forall x, ((x, y) \in f \text{ and } (x, z) \in f) \Rightarrow y = z)$
Alt function notation	$X \xrightarrow{f} Y \Leftrightarrow f : X \rightarrow Y$
Def of $f(x)$:	$f(x) = y \Leftrightarrow f : A \rightarrow B$ and $(x, y) \in f$
Domain:	$\text{Domain}(f) = A \Leftrightarrow f : A \rightarrow B$
Codomain:	$\text{Codomain}(f) = B \Leftrightarrow f : A \rightarrow B$
Image:	$f(S) = \{y : \exists x, x \in S \text{ and } y = f(x)\}$
Range:	$\text{Range}(f) = f(\text{Domain}(f))$
Identity Map:	$id_A : A \rightarrow A$ and $\forall x, id_A(x) = x$
Composition:	$f : A \rightarrow B$ and $g : B \rightarrow C \Rightarrow (g \circ f) : A \rightarrow C$ and $\forall x, (g \circ f)(x) = g(f(x))$
Injective (one-to-one):	f is injective $\Leftrightarrow \forall x, \forall y, f(x) = f(y) \Rightarrow x = y$
Surjective (onto):	f is surjective $\Leftrightarrow f : A \rightarrow B$ and $(\forall y, y \in B \Rightarrow \exists x, y = f(x))$
Bijjective:	f is bijective $\Leftrightarrow f$ is injective and f is surjective
Inverse:	$f^{-1} : B \rightarrow A \Leftrightarrow f : A \rightarrow B$ and $f \circ f^{-1} = id_B$ and $f^{-1} \circ f = id_A$
Inverse Image:	$f : A \rightarrow B$ and $S \subseteq B \Rightarrow f^{-1}(S) = \{x \in A : f(x) \in S\}$

Example Prove that if $A \subseteq B$ then $A \cap B = A$.

Example (left cancellation for injective functions) Let X, Y, Z be sets and $f : Y \rightarrow Z$. Show that if f is injective then for any functions $g, h : X \rightarrow Y$

$$(f \circ g = f \circ h) \Rightarrow g = h$$

Sequences

Definition A **finite sequence** is a function $t : \mathbb{I}_n \rightarrow A$ where n is a natural number and A is a set. An **infinite sequence** is a function $t : \mathbb{N}^+ \rightarrow A$ where A is a set. In either case, $t(k)$ is called the k^{th} term of the sequence.

Remark It is often convenient to say that t is a finite (resp infinite) sequence if $t : \mathbb{O}_n \rightarrow A$ (resp. $t : \mathbb{N} \rightarrow A$). In this case we say that $t(k)$ is the $k + 1^{\text{st}}$ term of the sequence.

Notation If $t : \mathbb{I}_n \rightarrow A$ is a finite sequence we write

$$t_1, t_2, t_3, \dots, t_n$$

as another notation for t , where $t_k = t(k)$ for all $k \in \mathbb{I}_n$. Similarly if $t : \mathbb{N}^+ \rightarrow A$ we write

$$t_1, t_2, t_3, \dots$$

for t where $t_k = t(k)$ for all $k \in \mathbb{N}^+$.

Remark Sometimes for readability we might want to enclose a sequence in parenthesis. For example, we might write “Let $t = (1, 2, 3, 4)$ ” instead of “Let $t = 1, 2, 3, 4$ ”. In this sense there is really no distinction between n -tuples and finite sequences.

Notation We use an overbar to indicate an infinite repeating sequence, i.e.

$$t_0, t_1, \dots, t_{k-1}, \overline{t_k, \dots, t_{k+n-1}}$$

denotes the sequence infinite sequence t such that $t_i = t_{k+((i-k) \bmod n)}$ for all $i \geq k+n$.

Example Write the first five terms of the sequence $\mathbb{N} \xrightarrow{a} \mathbb{N}$ by $a_n = n^2 + 1$.

Example What is the 1000th term in the sequence

$$9, 0, 8, \overline{3, 2, 4, 1, 5, 7, 6}$$

Example Write the first five terms of the sequence $\mathbb{N} \xrightarrow{a} \mathbb{N}$ given by

$$a(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot a(n-1) & \text{otherwise} \end{cases}$$

Some Facts from Number Theory

Theorem (Math Induction) Let $P(n)$ be any statement about a natural number variable n . Then

$$(P(0) \text{ and } \forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1)) \Rightarrow \forall n \in \mathbb{N}, P(n).$$

Theorem (Division Algorithm) Let $a, b \in \mathbb{Z}$, and $b > 0$. Then there exist unique integers $q, r \in \mathbb{Z}$ such that

$$a = qb + r \text{ and } 0 \leq r < b.$$

Remark In this theorem the number q is called the **quotient** and r is called the **remainder** when a is divided by b .

Definition Let $a, b \in \mathbb{Z}$ with $b > 0$. Then $a \bmod b$ is the remainder when a is divided by b . The quotient can be written as $\lfloor \frac{a}{b} \rfloor$ where $\lfloor x \rfloor$ is the greatest integer less than or equal to a real number x .

Definition Let $a, b \in \mathbb{Z}$. We say a **divides** b if $ak = b$ for some integer k . If a divides b we write $a \mid b$.

Definition Let $a, b \in \mathbb{Z}$ with $a \neq 0$ or $b \neq 0$. Then $\gcd(a, b)$ is the greatest positive integer which divides both a and b .

Example What is the quotient and remainder when $2^{1000} + 1$ is divided by 32?

Example What is the quotient and remainder when -100 is divided by 7?

Example True or False:

$$(a) 14 \mid 7 \quad (b) 7 \mid 14 \quad (c) 7 \mid -14 \quad (d) 7 \mid 0$$

Example What is the $\gcd(72, 60)$? $\gcd(295927, 304679)$?

Iteration

Discrete Dynamical Systems

Definition Let X be any set. Any function $f : X \rightarrow X$ is called a **set theoretic discrete dynamical system** (or simply **discrete dynamical system**).

Definition Let X be a set and $f : X \rightarrow X$. Define $f^0 = id_X$ and for all $k \geq 1$ define

$$f^k = f \circ f^{k-1}$$

Example Let $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = 2x + 1$. Find a nonrecursive formula for $f^k(x)$ for $k \geq 0$.

Theorem (Power Theorem) Let $f : X \rightarrow X$. For any $k, n \in \mathbb{N}$,

$$f^{k+n} = f^k \circ f^n$$

and

$$f^{kn} = \underbrace{f^n \circ f^n \circ \dots \circ f^n}_{k \text{ terms}} = (f^n)^k$$

Definition Let $f : X \rightarrow X$ and $x \in X$. The sequence

$$x, f(x), f^2(x), f^3(x), \dots$$

is called the **f -orbit** of x . The first term, x , is called the **seed** of the orbit. The $k + 1^{\text{st}}$ term is called the k^{th} **f -iterate** of x (or k^{th} **iterate** or k^{th} **iteration**). We write $\text{Orb}_f(x)$ for the f -orbit of x .

Remark $\text{Orb}_f(x) : \mathbb{N} \rightarrow X$ and $\text{Orb}_f(x)(n) = f^n(x)$ for all $n \in \mathbb{N}$.

Example (1) Find the complete f -orbit of 5 for $\mathbb{C} - \{0, 1\} \xrightarrow{f} \mathbb{C} - \{0, 1\}$ by $f(z) = \frac{1}{1-z}$. What is the f -orbit of 3? How about a ?

Definition Let X be a set, $x \in X$, and $f : X \rightarrow X$. Then the set of terms in the f -orbit of x is denoted $\mathcal{O}_f(x)$, i.e.

$$\mathcal{O}_f(x) = \{f^k(x) : k \in \mathbb{N}\}$$

We call $\mathcal{O}_f(x)$ the **set of f -iterates of x** (or simply **the set of terms in the f -orbit of x**).

Example What is $\mathcal{O}_f(5)$ in Example 1? How many elements are in $\mathcal{O}_f(5)$?

Types of Orbits

Definition Let $f : X \rightarrow X$ and $x \in X$. The f -orbit of x is **cyclic** if $f^n(x) = x$ for some $n \geq 1$. In this situation we say that x is a **cyclic point** (or **periodic point**) for f .

Example Is $\text{Orb}_f(5)$ cyclic in Example 1?

Definition Let $f : X \rightarrow X$ and $x \in X$. If $f^n(x) = x$ for some $n \geq 1$, we say x has **period**

n . If in addition $f^k(x) \neq x$ for all $1 \leq k < n$ then we say x has **minimum period** n . If x has period 1 we say x is a **fixed point** of f . If x has period n we also say that $\text{Orb}_f(x)$ has **period** n and if x has minimum period n we also say $\text{Orb}_f(x)$ has **minimum period** n as well.

Example What is the minimum period of 5 in Example 1?

Example Does f have any fixed points in Example 1?

Lemma Let $f : X \rightarrow X$, $x \in X$, and $n \in \mathbb{N}^+$.

$$x \text{ has minimum period } n \Rightarrow \#\mathcal{O}_f(x) = n$$

Example Why isn't it if and only if?

Definition Let $f : X \rightarrow X$ and $x \in X$. The f -orbit of x is **eventually cyclic** if $f^n(x) = f^m(x)$ for some n, m with $n \neq m$. In this situation we also say that x is an **eventually cyclic point** (or **eventually periodic point**) for f .

Definition Let $f : X \rightarrow X$ and $x \in X$ periodic point with period n . We say that $\mathcal{O}_f(x)$ is an **n -cycle** if and only if $\text{Orb}_f(x)$ is cyclic with minimum period n .

Definition Let $f : X \rightarrow X$ and $x \in X$. The f -orbit of x is **acyclic** if it is not eventually cyclic.

Example Can you come up with examples of each of these?

The Digraph

Definition A **directed graph** (or **digraph**) is a pair (V, E) where V is a set of elements called the **nodes** and $E \subseteq V \times V$ is the set of **directed edges**.

Definition Let $X \xrightarrow{f} X$ be a discrete dynamical system. The **digraph** of f is the directed graph (X, S) where $S = \{(a, f(a)) : a \in X\}$, i.e. the nodes are the elements of the domain and the directed edges connect each element a in the domain to $f(a)$.

Examples of Iteration

The Collatz Conjecture

Definition Define $T : \mathbb{Z} \rightarrow \mathbb{Z}$ by $\forall x \in \mathbb{Z}$

$$T(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{3x+1}{2} & \text{if } x \text{ is odd} \end{cases}$$

Conjecture (Collatz) For all $n \in \mathbb{N}^+$, $\exists k \geq 0, T^k(n) = 1$, i.e. the T -orbit of any positive integer contains one.

Remark Note that $\text{Orb}_T(1) = \overline{1, 2}$ so that the conjecture is equivalent to saying that the T -orbit of any positive integer is eventually periodic and enters the 2-cycle $\{1, 2\}$.

Sumerian Method for Computing Square Roots

Claim Let $a \in \mathbb{R}^+$ and $Root_a(x) = \frac{1}{2}(x + \frac{a}{x})$. For any $x \in \mathbb{R}^+$, the $Root_a$ -orbit of x converges to \sqrt{a} .

Example Find a fraction and a decimal that are a good approximation to $\sqrt{2}$ and $\sqrt{3}$ by the Sumerian Method.

Multiple Inputs: The Euclidean Algorithm

Remark If $f : A \times B \rightarrow C$ and $a \in A, b \in B$, we usually abbreviate $f((a, b))$ as $f(a, b)$.

Claim Define $Euc : \mathbb{N}^+ \times \mathbb{N} \rightarrow \mathbb{N}^+ \times \mathbb{N}$ by

$$Euc(n, m) = \begin{cases} (m, n) & \text{if } n < m \\ (n, 0) & \text{if } m = 0 \\ (m, n \text{ Mod } m) & \text{otherwise} \end{cases} .$$

for any $(n, m) \in \mathbb{N}^+ \times \mathbb{N}$. Then the Euc -orbit of any (n, m) is eventually fixed and contains the fixed point $(\gcd(n, m), 0)$.

Remark This method of computing $\gcd(n, m)$ is called the **Euclidean algorithm**.

Example Reduce the fraction $\frac{295927}{304679}$ by hand.

Non-numeric Inputs: Post's Tag Problem

Definition Let S be a set and let S^* be the set of words (finite sequences) which can be made from the alphabet S , i.e.

$$S^* = \bigcup_{n=0}^{\infty} \{f \mid f : \mathbb{I}_n \rightarrow S\} .$$

If $x \in S^*$ then $\#x$ is the number of letters in the word x . If $x, y \in S^*$ then $x \cdot y$ is the word formed by concatenating the words x and y . If $x = x_1x_2 \cdots x_n \in S^*$ then $x[a \dots b]$ is the word $x_a x_{a+1} x_{a+2} \cdots x_{b-1} x_b$ and $x[a] = x_a$.

Example What is $\{\star\}^* \cdot \{a, b\}^*$?

Definition Define $Tag : \{a, b\}^* \rightarrow \{a, b\}^*$ as follows. If $x \in \{a, b\}^*$ and $n = \#x$ then

$$Tag(x) = \begin{cases} x & \text{if } n < 3 \\ x[4 \dots n] \cdot aa & \text{if } x[1] = a \\ x[4 \dots n] \cdot bbab & \text{if } x[1] = b \end{cases}$$

In other words, if a word is less than three letters long, Tag returns it unchanged, if it is 3 or more letters and begins with the letter a then Tag deletes the first three letters and appends aa on the right, and if it is 3 or more letters and begins with b then Tag deletes the first three letters and appends $bbab$.

Problem (TAG) (Emil Post 1921) Are there any Tag -orbits which are not eventually cyclic?

Example What is the *Tag-orbit* of a ? $baba$? $bbbaa$?

Stick Figure Fractals

Definition Let A, B be any distinct points in the plane. Then \overline{AB} denotes the line segment with endpoints A and B (i.e. the set of all points in the plane which are on the line containing A and B and are either between A and B or are equal to A and B). The **directed segment from A to B** is a pair (\overline{AB}, A) , and is denoted \overrightarrow{AB} . In this case we say \overline{AB} is the segment associated with \overrightarrow{AB} (and can think of a directed segment as being a set of points in the plane in this sense).

Notation If $A = (a_1, a_2), B = (b_1, b_2)$ then \overrightarrow{AB} can be written $\text{dseg}([a_1, a_2], [b_1, b_2])$ and \overline{AB} can be written $\text{seg}([a_1, a_2], [b_1, b_2])$.

Remark A directed line segment can be thought of as a line segment with an arrow drawn on it in one of the two possible directions. Note that $\overline{AB} = \overline{BA}$ but $\overrightarrow{AB} \neq \overrightarrow{BA}$. A directed line segment can also be thought of as a set of points since the line segment associated with it is a set of points. So if we talk about a point being "on a directed segment" we mean that it is on the line segment associated with the directed segment and so on.

Definition A **stick** is either a line segment or a directed line segment. A **stick figure** is a finite set of sticks. Let U_{sf} be the set of all stick figures.

Remark Note that we can also consider a stick figure to be a set of points in the plane by considering the union of the points in the line segments and (line segments associated with) directed segments.

Definition Let $s = \text{dseg}([a, b], [c, d])$. Define $T_s : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$T_s(x, y) = ((c - a)x + (b - d)y + a, (d - b)x + (c - a)y + b).$$

T_s is called the **affine map induced by s** .

Remark We will show how to derive this map later in the course. Intuitively, it is the map that sends the directed segment from $(0, 0)$ to $(1, 0)$ to the directed segment $s = \text{dseg}([a, b], [c, d])$, and the directed segment from $(0, 0)$ to $(0, 1)$ to the directed segment obtained by rotating s by 90° CCW about (a, b) .

Example Find the affine map induced by the directed segment from $(1, 1)$ to $(2, 2)$.

Lemma Let s be a directed segment and t a line segment. Then $T_s(t)$ is a line segment.

Definition Let s, t be directed segments with $t = \overrightarrow{AB}$. Then $T_s(t) = (T_s(\overline{AB}), T_s(A))$.

Definition If G is a stick figure and s a directed segment then $T_s(G)$ is the stick figure $\bigcup_{x \in G} \{T_s(x)\}$.

Definition For each stick figure G define a dynamical system $\gamma_G : U_{sf} \rightarrow U_{sf}$ as follows. Let $S \in U_{sf}$ be a stick figure. For each $x \in S$, define

$$g(x) = \begin{cases} \{x\} & \text{if } x \text{ is a line segment} \\ T_x(G) & \text{if } x \text{ is a directed segment} \end{cases}$$

Then $\gamma_G(S) = \bigcup_{x \in S} g(x)$. The dynamical system γ_G is called the **stick figure iterator** associated with G . The figure G is called the **generator** for the stick figure iterator.

Claim In many cases the γ_G -orbit of a seed S converges to a fractal shape.

Example See my Maple worksheet [LectureExamples.mws](#) for examples.

GeeBees (Grid Based Fractals)

Algorithm Let $n \in \mathbb{N}^+$ and $a_1, \dots, a_k \in \mathbb{I}_{n^2}$. Define a dynamical system $GB(n; a_1, \dots, a_k)$ as follows. Let the seed be a set containing one uncolored square. The process is:

1. Subdivide each uncolored square in the input set into an $n \times n$ grid of congruent subsquares and number these subsquares from 1 to n^2 from left to right and bottom to top, starting in the lower left corner.
2. Color the subsquares numbered a_1, \dots, a_k .
3. Output the set of colored and uncolored subsquares.

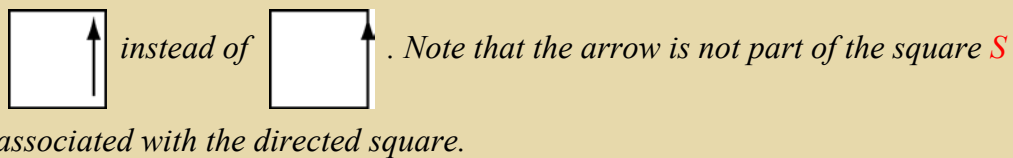
Remark The background (uncolored subsquares of the original square) of a GB converges to a fractal shape.

Example Plot the first few iterations of $GB(3; 2, 4)$.

HeeBGB's

Definition A **directed square** is a pair (S, \vec{AB}) where S is a square in the plane and \vec{AB} is a directed segment whose associated line segment is a side of S .

Notation When drawing a picture of a directed square we will draw the directed segment inside the square next to the edge instead of directly on top of the edge to avoid confusion when two directed squares share a common edge, i.e.



Definition A **labeled square** is a member of the following 9 families:

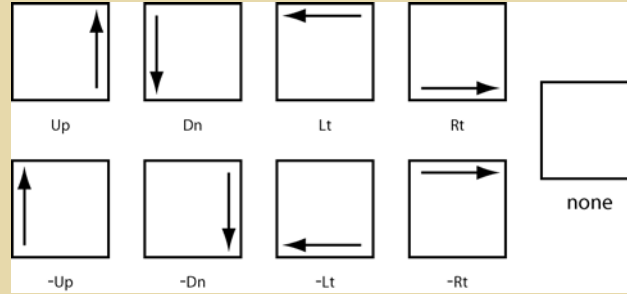


Figure 1: Labeled Squares

Each labeled square must have one of the orientations shown above, but can have any position or size. Let U_{LS} be the set of all labeled squares.

Remark Notice that every labeled square is a directed square except for the members of the family labeled **none**. Each labeled directed square is either positive or negative (the negative ones have the arrow on the left when viewed with the arrow pointing upwards).

Definition Define the **vertical mirror image** of each labeled square a to be \overleftarrow{a} as follows:

a	Up	-Up	Dn	-Dn	Lt	-Lt	Rt	-Rt	none
\overleftarrow{a}	-Up	Up	-Dn	Dn	-Rt	Rt	-Lt	Lt	none

i.e. the sign always changes and left and right are interchanged.

Remark Note that this is what is obtained if each of the images in Figure 1 above are reflected about the vertical line through the center of the square.

Definition A **GB figure** is a finite set of labeled squares. Let U_{GB} be the set of all GB figures.

Definition Let $n \in \mathbb{N}^+$ and $a_1, \dots, a_{n^2} \in \{Up, Dn, Lt, Rt, -Up, -Dn, -Lt, -Rt, none\}$ (the label set). Define a dynamical system $HeeBGB(a_1, \dots, a_{n^2}) : U_{GB} \rightarrow U_{GB}$ as follows. First define $g : U_{LS} \rightarrow U_{GB}$ as follows. For each $x \in U_{LS}$,

If x is labeled **none** then $g(x) = \{x\}$.

If x is a directed square then

1. Rotate x so its arrow points upwards.
2. Subdivide x into an $n \times n$ grid of congruent subsquares.
 - a. if x is positive, label these subsquares from a_1 to a_{n^2} from left to right and bottom to top, starting in the lower left corner.
 - b. if x is negative, label these subsquares from \overleftarrow{a}_1 to \overleftarrow{a}_{n^2} from right to left and bottom to top, starting in the lower right corner.
3. Undo the rotation from step number 1 to return the square (and all its new subsquares) to the original position and orientation. $g(x)$ is the set of these subsquares.

Now, let $S \in U_{GB}$. Define $HeeBGB(a_1, \dots, a_{n^2})(S) = \bigcup_{x \in S} g(x)$.

Algorithm To draw a *HeeBGB* fractal start with a seed consisting of a set containing a single labeled square whose label is *Up* and a choice of labels a_1, \dots, a_{n^2} . Compute the $HeeBGB(a_1, \dots, a_{n^2})$ -orbit of the seed, but color the squares labeled *none* as you iterate. The uncolored portion of the *HeeBGB*-orbit of this seed that is contained in the original square always converges to a fractal image (i.e. you are coloring the background, not the fractal).

Example Draw the first few iterations of $HeeBGB(Up, Up, Up, none)$.

Example Draw the first few iterations of $HeeBGB(Up, -Dn, Lt, none)$.

Example Draw the first few iterations of $HeeBGB(Dn, -Dn, Rt, none)$.

Newton's Method

Definition Let $R \subseteq \mathbb{R}$, $f : R \rightarrow \mathbb{R}$, and $r \in R$. We say r is a **root** of f if $f(r) = 0$.

Definition Let $R \subseteq \mathbb{R}$ and $f : R \rightarrow \mathbb{R}$ a differentiable function. Define $Newt_f : \tilde{R} \rightarrow \mathbb{R}$ by

$$Newt_f(x) = x - \frac{f(x)}{f'(x)}$$

for all $x \in \tilde{R}$ where $\tilde{R} = \{x \in R : f'(x) \neq 0\}$.

Remark $Newt_f(a)$ is the x coordinate of point where the tangent line to the graph of f at $(a, f(a))$ meets the x -axis.

Theorem (Newton's Method) Let $R \subseteq \mathbb{R}$, $f : R \rightarrow \mathbb{R}$ a differentiable function, and $r \in R$ a root of f . If $f'(r) \neq 0$ then there exists an interval $I \subseteq R$ such that $r \in I$, $Newt_f : I \rightarrow I$ and for all $x \in I$ the $Newt_f$ -orbit of x converges to r .

Example See my Maple worksheet [LectureExamples.mws](#) for an example.

Changing Integer Base

Theorem (Base b representation) Let $b, n \in \mathbb{N}$, $b > 1$. There are unique integers $d_0, d_1, d_2, \dots \in \mathbb{O}_{b-1}$ such that

$$n = \sum_{i=0}^{\infty} d_i b^i.$$

Definition The sequence $\dots d_2 d_1 d_0$ is called the **base b representation** of n . If b is not clear from context we may write $\dots d_2 d_1 d_0 (b)$ to indicate the base.

Remark Since the sum is finite, there must exist $k \in \mathbb{N}$ such that $d_i = 0$ for all $i > k$. Thus we often abbreviate $d_0 d_1 d_2 \dots$ by $d_0 d_1 \dots d_k$ (i.e. drop the trailing zeros).

Definition Let $b \in \mathbb{N}$ and $b > 1$. Define $Base_b : \mathbb{N} \rightarrow \mathbb{N}$ by

$$\text{Base}_b(x) = \frac{x - (x \bmod b)}{b}.$$

Example If $b = 2$ then

$$\text{Base}_2(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x-1}{2} & \text{if } x \text{ is odd} \end{cases}.$$

Example If $b = 3$ then

$$\text{Base}_3(x) = \begin{cases} \frac{x}{3} & \text{if } x \bmod 3 \text{ is } 0 \\ \frac{x-1}{3} & \text{if } x \bmod 3 \text{ is } 1 \\ \frac{x-2}{3} & \text{if } x \bmod 3 \text{ is } 2 \end{cases}.$$

Theorem (Base Conversion) Let $n \in \mathbb{N}$. If each term in the Base_b -orbit of n is replaced by its value $\bmod b$, the sequence produced will be the base b representation of n (with the digits listed from left to right from least significant to most significant.).

Example What happens if we apply this to a base ten number?

Example Convert 314 to base 2 by this method.

Conway's Fractran

Definition A **Fractran** program consists of a finite sequence of positive rational numbers

$$F = [r_1, r_2, \dots, r_k]$$

with r_k an integer. Each such sequence defines a dynamical system $f_F : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ by

$$f_F(n) = r_i n \quad \text{where } i = \min\{j : r_j n \in \mathbb{Z}\}$$

i.e. f_F multiplies n by the first rational number in the sequence for which the product is an integer.

Remark To “compute” with a Fractran program we simply compute the f_F -orbit of some seed and look for certain terms in the orbit for the answers. For example, we might look at the exponents of the powers of two that appear in the orbit for the Fractran program's output.

Example: (Conway) Let

$$\text{PrimeGame} = \left[\frac{17}{91}, \frac{78}{85}, \frac{19}{51}, \frac{23}{38}, \frac{29}{33}, \frac{77}{29}, \frac{95}{23}, \frac{77}{19}, \frac{1}{17}, \frac{11}{13}, \frac{13}{11}, \frac{15}{2}, \frac{1}{7}, \frac{55}{1} \right]$$

and define $f = f_{\text{PrimeGame}}$. The powers of 2 which occur in the f -orbit of 2 are

$$2^2, 2^3, 2^5, 2^7, 2^{11}, 2^{13}, 2^{17}, 2^{19}, \dots$$

in exactly that order, i.e. **PrimeGame** computes the prime numbers in order. See [LectureExamples.mws](#) to try it.

Example: (Monks) Let

$$\text{CollatzGame} = \left[\frac{1}{11}, \frac{136}{15}, \frac{5}{17}, \frac{4}{5}, \frac{26}{21}, \frac{7}{13}, \frac{1}{7}, \frac{33}{4}, \frac{5}{2}, 7 \right]$$

and define $f = f_{\text{CollatzGame}}$. The powers of 2 which occur in the f -orbit of 2^n are

$$2^n, 2^{T(n)}, 2^{T^2(n)}, 2^{T^3(n)}, \dots$$

in exactly that order, i.e. *CollatzGame* computes the Collatz orbits of natural numbers. See [LectureExamples.mws](#) to try it.

Example: (Conway) Let

$$\text{PolyGame} = \left[\frac{583}{559}, \frac{629}{551}, \frac{437}{527}, \frac{82}{517}, \frac{615}{329}, \frac{371}{129}, \frac{1}{115}, \frac{53}{86}, \frac{43}{53}, \frac{23}{47}, \frac{341}{46}, \frac{41}{43}, \frac{47}{41}, \frac{29}{37}, \frac{37}{31}, \frac{299}{29}, \frac{47}{23}, \frac{161}{15}, \frac{527}{19}, \frac{159}{7}, \frac{1}{17}, \frac{1}{13}, \frac{1}{3} \right]$$

and define $f = f_{\text{PolyGame}}$. Define $f_c(n) = m$ if the f_{PolyGame} -orbit of $c2^{2^n}$ stops at 2^{2^m} and otherwise leave $f_c(n)$ undefined. Then every computable function appears among f_0, f_1, f_2, \dots .

Cellular Automata

Definition An n -dimensional k -state cellular automaton is a discrete dynamical system $G \xrightarrow{c} G$ where G is the set of all functions from \mathbb{Z}^n to \mathbb{I}_k . Each element of \mathbb{Z}^n is called a **cell**. Each element $f \in G$ is called a **state** and its value on a particular cell is called the **state** of that cell. The set G is called the **state space**. To each cell we assign a finite **neighborhood** of cells such that the neighborhood of the translation of a cell is the translation of the neighborhood of the original cell. The map c must be completely determined by a single rule that determines $c(f)(p)$ from the values of $f(q)$ for all q in the neighborhood of p , i.e. the state of a cell after iterating is completely determined by the states of its neighbors before iterating.

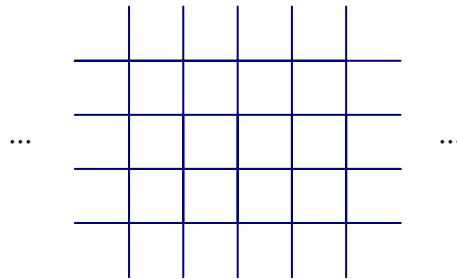
Remark A cellular automaton (CA) is usually represented by a string or grid of squares, where each square in the grid represents a cell, and the states of each cell are represented by colors.

Example A one dimensional CA can be represented as a row of cells.

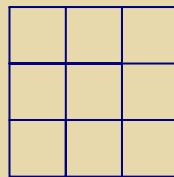


The states of the cells can be represented by coloring the cells different colors corresponding to the current state of that cell. The most common neighborhood to consider for a cell consists of the cell itself, the cell immediately to its left, and the cell immediately to its right (though others are possible).

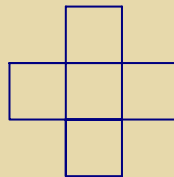
Example A two dimensional CA can be represented as a grid of cells:



Definition There are two commonly used neighborhoods. The **Moore neighborhood** is a square shaped neighborhood centered at the cell. The most commonly used one consists of a cell and all of the cells that share a boundary point in common with that cell:



The **von Neumann neighborhood** consists of a diamond shaped neighborhood centered at the cell. The most commonly used one consists of the cell and its neighbors immediately to the left, right, above, and below the cell:

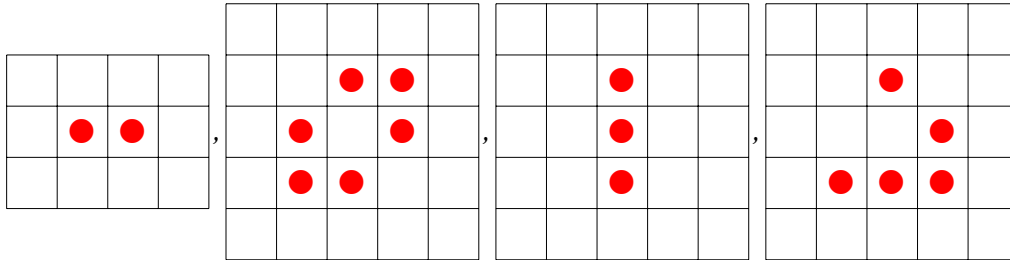


Definition An **outer totalistic** (or simply **totalistic**) CA is one whose rule (map) is completely determined by the sum of the state values of the neighbors of each cell.

Definition **Binary cellular automata** are those with only two states for each cell. In this situation we say that a cell is **alive** if its state is 1 and **dead** if its state is 0.

Example The most famous CA is **Conway's Game of Life**. It is a 2 dimensional binary (=2 state) CA whose rule is given as follows. A dead cell becomes alive if exactly three of its Moore neighbors are alive, and a live cell stays alive if either two or three of its Moore neighbors (other than itself) are alive. Otherwise the cell becomes dead.

Example Compute the orbits of the following seed states in Conway's Game of Life (assuming that all cells other than the red ones shown are dead).



Definition A fixed point of the Game of Life cellular automaton is called a **Still Life**.

Example See Mirek's Celebration or Life32 for interesting examples.

Example In 1999 Paul Rendell implemented a Turing Machine in Life. In 2002, Paul Chapman extended this to construct a universal Turing Machine in Life.

Introduction to Maple

Maple Basics

- Go through the *New User's Tour* in Maple. From the Maple help menu select *New User's Tour* and work through topic numbers 1,2 (briefly), 3, 4, 5, 10, 11.

Maple Language

- Every Maple statement ends with either ';' or ':'. Statements that end with ';' will display their results, statements that end with ':' do not.

- **Assignment statement:**

```
[variable]:= [expression]
```

Examples: `x:=2; f:="Hello world"; a[0]:=x+2;`

- **Conditional Statement:**

```
if [B1] then
  [S1]
elif [B2]
  [S2]
:
elif [Bn]
  [Sn]
else
  [Sn+1]
end if;
```

where B_1, B_2, \dots, B_n are Boolean expressions and S_1, S_2, \dots, S_{n+1} are sequences of statements..

Examples:

```
if 0<x then
  print("It is positive.");
end if;
if x=0 then
```

```

    print("It is zero.");
else
    print("It's not zero.");
end if;
if 0<x then
    print("It is positive.");
elif x=0 then
    print("It is zero.");
else
    print("It is negative.");
end if;

```

● Loop statement

```

for [var] from [start] to [end] by [inc] while [B] do
[S] end do;

```

where [var] is a variable, [start], [end], [inc] are numbers, [B] a Boolean expression and [S] a sequences of statements. Note that any of the numbered pieces:

```

for [var] from [start] to [end] by [inc] while [B] do
  1      2      3      4      5
[S] end do;

```

are optional. The default value for [start] and [inc] is 1. The default for [end] is infinity. The default for [B] is true.

Examples:

```

for i from 0 to 10 by 2 do
    print(i);
end do;
for n while n^2<100 do
    print(n^2);
end do;

```

● Procedures

```

proc([args])
    [S]
end proc;

```

where [args] is a sequence of zero or more variables and [S] is a sequence of statements. To return a value from a procedure use the `return [values]` command in the procedure. A procedure also returns the value of the last statements executed by default. See `?proc` for more details about procedures.

Examples:

```

Square:=proc(n)
    return n^2
end proc;
CollatzFunction:=proc(n)

```

```

    if n mod 2=0 then
        return n/2
    else
        return (3*x+1)/2
    end if;
end proc;

```

To call any of the procedures above, simply use it like a function, e.g.

```

CollatzFunction(3);
Square(7);

```

Maple Data types

- **Expressions**

Maple has all the usual mathematical expressions built in.

Examples:

```

(x^2+3*x)/(x-3);
sqrt(2);
Pi*exp(x)-ln(3)+sin(x);
x mod 6;
etc.

```

- **Functions**

While you can always use a proc to define a function, for simple functions it is easier to use the $x \rightarrow$ (think of this as “x maps to”) notation as indicated in the following example.

Ex:

```

Square:=proc(x) RETURN(x^2) end proc;
Square:=x->x^2;

```

These statements both define the same function.

- **Sequences, Sets, and Lists**

A Sequence: a_0, \dots, a_n ;

A List: $[a_0, \dots, a_n]$;

A Set: $\{a_0, \dots, a_n\}$;

The `seq()` command can construct sequences. To convert a sequence to a list or set enclose it in `[]` or `{}`. To convert a list to a sequence use `op()`. The `nops()` command tells you how many elements are in a list or set. The operators `member`, `union`, and `intersect` work with sets as you would expect.

Examples:

```

S:=1,2,3,4; # S is a sequence
L:=[1,2,3,4]; # L is a list
A:={1,2,3,4}; # A is a set
T:=seq(i,i=1..4); # T is the same sequence as S
M:=[T]; # M is the same list as L
B:={T}; # B is the same set as A

```

```
Z:={op(M)}; # Z is the same set as A and B
n:=nops(L); # n is assigned 4
```

● Subscripts (indexes)

The i^{th} element in a list L is $L[i]$.

Example:

```
L:=[2,4,6,8];
n:=L[3]; # n is assigned 6
```

To access a sublist, use a range:

Example:

```
L:=[2,4,6,8,10];
n:=L[2..4]; # n is assigned [4,6,8]
```

Metric Spaces

Definition A *metric space* is a pair (X, d) where X is a set and $d : X \times X \rightarrow \mathbb{R}$ such that for all $x, y, z \in X$:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0 \Leftrightarrow x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) + d(y, z) \geq d(x, z)$

In this situation, d is called a **metric** (or distance function) on X , and the elements of X are called the **points** in the metric space.

Examples of Metric Spaces

Example $(\mathbb{R}, d_{\text{Euc}})$ is a metric space where $d_{\text{Euc}}(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$.

Notice this is just a special case of the more general theorem:

Theorem $(\mathbb{R}^n, d_{\text{Euc}})$ is a metric space where

$$d_{\text{Euc}}((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

d_{Euc} is called the **Euclidean metric** on \mathbb{R}^n .

Definition Let $d_{\text{Taxi}} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$d_{\text{Taxi}}((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sum_{i=1}^n |x_i - y_i|$$

The map d_{Taxi} is called the **lattice metric**, the **Manhattan metric**, or the **taxicab metric**.

Definition Let $d_{\text{max}} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$d_{\text{max}}((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max\{|x_i - y_i| : i \in \{1, \dots, n\}\}$$

The map d_{max} is called the **maximum metric**.

Definition The set of **2-adic integers**, denoted \mathbb{Z}_2 , is the set of all infinite sequences of 0's and 1's, i.e.

$$\mathbb{Z}_2 = \{(s_0, s_1, \dots) : \forall i \in \mathbb{N}, s_i \in \{0, 1\}\}$$

Definition Let $d_2 : \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow \mathbb{R}$ by

$$d_2((s_0, s_1, \dots), (t_0, t_1, \dots)) = \frac{1}{2^k}$$

where $k = \min\{i \mid s_i \neq t_i\}$ if $(s_0, s_1, \dots) \neq (t_0, t_1, \dots)$ and

$$d_2((s_0, s_1, \dots), (t_0, t_1, \dots)) = 0$$

if $(s_0, s_1, \dots) = (t_0, t_1, \dots)$. The map d_2 is called the **2-adic metric**.

Theorem $(\mathbb{R}^n, d_{\text{Taxi}})$, $(\mathbb{R}^n, d_{\text{max}})$, and (\mathbb{Z}_2, d_2) are metric spaces.

Remark It is a fact that (\mathbb{Z}_2, d_2) cannot be embedded in $(\mathbb{R}^n, d_{\text{Euc}})$ for any n . The 2-adic metric is simple to compute and work with, but the geometry of (\mathbb{Z}_2, d_2) is very strange.

Properties of Metric Spaces

Definition Let (X, d) be a metric space, $\delta \in \mathbb{R}^+$, and $x \in X$. Then

$$B(x; \delta) = \{y \in X \mid d(x, y) < \delta\} \text{ and}$$

$$\bar{B}(x; \delta) = \{y \in X \mid d(x, y) \leq \delta\}$$

$B(x; \delta)$ is called the **open ball of radius δ** centered at x , and $\bar{B}(x; \delta)$ is called the **closed ball of radius δ** centered at x .

Definition Let (X, d) be a metric space and $U \subseteq X$. Then U is **open** if and only if

$$\forall x \in U, \exists \delta \in \mathbb{R}^+ \text{ such that } B(x; \delta) \subseteq U$$

Definition Let (X, d) be a metric space and $U \subseteq X$. Then U is **closed** if and only if $X - U$ is open.

Remark There are sets which are neither open nor closed.

Definition Let (X, d) be a metric space and $U \subseteq X$. Then U is **bounded** if and only if $\exists \delta \in \mathbb{R}^+, \exists x \in X$, such that $U \subseteq B(x; \delta)$.

Definition Let $x_0, x_1, x_2, \dots \in X$ and (X, d) a metric space. Let $x \in X$. Then

$$\lim_{n \rightarrow \infty} x_n = x \Leftrightarrow \forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{N}^+ \text{ such that } \forall n \in \mathbb{N}, n > N \Rightarrow d(x_n, x) < \varepsilon$$

In this case we say that the sequence x_0, x_1, x_2, \dots **converges** to the limit x in (X, d) .

Definition Let $x_0, x_1, x_2, \dots \in X$ and (X, d) a metric space. Then the sequence x_0, x_1, x_2, \dots is called a **Cauchy Sequence** if and only if

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{N}^+ \text{ such that } \forall i > N, \forall j > N, d(x_i, x_j) < \varepsilon$$

i.e., the terms of the sequence get arbitrarily close to each other.

Definition Let (X, d) be a metric space. Then (X, d) is a **complete** metric space if and only if every Cauchy sequence in (X, d) converges to a limit $x \in X$.

Example $(\mathbb{R}, d_{\text{Euc}})$ is complete. In fact, this is one of the axioms that define the real

numbers.

Example $(\mathbb{Q}, d_{\text{Euc}})$ is not complete.

Definition Let (X, d) be a metric space and $U \subseteq X$. U is **compact** if and only if every open cover has a finite subcover, i.e. whenever $\{U_i\}_{i \in I}$ satisfies $U \subseteq \bigcup_{i \in I} U_i$ and $\forall i \in I, U_i$ is open, then $\exists i_1, i_2, \dots, i_k \in I$ for which $U \subseteq U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_k}$.

Theorem (Heine-Borel) Let $A \subseteq (\mathbb{R}^n, d_{\text{Euc}})$. Then A is compact if and only if A is closed and A is bounded.

Continuity

Definition Let (X, d) and (Y, d') be metric spaces and $f : X \rightarrow Y$. Then f is **continuous** with respect to the metrics d and d' if and only if $\forall U \subseteq Y, U$ is open in $(Y, d') \Rightarrow f^{-1}(U)$ is open in (X, d) .

Remark In other words a function between metric spaces is continuous if and only if the inverse image of every open set is open.

Theorem Let (X, d) be a complete metric space, $f : X \rightarrow X$ a continuous map, and x_0, x_1, x_2, \dots a convergent sequence in X with $\lim_{n \rightarrow \infty} x_n = x$. Then $f(x_0), f(x_1), f(x_2), \dots$ is a convergent sequence and $\lim_{n \rightarrow \infty} f(x_n) = f(\lim_{n \rightarrow \infty} x_n)$, i.e., limits commute with continuous maps.

The Metric Space of Shapes

Definition Let $n \in \mathbb{N}^+$. Define

$$K_n = \{A \subseteq \mathbb{R}^n \mid A \text{ is compact}\}.$$

Example K_2 is the set of all compact subsets in the plane.

Definition Let (X, d) be a metric space, $S \subseteq X$, and $\delta \in \mathbb{R}^+$. The **open collar of radius δ** about S is the set $B(S; \delta) = \bigcup_{\alpha \in S} B(\alpha; \delta)$ and the **closed collar of radius δ** about S is the set $\bar{B}(S; \delta) = \bigcup_{\alpha \in S} \bar{B}(\alpha; \delta)$.

Example Sketch $B(\text{MrFace}; \delta)$ for various values of δ .

Definition Let $S \subseteq \mathbb{R}$ and $t \in \mathbb{R} \cup \{\infty\}$. Then $t = \sup(S)$ if and only if $\forall x \in S, x \leq t$ and $\forall u \in \mathbb{R}$, if $\forall x \in S, x \leq u$ then $u \geq t$. $\sup(S)$ is called the **supremum** of S .

Example $\sup(0 \dots 1) = 1$ but $(0 \dots 1)$ has no maximum value.

Definition Let $S \subseteq \mathbb{R}$ and $t \in \mathbb{R} \cup \{-\infty\}$. Then $t = \inf(S)$ if and only if $\forall x \in S, x \geq t$ and $\forall u$, if $\forall x \in S, x \geq u$ then $u \leq t$. $\inf(S)$ is called the **infimum** of S .

Remark If a set is closed and bounded, then $\sup(S) = \max(S)$ and $\inf(S) = \min(S)$.

Definition Let $d_H : K_n \times K_n \rightarrow \mathbb{R}$ by $d_H(S, T) = \inf\{\delta \mid S \subseteq \bar{B}(T; \delta) \text{ and } T \subseteq \bar{B}(S; \delta)\}$. d_H is called the **Hausdorff metric**.

Remark Pages 267-269 of your textbook has a good explanation of this metric.

Example Let $S = \{(x, x) : x \in [0...1]\}$ and $T = \{p : d_{\text{Euc}}(p, (1, 0)) \leq 0.2\}$ be elements of K_2 . Compute $d_H(S, T)$.

Theorem (K_n, d_H) (or the metric space where fractals live) is a complete metric space.

Chaos

Dynamical Systems - Take 2

Definition Let (X, d) be a metric space (or a topological space). Any function $f : X \rightarrow X$ is called a **discrete dynamical system**. To indicate the metric we sometimes write $f : (X, d) \rightarrow (X, d)$

Definition Let $f : (X, d) \rightarrow (X, d)$, $g : (Y, d') \rightarrow (Y, d')$. We say the dynamical systems f, g are **conjugate** if and only if there exists $h : X \rightarrow Y$ such that

1. h is a **homeomorphism** (a continuous bijection with continuous inverse) and
2. $h \circ f = g \circ h$

In this situation h is called a **topological conjugacy** (or simply **conjugacy**) between f and g .

Remark The study of discrete dynamical systems is the study of those properties which are preserved by conjugacy.

Definition Let (X, d) be a metric space, $f : X \rightarrow X$, and let q be a fixed point of f . Then q is an **attracting fixed point** if and only if

$$\exists \delta \in \mathbb{R}^+, \forall x \in B(q; \delta), \lim_{n \rightarrow \infty} f^n(x) = q$$

i.e. if the f -orbit of every point in some ball centered at q converges to q .

Definition Let (X, d) be a metric space, $f : X \rightarrow X$, and let q be a fixed point of f . Then q is a **repelling fixed point** if and only if

$$\exists \delta \in \mathbb{R}^+, \forall x \in B(q; \delta) - \{q\}, \exists N \in \mathbb{N}, f^N(x) \notin B(q; \delta)$$

i.e. if the f -orbit of every point other than q in some ball centered at q contains a point outside the ball.

Definition Let (X, d) be a metric space, $f : X \rightarrow X$, and let q be a periodic point of f with period n . We say the n -cycle containing q is an **attracting cycle** (resp. **repelling cycle**) if and only if q is an attracting (resp. repelling) fixed point of f^n .

Example Classify the fixed points of $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = 3x$.

Example Classify the fixed points of $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = \frac{1}{2}x$.

Theorem Attracting and repelling fixed points are preserved by topological conjugacy.

Graphical Analysis and Time Series Plots

Definition Let $f : \mathbb{R} \rightarrow \mathbb{R}$, and $x \in X$. Then the **time series plot** of the orbit of x is the

graph of the points

$$(0, x), (1, f(x)), (2, f^2(x)), \dots, (k, f^k(x)), \dots$$

Remark Sometimes we connect the points with line segments to make them more visible.

Definition Let $f : \mathbb{R} \rightarrow \mathbb{R}$, and $x \in X$. Then the **graphical analysis** of the orbit of x is the graph consisting of:

- (a) the graph of f
 - (b) the graph of $y = x$
 - (c) a line segment connecting (x_k, x_{k+1}) to (x_{k+1}, x_{k+1}) for each $k \in \mathbb{N}$
 - (d) a line segment connecting (x_{k+1}, x_{k+1}) to (x_{k+1}, x_{k+2}) for each $k \in \mathbb{N}$
- where $x_k = f^k(x)$.

Remark Usually we connect these line segments in order starting from $k = 0$, drawing the segment in part (c) before part (d). It is often customary to add the segment from $(x, 0)$ to (x, x_1) as an initial segment.

Example Draw the graphical analysis for the f -orbit of seeds 0.23 and 0.230001 for

- (a) $f(x) = \frac{1-x}{2}$
- (b) $g(x) = 2x$
- (c) $h(x) = x^2 - 2$

Devaney's Definition

Definition A dynamical system $f : (X, d) \rightarrow (X, d)$ is said to be **transitive** if and only if

$$\forall x, y \in X, \forall \varepsilon \in \mathbb{R}^+, \exists z \in B(x; \varepsilon), \exists k \in \mathbb{N}, f^k(z) \in B(y; \varepsilon)$$

i.e. for any $\varepsilon \in \mathbb{R}^+$ and for any two points in X there is a third point whose orbit passes within ε of both points.

Remark Sometimes this property is called **mixing**.

Definition A dynamical system $f : (X, d) \rightarrow (X, d)$ is said to have **sensitive dependence on initial conditions** if and only if

$$\exists \delta \in \mathbb{R}^+, \forall x \in X, \forall \varepsilon \in \mathbb{R}^+, \exists y \in B(x; \varepsilon) - \{x\}, \exists k \in \mathbb{N}, d(f^k(x), f^k(y)) > \delta$$

i.e. there is a positive constant so that for any x there is a point y arbitrarily close to x such that the orbits of x and y will eventually be separated by at least the constant.

Definition Let (X, d) be a metric space and $A \subseteq X$. Then A is **dense** in X if and only if

$$\forall x \in X, \forall \varepsilon \in \mathbb{R}^+, B(x; \varepsilon) \cap A \neq \emptyset$$

i.e. A is dense if every open ball contains a point of A .

Definition (Devaney) A discrete dynamical system is **chaotic** if and only if

1. it has dense periodic points,
2. it is transitive and

3. it has sensitive dependence on initial conditions.

Remark Chaotic maps give us a model for unpredictable deterministic systems.

Touhey's Definition

Theorem (Touhey 1997) A discrete dynamical system (on an infinite set) is chaotic if and only if every finite collection of open sets shares infinitely many periodic orbits.

Chaotic Maps

The following are examples of chaotic maps:

1. Quadratic maps

For each $c \in \mathbb{C}$ define $Q_c(x) = x^2 + c$.

- $Q_{-2}(x) = x^2 - 2$ is chaotic on $[-2 \dots 2]$.
- $Q_0(z) = z^2$ is chaotic on the unit circle.
- Q_c is chaotic on a fractal set called J_c (more later).
- The Logistic Map $Q(x) = 4x(1-x)$ is chaotic on $[0 \dots 1]$.

2. The Doubling Map

$$D(x) = \begin{cases} 2x & \text{if } x \in [0 \dots 1/2) \\ 2x - 1 & \text{if } x \in [1/2 \dots 1] \end{cases}$$

is chaotic on $[0 \dots 1]$

3. The Tent Map

$$T(x) = 1 - |2x - 1|$$

is chaotic on $[0 \dots 1]$.

4. (J. Joseph) The Extended Collatz Map

$$T(z) = \begin{cases} \frac{1}{2}z & \text{if } z \equiv 0 \\ \frac{3z+1}{2} & \text{if } z \equiv 1 \\ \frac{3z+i}{2} & \text{if } z \equiv i \\ \frac{3z+1+i}{2} & \text{if } z \equiv 1+i \end{cases}$$

is chaotic on $\mathbb{Z}_2[i]$.

The Contraction Mapping Theorem

Definition Let (X, d) be a metric space and $f : X \rightarrow X$. Then f is called a **contraction mapping** if and only if $\exists s \in (0 \dots 1), \forall x, y \in X, d(f(x), f(y)) \leq sd(x, y)$. In this situation s is called a **contraction factor** of f .

Theorem Every contraction mapping is continuous.

Theorem (The Derivative Test) Let $I = (a \dots b) \subseteq \mathbb{R}$ and $f : I \rightarrow I$ differentiable

on I . If $\exists s \in (0 \dots 1)$ such that $\forall x \in I, |f'(x)| \leq s < 1$, then f is a contraction mapping with contraction factor s .

Theorem (The Contraction Mapping Theorem) Let $f : X \rightarrow X$ be a contraction mapping on a complete metric space (X, d) with contraction factor s .

- (1) f has a unique fixed point, q .
- (2) The f -orbit of every element of X converges to q
(i.e., $\forall x \in X, \lim_{n \rightarrow \infty} f^n(x) = q$).
- (3) If x_0, x_1, x_2, \dots is the f -orbit of $x_0 \in X$ then

$$d(x_n, q) \leq \frac{s^n}{1-s} d(x_0, x_1)$$

for all $n \in \mathbb{N}$.

Remark Every contraction map has an attracting fixed point.

Hutchinson Operators

Definition Let w_0, w_1, \dots, w_k be contraction mappings on \mathbb{R}^n with contraction factors c_0, c_1, \dots, c_k respectively and define $W : K_n \rightarrow K_n$ by

$$W(A) = w_0(A) \cup w_1(A) \cup \dots \cup w_k(A).$$

W is called the **Hutchinson operator** associated with w_0, w_1, \dots, w_k and we write $W = \text{Hutch}(w_0, w_1, \dots, w_k)$.

Theorem (Hutchinson) W is a contraction mapping on (K_n, d_H) with contraction factor $c = \max\{c_0, c_1, \dots, c_k\}$.

Definition If W is a Hutchinson operator then the unique fixed point of W is called the **attractor of W** and is denoted F_W .

Remark Applying the three parts of the contraction mapping theorem to W gives us a lot of information about producing fractals with Hutchinson operators.

Iterated Function Systems

Complex Numbers

Definition Let $\mathbb{C} = \mathbb{R}^2$. For each $(x, y) \in \mathbb{C}$ we formally write $(x, y) = x + yi$. This form, $x + yi$, is called the **standard form** of the complex number (x, y) .

Definition Let $x + yi, a + bi \in \mathbb{C}$, then:

1. $\overline{x + yi} = x - yi$. (This is called the **complex conjugate**.)
2. $|x + yi| = \sqrt{x^2 + y^2}$. (This is called the **complex norm**.)
3. $\text{Arg}(x + yi)$ = the angle in $[0 \dots 2\pi)$ of (x, y) in polar form (not defined for $x = y = 0$). (This is called the **Argument** of $x + yi$.)
4. $\text{Re}(x + yi) = x$. (This is called the **real part** of $x + yi$.)
5. $\text{Im}(x + yi) = y$. (This is called the **imaginary part** of $x + yi$.)
6. $(x + yi) + (a + bi) = (x + a) + (y + b)i$. (This is the definition of **addition** in \mathbb{C} .)
7. $(x + yi)(a + bi) = (xa - yb) + (ya + xb)i$. (This is the definition of **multiplication** in \mathbb{C} .)

Notation We can abbreviate $0 + yi$ as yi , $x + 0i$ as x , $x + 1i$ as $x + i$, and $x - 1i$ as $x - i$ with no ambiguity in the above definitions. With this notation $i = (0, 1)$ and $i^2 = -1$. It is easy to verify that the usual laws of addition and multiplication (associative, commutative, distributive, identity, etc.) hold for the complex numbers as well.

Definition Let $\theta \in \mathbb{R}$. Then $e^{i\theta} = \cos \theta + i \sin \theta$

Definition Let $x + yi \in \mathbb{C} - \{0\}$. The **standard polar form** of $x + yi$ is $re^{i\theta}$ where $r = |x + yi|$ and $\theta = \text{Arg}(x + yi)$.

Theorem $e^{i\pi} + 1 = 0$ (The most beautiful theorem in mathematics?)

Theorem Let $\theta, \gamma \in \mathbb{R}$

1. $e^{i\theta}e^{i\gamma} = e^{i(\theta+\gamma)}$.
2. $|e^{i\theta}| = 1$.
3. $e^{i\theta} = e^{i(-\theta)}$.

Theorem Let $z, z_1, z_2 \in \mathbb{C}$. Then:

1. $|z_1 z_2| = |z_1| |z_2|$
2. $d_{\text{Euc}}(z_1, z_2) = |z_2 - z_1|$
3. $\overline{z_1 z_2} = \overline{z_1} \overline{z_2}$ i.e. the conjugate of a product is the product of conjugates.
4. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ i.e. the conjugate of a sum is the sum of the conjugates.
5. $z \overline{z} = |z|^2$
6. $|z| = |\overline{z}|$
7. If $z = re^{i\theta}$ in polar form, then $\overline{z} = re^{i(-\theta)}$

Affine Maps

Definition Define $M_{m,n}(\mathbb{R})$ to be the set of all $m \times n$ matrices with real number entries. Let $A \in M_{m,n}(\mathbb{R})$ and $i \in \mathbb{I}_m, j \in \mathbb{I}_n$. Then $A\langle i, j \rangle$ is the entry in the i^{th} row and j^{th} column of A . If $c \in \mathbb{R}$ then $cA \in M_{m,n}(\mathbb{R})$ and

$$(cA)\langle i, j \rangle = c(A\langle i, j \rangle)$$

If $B \in M_{m,n}(\mathbb{R})$ then $A + B \in M_{m,n}(\mathbb{R})$ and

$$(A + B)\langle i, j \rangle = A\langle i, j \rangle + B\langle i, j \rangle.$$

If $B \in M_{n \times p}(\mathbb{R})$ then $AB \in M_{m,p}(\mathbb{R})$ and for all $i \in \mathbb{I}_m, j \in \mathbb{I}_p$

$$AB\langle i, j \rangle = \sum_{k=1}^n A\langle i, k \rangle B\langle k, j \rangle.$$

Remark We often identify elements of \mathbb{R}^n with elements in $M_{1,n}(\mathbb{R})$ and $M_{n,1}(\mathbb{R})$ when appropriate.

Definition $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called an **affine map** or **affine transformation** if and only if $T(x) = Mx + B$ for some $n \times n$ matrix M and $B \in \mathbb{R}^n$.

Remark We will mostly restrict our attention to affine maps on \mathbb{R}^2 in this course.

Theorem An affine transformation on \mathbb{R}^2 is completely determined by where it maps

3 non-collinear points.

Representations of Affine Maps on \mathbb{R}^2

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an affine map, and let $p \in \mathbb{R}^2$. Let $x, y \in \mathbb{R}$ such that $p = (x, y)$ and let $z = x + yi$. Then there exist a 2×2 real matrix M , $B \in \mathbb{R}^2$, $\alpha, \beta, \gamma \in \mathbb{C}$, and $a, b, c, d, e, f, r, s, \theta, \phi \in \mathbb{R}$ such that:

Form Name	Math Notation
Matrix	$T(p) = Mp + B$
Standard	$T(x, y) = (ax + by + e, cx + dy + f)$
Geometric	$T(x, y) = (r \cos(\theta)x - s \sin(\phi)y + e, r \sin(\theta)x + s \cos(\phi)y + f)$
Complex	$T(z) = \alpha z + \beta \bar{z} + \gamma$

These are expressed in the *chaos* Maple package in the following notation:

Form Name	Maple Notation
Matrix	$\text{affineM}(M, B)$
Standard	$\text{affine}(a, b, c, d, e, f)$
Geometric	$\text{Affine}(r, s, \theta, \phi, e, f)$
Complex	$\text{affineC}(\alpha, \beta, \gamma)$

An IFS (see below) in Maple, is a Maple list of one or more of these affine maps. Note that in the Geometric Form, the Maple program assumes that θ, ϕ are in degrees, not radians.

Each form has its own advantages. We can convert from any form to any other form. It suffices to give the formulas for converting between any form and standard form. Thus, if you are given Matrix form and want to convert to Complex form, first convert to standard and then to Complex.

Theorem *In the above notation:*

Converting Standard form to Matrix form and vice-versa:

$$\text{affine}(a, b, c, d, e, f) = \text{affineM}(M, B) \Leftrightarrow$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} e \\ f \end{pmatrix}$$

Converting Standard form to Complex form and vice-versa:

$$\text{affine}(a, b, c, d, e, f) = \text{affineC}(A + Bi, C + Di, E + Fi) \Leftrightarrow$$

$$\begin{aligned}
 a &= A + C & \text{and} & & A &= \frac{1}{2}(a + d) \\
 b &= D - B & & & B &= \frac{1}{2}(c - b) \\
 c &= B + D & & & C &= \frac{1}{2}(a - d) \\
 d &= A - C & & & D &= \frac{1}{2}(c + b) \\
 e &= E & & & E &= e \\
 f &= F & & & F &= f
 \end{aligned}$$

Converting Standard form to Geometric form and vice-versa:
 $\text{affine}(a, b, c, d, e, f) = \text{Affine}(r, s, \theta, \phi, E, F) \Leftrightarrow$

$$\begin{aligned}
 a &= r \cos(\theta) & \text{and} & & r &= \sqrt{a^2 + c^2} \\
 b &= -s \sin(\phi) & & & s &= \sqrt{b^2 + d^2} \\
 c &= r \sin(\theta) & & & \theta &= \arctan\left(\frac{c}{a}\right) \\
 d &= s \cos(\phi) & & & \phi &= \arctan\left(\frac{d}{b}\right) - 90^\circ \\
 e &= E & & & E &= e \\
 f &= F & & & F &= f
 \end{aligned}$$

Remark The effect of the affine map $\text{Affine}(r, s, \theta, \phi, E, F)$ on a geometric figure is as follows:

r	scales the figure horizontally by a factor of $ r $ (if r is negative, it also reflects the figure across the y -axis)
s	scales the figure vertically by a factor of $ s $ (if s is negative, it also reflects the figure across the x -axis)
θ	rotates horizontal lines by θ degrees CCW about the point where they intersect the y -axis
ϕ	rotates vertical lines by ϕ degrees CCW about the point where they intersect the x -axis
e	translates the figure horizontally by an amount e
f	translates the figure vertically by an amount f

Note that if $\theta = \phi$, then the effect of both numbers combined is to rotate the entire figure about the origin by an angle θ counterclockwise (CCW). Negative angles rotate clockwise (CW) instead of counterclockwise. Also note that $\text{Affine}(r, s, \theta, \phi, e, f)$ always sends the origin, $(0, 0)$, to the point (e, f) .

Contraction Factor for Affine Maps

Theorem Let $\alpha, \beta, \gamma \in \mathbb{C}$ and $c = |\alpha| + |\beta|$. Then the map $T = \text{affineC}(\alpha, \beta, \gamma)$ is a

contraction mapping if and only if $c < 1$. Further if T is a contraction mapping then c is a contraction factor for T .

IFS's

Definition A Hutchinson operator $W = Hutch(w_0, w_1, \dots, w_k)$ such that w_0, w_1, \dots, w_k are all affine maps is called an **iterated function system** or IFS. We write $W = [w_0, w_1, \dots, w_k]$ in this case.

Remark $\{\text{contraction affine mappings}\} \subseteq \{\text{affine mappings}\}$. But there are many contraction mappings that aren't affine. For example, $f(x) = \frac{1}{2} \cos(x)$ on \mathbb{R} , or $f(x) = \sqrt{x}$ on $[1 \dots \infty)$.

Remark A stick figure generator that contains finitely many directed segments whose associated affine maps are contraction maps (and no ordinary segments) is an example of a IFS. The attractor of the IFS can be obtained by iterating the stick figure dynamical system starting with a single directed segment as the seed.

Remark Both HeeBGB's and GB's are IFS's which map the original square onto the appropriate subsquares in the given manner. Coloring a HeeBGB is just producing the attractor of the IFS by the Deterministic Method (see below) where we color the background, not the image itself.

The Deterministic Method

Algorithm To draw the attractor of an IFS, W , simply compute the terms in the W -orbit of any seed in K_n until the image is as close to the attractor as you desire.

Remark By the contraction mapping theorem, the IFS W has an attractor, no matter what shape we start with for a seed, its orbit will converge to the attractor, and we can compute the number of iterations required to obtain an image that is within any desired accuracy of the attractor.

Guess My IFS

Remark Let $W = [w_0, w_1, \dots, w_k]$ be an IFS. By the contraction mapping theorem, F_W is the unique fixed point of W , i.e. F_W is the only element of K_n such that

$$W(F_W) = F_W.$$

By the definition of W this means that F_W is the unique solution A in K_n to the equation:

$$A = w_0(A) \cup w_1(A) \cup \dots \cup w_k(A).$$

In particular,

$$F_W = w_0(F_W) \cup w_1(F_W) \cup \dots \cup w_k(F_W)$$

so that the attractor is a union of finitely many affine images of itself (each of which must therefore be a finite union of strictly smaller affine images of itself, and so on ad infinitum).

Thus, given the attractor of an IFS, we can determine an IFS that produces it (not unique!) by identifying a finite number of strictly smaller affine images if the

attractor whose union is the entire shape.

Address My IFS

Definition Let $n \geq 2$. Define Σ_n to be the set of all infinite sequences whose terms are in $\{0, 1, 2, \dots, n-1\}$, i.e.

$$\Sigma_n = \{s_0, s_1, s_2, \dots \mid s_i \in \{0, 1, 2, \dots, n-1\}\}$$

Σ_n is called the **sequence space on n letters**.

Example Σ_2 is the set of 2-adic integers \mathbb{Z}_2 .

Definition Let $W = [w_0, w_1, \dots, w_{n-1}]$ be an IFS and let F_w be the attractor of W . Define the **address map** $\Phi : \Sigma_n \rightarrow F_w$ by

$$\Phi(s_0 s_1 s_2 \dots) = \bigcap_{i=0}^{\infty} w_{s_0} \circ w_{s_1} \circ \dots \circ w_{s_i}(F_w)$$

Theorem Let $W = [w_0, w_1, \dots, w_{n-1}]$ be an IFS.

1. $\Phi(s)$ is a single point in F_w for any $s = s_0 s_1 s_2 \dots \in \Sigma_n$.
2. Φ is onto.
3. $\lim_{i \rightarrow \infty} w_{s_0} \circ w_{s_1} \circ \dots \circ w_{s_i}(F_w) = \{\Phi(s)\}$ in (K_n, d_H)
4. $\lim_{i \rightarrow \infty} w_{s_0} \circ w_{s_1} \circ \dots \circ w_{s_i}(x) = \Phi(s)$ for any $x \in \mathbb{R}^m$ in $(\mathbb{R}^m, d_{\text{Euc}})$

Definition s is called an **address** of the point $\Phi(s)$ in the attractor.

Definition Let W be an IFS and F_w its attractor. Then W is said to be **totally disconnected** \Leftrightarrow every point in F_w has a unique address, i.e. Φ is bijective.

How big is the Attractor?

Definition Let $S \in K_2$ be a compact set. Then the **diameter** of S is the real number

$$\text{diam}(S) = \sup\{d_{\text{Euc}}(x, y) \mid x, y \in S\}.$$

Example The diameter of a circle is a special case of this definition.

Remark You will prove the following for homework: Let $W = [w_0, \dots, w_{n-1}]$ be an IFS and c_0, \dots, c_{n-1} the contraction factors of w_0, \dots, w_{n-1} respectively. Let $a, b \in \mathbb{R}^k$ and $c = \max\{c_0, \dots, c_{n-1}\}$. Let $a = \Phi(t_1, \dots, t_m, t_{m+1}, \dots)$ and $b = \Phi(t_1, \dots, t_m, t'_{m+1}, \dots)$ then

$$d_{\text{Euc}}(a, b) \leq c^m \text{diam}(F_w).$$

i.e. if two points have addresses that agree in the first m digits, then they will be no further than $c^m \text{diam}(F_w)$ apart.

Theorem (Monks) Let $W = [w_0, \dots, w_n]$ be an IFS, c_0, \dots, c_n the contraction factors of w_0, \dots, w_n respectively, and q_0, \dots, q_n the fixed points of w_0, \dots, w_n respectively. Define $c = \max\{c_0, \dots, c_n\}$ and $r = \max\{d(q_i, q_j) : i, j \in \mathbb{O}_n\}$. Then for any $a \in F_w$ and any $i \in \mathbb{O}_n$,

$$d_{\text{Euc}}(a, q_i) \leq \frac{1}{1-c} r.$$

Corollary $F_W \subseteq \bigcap_{i=0}^n \bar{B}(q_i; \frac{r}{1-c})$

The Shift Map

Definition Let $W = [w_0, \dots, w_{n-1}]$ be a totally disconnected IFS. Define $\sigma : F_W \rightarrow F_W$ by $\sigma(\Phi(s_0s_1s_2\dots)) = \Phi(s_1s_2s_3\dots)$. Then σ is called the **shift map** on F_W .

Theorem A shift map is chaotic!

The Address Method

Algorithm To draw the attractor of an IFS, $W = [w_0, \dots, w_n]$, start with a fixed point, q , of one of the maps w_0, \dots, w_n . Use the preceding theorem and remark to determine m such that any two points with addresses that agree on the first m digits will be less than a pixel width apart. For each finite sequence t_1, \dots, t_m in \mathbb{O}_n^m plot the point

$$w_{t_1} \circ w_{t_2} \circ \dots \circ w_{t_m}(q).$$

Remark There are $(n+1)^m$ sequences in \mathbb{O}_n^m so that sometimes this method may be limited by the number of points you can compute and plot.

The Random Iteration Method

Remark If we choose t_1, t_2, t_3, \dots at random from \mathbb{O}_n , then it is very likely that every finite sequence of any given length will eventually occur as a subsequence of our choices.

Algorithm To draw the attractor of an IFS, $W = [w_0, \dots, w_n]$. Start with a fixed point, q , of one of the maps w_0, \dots, w_n . This is your current point.

1. Choose a random number i from \mathbb{O}_n and plot w_i of the current point. This becomes the new current point.
2. Iterate!

Remark You can actually start with any point you like, not necessarily a fixed point, but starting with the fixed point guarantees that all points you plot will be in the attractor, not just near to it after a sufficient number of iterations.

Applications

Fractal Randomness Testing

Algorithm Given a sequence of values s whose terms are in \mathbb{O}_3 , draw the attractor of $\text{HeeBGB}(\text{Up}, \text{Up}, \text{Up}, \text{Up})$ by the random iteration method, using s as the source of the "random" numbers.

Remark Since the attractor of $\text{HeeBGB}(\text{Up}, \text{Up}, \text{Up}, \text{Up})$ is the filled-in unit square, if the sequence is missing any addresses there will be holes in the attractor where the points having those addresses would be.

Fractal Curves

Definition Let b be an integer greater than 1. Then the **Base b Ruler IFS** is

$W = [w_0, w_1, \dots, w_{b-1}]$ where $w_i : \mathbb{R} \rightarrow \mathbb{R}$ by

$$w_i(x) = \frac{1}{b}x + \frac{i}{b}$$

for $i \in \mathbb{O}_{b-1}$.

Theorem The attractor of any Base b Ruler IFS is $[0 \dots 1]$. For each $t \in [0 \dots 1]$, $t = 0.t_1t_2t_3 \dots_{(b)}$ if and only if $t = \Phi(t_1t_2t_3 \dots)$, i.e. the digits in the base b representation of t and an address of t are the same sequence.

Example Determine the points in the Middle Thirds Cantor Set.

Example Determine the coordinates of all points in the right Sierpinski triangle.

Theorem Let $W = [w_0, \dots, w_{n-1}]$ be an IFS and define

$$f_\Phi(0.t_1t_2t_3 \dots_{(n)}) = \Phi(t_1t_2t_3 \dots)$$

Then f_Φ will be a function from $[0..1] \rightarrow F_w$ if and only if $\Phi(s_1s_2 \dots) = \Phi(t_1t_2 \dots)$ whenever $0.s_1s_2s_3 \dots_{(n)} = 0.t_1t_2t_3 \dots_{(n)}$.

Theorem If f_Φ is a function, then it is continuous.

Theorem (Barnsley): Let $W = [w_0, \dots, w_{n-1}]$ be an IFS and F_w its attractor. If there exist distinct points $\{(s_i, t_i) \in F_w \mid i \in \{0 \dots n\}\}$ such that for $i \in \mathbb{O}_{n-1}$

- $w_i(s_0, t_0) = (s_i, t_i)$

and

- $w_i(s_n, t_n) = (s_{i+1}, t_{i+1})$

then the map f_Φ is a continuous map from $f : [0..1] \rightarrow F_w$.

Example See Lecture Examples Maple worksheet.

Fractal Interpolation

Definition A set of **data** is a collection $\{(x_i, y_i) \in \mathbb{R}^2 \mid i \in \{0, 1, \dots, n\}\}$ and $x_0 < x_1 < \dots < x_n$. An **interpolation function** for a given set of data is a continuous map $f : [x_0 \dots x_n] \rightarrow \mathbb{R}$ such that $f(x_i) = y_i$ for all $i \in \{0, 1, \dots, n\}$, i.e its graph passes through all of the data points.

Example Linear interpolation, cubic spline, polynomial, etc.

Definition Let $(x_0, y_0), \dots, (x_n, y_n)$ be a set of data with $n > 2$. Then a **Barnsley Interpolation Function** is an IFS $W = [w_0, w_2, \dots, w_{n-1}]$ such that for all $i \in \mathbb{O}_{n-1}$,

- $w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ c_i & d_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix}$
- $w_i \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$ and $w_i \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix}$

Remark Simply put, we connect the data points with the chins of Mr Face, keeping

the sides of his head vertical. The seed is the cousin of Mr Face whose chin connects the first and last data points. We choose the d_i 's with $|d_i| < 1$ to vary the fractal dimension (ruggedness) of the interpolation graph. Choosing $|d_i| < 1$ guarantees that our function is a contraction mapping.

Theorem Let W be the IFS in the previous definition. Then for each $i \in \mathbb{O}_{n-1}$,

$$w_i = \text{affine}\left(\frac{x_{i+1} - x_i}{x_n - x_0}, 0, \frac{y_{i+1} - y_i}{x_n - x_0} - d_i \frac{y_n - y_0}{x_n - x_0}, d_i, \frac{x_n x_i - x_0 x_{i+1}}{x_n - x_0}, \frac{x_n y_i - x_0 y_{i+1}}{x_n - x_0} - d_i \frac{x_n y_0 - x_0 y_n}{x_n - x_0}\right)$$

Furthermore, F_W is the graph of an interpolation function.

Dimension

Topological Dimension

Topological Background

Definition Let (X, d) and (Y, d') be metric spaces. Then (X, d) is said to be **homeomorphic** to (Y, d') if $\exists f : X \rightarrow Y$ such that f is bijective, continuous, and its inverse is continuous. In this case, f is said to be a **homeomorphism**.

Definition Any property of a metric space which is preserved by homeomorphisms is called a **topological invariant**, i.e. if P is a topological invariant, and if (X, d) and (Y, d') are homeomorphic, then $P(X, d) \Leftrightarrow P(Y, d')$.

Definition Let (X, d) be a metric space and $U \subseteq X$. Then the **interior** of U is $U^\circ = \{x \in U \mid \exists \delta > 0, B(x; \delta) \subseteq U\}$. Also, the **boundary** of U is $\partial U = X - U^\circ - (X - U)^\circ$, i.e. we take away the interior of the set and the interior of the complement to get the boundary.

Theorem U° is open for any set.

Theorem A set S is open $\Leftrightarrow S = S^\circ$.

Topological Dimension

Definition Let $U \subseteq \mathbb{R}^m$. Define the **topological dimension** of U , denoted $\dim_T(U)$, to be the integer given by

- $\dim_T(\emptyset) = -1$ and \emptyset is the only subset A of \mathbb{R}^m for which $\dim_T(A) = -1$.
- $\dim_T(U) \leq n$ if and only if for any $x \in U$ and any open set $W \subseteq U$ containing x , there exists an open set V with $x \in V \subseteq W$ such that the topological dimension $\dim_T(\partial V) \leq n - 1$. [Note: V, W must be open in U as a metric subspace of \mathbb{R}^m , but not necessarily open as a subset of \mathbb{R}^m . Similarly, ∂V refers to the boundary of V in U , not in \mathbb{R}^m .]

Then $\dim_T(U) = n$ if and only if $\dim_T(U) \leq n$ but $\dim_T(U) \not\leq n - 1$.

Remark $\dim_T(U)$ is always an integer by definition.

Hausdorff Dimension

Example Consider the following

Figure:	a point	unit segment	Sierpinski triangle	unit square	unit cube
# of pts:	1	∞	∞	∞	∞
Length:	0	1	∞	∞	∞
Area:	0	0	0	1	∞
Volume:	0	0	0	0	1

Notice that none of these standard measures give a nontrivial value for the Sierpinski triangle.

Definition Let $\varepsilon \in \mathbb{R}^+$, $0 \leq p < \infty$, and $A \subseteq \mathbb{R}^2$ a bounded subset of the plane. Then

$$M_p(A; \varepsilon) = \inf \left\{ \sum_{i=1}^{\infty} \text{diam}(A_i)^p : A = \bigcup_{i=1}^{\infty} A_i \text{ and } \forall i, \text{diam}(A_i) < \varepsilon \right\}$$

and

$$M_p(A) = \sup \{ M_p(A; \varepsilon) : \varepsilon \in \mathbb{R}^+ \}.$$

$M_p(A)$ is called the Hausdorff p -measure of A .

Remark $M_p(A)$ can equal ∞ . $M_p(A; \varepsilon)$ is obviously a nonincreasing function of ε , so $M_p(A)$ is the same as $\lim_{\varepsilon \rightarrow 0} M_p(A; \varepsilon)$.

Theorem For each A , $\exists d \in \mathbb{R}$ such that $M_p(A) = \infty$ for $p < d$ and $M_p(A) = 0$ for $p > d$.

Definition The number d in the previous theorem is called the **Hausdorff dimension** of A , and written $d = \dim_H(A)$.

Similarity and Congruence

Definition Let (X, d) be a metric space. A **similitude** (or **similarity map**) is a surjective map $f : X \rightarrow X$ such that

$$\exists c \in \mathbb{R}^+, \forall x, y \in X, d(f(x), f(y)) = cd(x, y).$$

In this case c is called the **similarity factor** (or **scaling factor** or **ratio of similarity**).

Definition Let (X, d) be a metric space and $A, B \subseteq X$. Then A is **similar** to B if $B = f(A)$ for some similitude f .

Theorem $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a similitude with scaling factor c if and only if $f = \text{Affine}(r, s, \theta, \phi, e, f)$ with $|r| = |s| = c$ and $\theta = \phi + \pi k$ for some $k \in \mathbb{Z}$.

Theorem If f is a similitude with scaling factor c , then f is bijective and f^{-1} is a similitude with scaling factor $1/c$.

Definition Let (X, d) be a metric space. Then $f : X \rightarrow X$ is called an **isometry** if and only if f is a similitude with scaling factor equal to 1.

Remark In other words f is an isometry if

1. f is surjective and
2. $\forall x, y \in X, d(f(x), f(y)) = d(x, y)$, i.e. f preserves all distances.

Definition Let (X, d) be a metric space and $A, B \subseteq X$. Then A is **congruent** to B if $B = f(A)$ for some isometry f .

Corollary If f is an isometry, then f is bijective and f^{-1} is an isometry.

Theorem $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is an isometry iff $f = \text{Affine}(r, s, \theta, \phi, e, f)$ with $|r| = |s| = 1$ and $\theta = \phi + \pi k$ for some $k \in \mathbb{Z}$.

Self-similarity and Dimension

Definition Let $W = [w_0, w_1, \dots, w_{n-1}]$ be an IFS and F_w its attractor. Then W is said to be **just touching** \Leftrightarrow it is not totally disconnected and there exists an open set $U \in \mathbb{R}^2$ such that

1. $W(U) \subseteq U$ and
2. $w_i(U) \cap w_j(U) = \emptyset$ for all $i, j \in [0, 1, \dots, n-1]$ for $i \neq j$.

Definition Let W be an IFS. Then W is said to be **overlapping** \Leftrightarrow it is not totally disconnected and not just touching.

Example The middle thirds Cantor set is totally disconnected.

Example The Sierpinski triangle is just touching.

Example A trivial example of an overlapping IFS is an IFS containing the same map twice.

Definition Let $W = [w_0, w_1, \dots, w_{n-1}]$ be an IFS. F_w is said to be **self-similar** $\Leftrightarrow F_w$ is not overlapping and each w_i is a similarity. F_w is **strictly self-similar** if and only if the similarity factors are all equal.

Definition If F_w is a strictly self-similar attractor of a non-overlapping IFS $W = [w_0, w_1, \dots, w_{n-1}]$, then the **similarity dimension** of F_w is defined to be the unique number d such that $N = (\frac{1}{r})^d$ where r is the similarity factor (= contraction factor for the affine maps). We write $\dim_S(F_w) = d$ in this case.

Remark If we solve the defining equation for d we obtain

$$\dim_S(F_w) = \frac{\ln N}{\ln(\frac{1}{r})}$$

Theorem For attractors of non-overlapping IFS's, $\dim_S(F_w) = \dim_H(F_w)$.

Theorem (Moran) Let F_w be the attractor of a non-overlapping IFS, $W = [w_0, w_1, \dots, w_{n-1}]$ such that each w_i is a similitude with similarity factor c_i respectively. Then $\dim_H(F_w)$ is the unique number d such that

$$c_0^d + c_1^d + \dots + c_n^d = 1.$$

If W is overlapping then $\dim_H(F_w) \leq d$.

Theorem Let W be the Barnsley interpolation function IFS given in the definition. If $\sum_{k=0}^{n-1} |d_k| > 1$ and the interpolation points do not lie on a straight line, then the fractal dimension of F_w is the unique real number D such that

$$\sum_{k=0}^{n-1} |d_k| a_k^{D-1} = 1.$$

Approximating the Hausdorff Dimension

Grid Dimension

Algorithm To estimate the Grid dimension of a shape:

1. Cover the shape with grids of size l_1, l_2, \dots, l_k
2. For each grid, count the number N_i of grid boxes whose interior intersects the shape
3. Plot $\ln(N_i)$ vs. $\ln(\frac{1}{l_i})$ and compute the least squares linear regression line through the points $\{(\ln(\frac{1}{l_1}), \ln(N_1)), \dots, (\ln(\frac{1}{l_k}), \ln(N_k))\}$
4. The slope is an estimate of the Grid dimension (and is also an estimate of the Hausdorff dimension).

Remark The slope is independent of the units used.

Complex Fractals

Julia Sets

Definition If $z \in \mathbb{C}$ and $z = re^{i\theta}$ where $r = |z|$ and $\theta \in [0 \dots 2\pi]$ then $\sqrt{z} = z^{\frac{1}{2}}$ is $(re^{i\theta})^{\frac{1}{2}} = r^{\frac{1}{2}} e^{\frac{i\theta}{2}}$. \sqrt{z} is the **principal square root** of a complex number.

Example Let $c \in \mathbb{C}$. Let $w_0(z) = \sqrt{z-c}$, $w_1(z) = -\sqrt{z-c}$, and $W = [w_0, w_1]$. Since w_0, w_1 are not affine, W is not an IFS.

Is W a Hutchinson operator? Not quite. But it behaves like a Hutchinson Operator in the sense that $\exists \gamma \subseteq \mathbb{R}^2$ such that for any $A \in K_2$, $W^n(A \cap (\mathbb{R}^2 - \gamma))$ converges to a unique set F_w such that $W(F_w) = F_w$.

Definition That unique set F_w is called the **Julia Set** associated with c and is denoted J_c .

Definition Let $c \in \mathbb{C}$. Let $Q_c(z) = z^2 + c$. The **filled in Julia Set**, K_c , is $K_c = \{z : \text{the } Q_c \text{ orbit of } z \text{ is bounded}\}$

Facts about Julia Sets:

1. $J_c = \partial K_c$
2. $J_c = K_c$ if J_c is totally disconnected
3. All Julia Sets fit into a closed ball of radius two centered at the origin, i.e. if $|Q_c^n(z)| > 2$ for any n then $z \notin K_c$.

Remark For each $c \in \mathbb{C}$, there is a Julia Set J_c .

Theorem J_c is connected if and only if the Q_c orbit of 0 is bounded, i.e. if and only if 0 is in the filled in Julia Set K_c .

The Mandelbrot Set

Definition The **Mandelbrot Set**, M , is the set of all $c \in \mathbb{C}$ such that J_c is connected.

Because of the previous theorem, we write

$$M = \{c \in \mathbb{C} : \text{the } Q_c\text{-orbit of } 0 \text{ is bounded}\}$$

Remark While there are infinitely many Julia Sets, there is only one Mandelbrot Set.

Facts about the Mandelbrot Set:

1. It is symmetric with respect to the x-axis
2. It is connected
3. Every open set containing any point on the boundary of M contains infinitely many “baby M ’s”. Note: The babies are not similar to M .
4. K_c “looks like” M near c .
5. Every bulb, B , has the property that $\exists n \in \mathbb{N}^+$ such that $\forall c \in B$, the Q_c orbit of 0 converges to an n -cycle. The integer n is called the **period of the bulb**.

The Escape Time Algorithm

Algorithm Assign to each screen pixel a representative complex number and color the pixel based on the number of iterations it required for a term in the Q_c -orbit to have absolute value greater than 2 (or some particular color if no such term is obtained after a predetermined number of iterations).

Proofs

The Power Theorem

Theorem Let $f : X \rightarrow X$. For any $k, n \in \mathbb{N}$,

$$f^{k+n} = f^k \circ f^n.$$

and

$$f^{kn} = \underbrace{f^n \circ f^n \circ \dots \circ f^n}_{k \text{ terms}}.$$

(where “ 0 terms” means the identity map).

Pf.

1. ★We proceed by induction on k ★

2. Let $f : X \rightarrow X$

3. Let $n \in \mathbb{N}$

★Basis step ★

4. $f^{0+n} = f^n$

arithmetic

5. $= id_X \circ f^n$

(homework)

6. $= f^0 \circ f^n$

def f^0

7. $\forall n \in \mathbb{N}, f^{0+n} = f^0 \circ f^n$

$\forall +$

★ inductive hypothesis ★

8. Let $k \in \mathbb{N}$

9. Assume $f^{k+n} = f^k \circ f^n$

10. $f^{k+1+n} = f^{1+k+n}$

arithmetic

11. $= f \circ f^{k+n}$

def f^k

- 12. $= f \circ (f^k \circ f^n)$ substitution
- 13. $= (f \circ f^k) \circ f^n$ ◦ is associative (homework)
- 14. $= f^{1+k} \circ f^n$ def f^k
- 15. \leftarrow
- 16. $f^{k+n} = f^k \circ f^n \Rightarrow f^{(k+1)+n} = f^{k+1} \circ f^n$ $\Rightarrow +$
- 17. $\forall k \in \mathbb{N}, \forall n \in \mathbb{N}, f^{k+n} = f^k \circ f^n \Rightarrow f^{(k+1)+n} = f^{k+1} \circ f^n$ $\forall +$ (twice)
- 18. $\forall k \in \mathbb{N}, \forall n \in \mathbb{N}, f^{k+n} = f^k \circ f^n$ induction

★ the proof of the second equation is similar ★

QED

Lemma Let $f : X \rightarrow X, x \in X,$ and $n \in \mathbb{N}^+.$
 x has minimum period $n \Rightarrow \#O_f(x) = n$

Pf:

- 1. Let $f : X \rightarrow X, x \in X,$ and $n \in \mathbb{N}^+.$
 (\Rightarrow)
- 2. Assume x has minimum period n
- 3. $f^n(x) = x$ and $\forall k \in \mathbb{I}_{n-1}, f^k(x) \neq x$ def min period
- 4. Define $S = \{x, f(x), \dots, f^{n-1}(x)\}$
 ★Let's first show that $O_f(x) = S$ ★
 ★First we show $O_f(x) \subseteq S$ ★
- 5. Let $y \in O_f(x)$
- 6. $y = f^j(x)$ for some $j \in \mathbb{N}$ def O
- 7. $j = nq + r$ and $0 \leq r < n$ for some $q, r \in \mathbb{N}$ division algorithm
- 8. $y = f^j(x)$ line 6
- 9. $= f^{nq+r}(x)$ substitution
- 10. $= f^r(f^{nq}(x))$ Power Thm
- 11. $= f^r(f^n(f^n(\dots f^n(x))))$ (with q f^n 's) Power Thm
- 12. $= f^r(x)$ substitution (q times)
- 13. $\in S$ def S
- 14. $O_f(x) \subseteq S$ def subset
 ★ Now we show $S \subseteq O_f(x)$ ★
- 15. Let $z \in S$
- 16. $z = f^i(x)$ for some $i \in \mathbb{O}_{n-1}$ def S
- 17. $\in O_f(x)$ def O
- 18. $S \subseteq O_f(x)$ def subset
- 19. $O_f(x) = S$ def set =

★ Now let's show that the n elements of S are distinct. Assume two powers are equal and show that only happens if they are the same power ★

- 20. Let $a, b \in \mathbb{O}_{n-1}$ and $a \geq b$
- 21. Assume $f^a(x) = f^b(x)$

22. $f^{a-b}(x) = f^{a-b}(f^n(x))$ subst
 23. $= f^{n+a-b}(x)$ Power Thm
 24. $= f^{n-b}(f^a(x))$ Power Thm
 25. $= f^{n-b}(f^b(x))$ subst
 26. $= f^n(x)$ Power Thm
 27. $= x$ subst
 28. $a - b \notin \mathbb{I}_{n-1}$ by lines 3,22,27
 29. $a - b \in \mathbb{O}_{n-1}$ since $a, b \in \mathbb{O}_{n-1}$
 30. $a - b \in \mathbb{O}_{n-1} - \mathbb{I}_{n-1}$ def set -
 31. $= \{0\}$ def \mathbb{O}_{n-1} and \mathbb{I}_{n-1}
 32. $a - b = 0$ def set notation
 33. $a = b$ arithmetic
 34. \leftarrow
 35. the elements of S are distinct def distinct
 36. $\#S = n$ def #
 37. $\#\mathcal{O}_f(x) = n$ subst
 38. \leftarrow
 39. x has minimum period $n \Rightarrow \#\mathcal{O}_f(x) = n \Rightarrow +$

QED

Change of Basis

Theorem Let $n \in \mathbb{N}$. If each term in the $Base_b$ -orbit of n is replaced by its value Mod b , the sequence produced will be the base b representation of n (with the least significant digit on the left).

Pf:

- ★ We proceed by induction on n ★
- Let $b \in \mathbb{N}, b > 1$.
- Define $Orb = Orb_{Base_b}$
★ basis step ★
- $Base_b(1) = \frac{1 - (1 \text{ Mod } b)}{b} = \frac{1-1}{b} = 0$ by def of $Base_b$
- $Base_b(1) = \frac{0 - (0 \text{ Mod } b)}{b} = \frac{0-0}{b} = 0$ by def of $Base_b$
- $Orb(1) = 1, 0, 0, 0, \dots$ by definition of orbit
- For any sequence s , define $s \text{ Mod } b$ to be the sequence whose i^{th} term is $s_i \text{ Mod } b$.
- $1 = 100000\dots_{(b)} = Orb(1) \text{ Mod } b$
★ inductive step
- Let $n \in \mathbb{N}^+$
- Assume $Orb(m) \text{ Mod } b = m_0 m_1 m_2 \dots_{(b)} = m$ for all $m < n$
- $n = (n_0 n_1 \dots n_k)_{(b)}$ for some $n_i \in \mathbb{O}_{b-1}$ and some $k \in \mathbb{N}$ by the representation theorem
★ Let's calculate $Base_b(n)$

$$\begin{aligned}
12. \quad & \text{Base}_b(n) = \frac{n - (n \bmod b)}{b} \\
13. \quad & = \frac{(n_0 n_1 \dots n_k)_{(b)} - n_0}{b} \\
14. \quad & = \frac{(n_0 n_1 \dots n_k)_{(b)} - n_0}{b} \\
15. \quad & = \frac{(0 n_1 n_2 \dots n_k)_{(b)}}{b} \\
16. \quad & = (n_1 n_2 \dots n_k)_{(b)} \\
17. \quad & \leq n
\end{aligned}$$

★ Since its less than n the assumption holds for $\text{Base}_b(n)$ ★

$$\begin{aligned}
18. \quad & \text{Orb}(\text{Base}_b(n)) \bmod b = n_1 n_2 \dots n_k \bar{0} \\
19. \quad & \text{Orb}(n) = n, \text{Orb}(\text{Base}_b(n)) \text{ by definition of orbit} \\
20. \quad & \text{Orb}(n) \bmod b = n \bmod b, \text{Orb}(\text{Base}_b(n)) \bmod b \\
21. \quad & = n_0, n_1 n_2 \dots n_k \bar{0} \\
22. \quad & = n_0, n_1 n_2 \dots n_k \bar{0}_{(b)} \\
23. \quad & = n \\
24. \quad & \leftarrow \\
25. \quad & \forall n \in \mathbb{N}, \text{Orb}(n) \bmod b = n_0 n_1 n_2 \dots_{(b)} = n
\end{aligned}$$

QED

Triangle Inequality and the Euclidean Metric

Lemma (Euclidean Triangle Inequality) Let $x, y, z \in \mathbb{R}^n$. Then

$$d_{\text{Euc}}(x, z) \leq d_{\text{Euc}}(x, y) + d_{\text{Euc}}(y, z).$$

Pf:

1. Let $x, y, z \in \mathbb{R}^n$.
2. $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n), z = (z_1, \dots, z_n)$ for some $x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n \in \mathbb{R}$ by def of \mathbb{R}^n
3. Define $a = d_{\text{Euc}}(x, z), b = d_{\text{Euc}}(x, y), c = d_{\text{Euc}}(y, z)$
 ★ in this notation we are trying to show that $a \leq b + c$
 ★ we will prove the case where $b \neq 0$ and $c \neq 0$. The other cases are for homework.
4. For $i \in \mathbb{I}_n$ define $a_i = x_i - z_i, b_i = x_i - y_i, c_i = y_i - z_i$
5. $\forall i \in \mathbb{I}_n, a_i = b_i + c_i$
6. $a, b, c \geq 0$
7. $a^2 = (d_{\text{Euc}}(x, z))^2$
8. $= \left(\sqrt{\sum_{i=1}^n (x_i - z_i)^2} \right)^2$
9. $= \sum_{i=1}^n (x_i - z_i)^2$
10. $= \sum_{i=1}^n a_i^2$
 (★ Note that a similar argument shows $b^2 = \sum_{i=1}^n b_i^2$ and $c^2 = \sum_{i=1}^n c_i^2$)
11. $= \sum_{i=1}^n (b_i + c_i)^2$
12. $= \sum_{i=1}^n (b_i^2 + 2b_i c_i + c_i^2)$
13. $= \sum_{i=1}^n b_i^2 + \sum_{i=1}^n (2b_i c_i) + \sum_{i=1}^n c_i^2$

14. $= b^2 + c^2 + \sum_{i=1}^n (2b_i c_i)$
15. $= b^2 + 2bc + c^2 + \sum_{i=1}^n (2b_i c_i) - 2bc$
16. $= (b + c)^2 + \sum_{i=1}^n (2b_i c_i) - 2bc$
17. $= (b + c)^2 + \sum_{i=1}^n (2b_i c_i) - (1 + 1)bc$
18. $= (b + c)^2 + \sum_{i=1}^n (2b_i c_i) - \left(\frac{b^2}{b^2} + \frac{c^2}{c^2}\right)bc$
19. $= (b + c)^2 + \frac{bc}{bc} \sum_{i=1}^n (2b_i c_i) - \left(\frac{1}{b^2} \sum_{i=1}^n b_i^2 + \frac{1}{c^2} \sum_{i=1}^n c_i^2\right)bc$
20. $= (b + c)^2 + \left(\sum_{i=1}^n \frac{2b_i c_i}{bc} - \sum_{i=1}^n \frac{b_i^2}{b^2} - \sum_{i=1}^n \frac{c_i^2}{c^2}\right)bc$
21. $= (b + c)^2 - \left(-\sum_{i=1}^n \frac{2b_i c_i}{bc} + \sum_{i=1}^n \frac{b_i^2}{b^2} + \sum_{i=1}^n \frac{c_i^2}{c^2}\right)bc$
22. $= (b + c)^2 - \sum_{i=1}^n \left(\frac{b_i^2}{b^2} - \frac{2b_i c_i}{bc} + \frac{c_i^2}{c^2}\right)bc$
23. $= (b + c)^2 - \left(\sum_{i=1}^n \left(\frac{b_i}{b} - \frac{c_i}{c}\right)^2\right)bc$
24. $\leq (b + c)^2$
25. $a^2 \leq (b + c)^2$
26. $a \leq b + c$

QED

Theorem $(\mathbb{R}^n, d_{\text{Euc}})$ is a metric space.

Pf:

1. $d_{\text{Euc}} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$
2. Let $x, y, z \in \mathbb{R}^n$
3. $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n), z = (z_1, \dots, z_n)$ for some $x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n \in \mathbb{R}$
4. $d_{\text{Euc}}(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$
5. ≥ 0
6. $d_{\text{Euc}}(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$
7. $= \sqrt{\sum_{i=1}^n (y_i - x_i)^2}$
8. $= d_{\text{Euc}}(y, x)$
9. $d_{\text{Euc}}(x, z) \leq d_{\text{Euc}}(x, y) + d_{\text{Euc}}(y, z)$
10. $d_{\text{Euc}}(x, x) = \sqrt{\sum_{i=1}^n (x_i - x_i)^2}$
11. $= \sqrt{\sum_{i=1}^n 0^2}$
12. $= 0$
13. Assume $d_{\text{Euc}}(x, y) = 0$
14. $\sum_{i=1}^n (x_i - y_i)^2 = \left(\sqrt{\sum_{i=1}^n (x_i - y_i)^2}\right)^2$
15. $= (d_{\text{Euc}}(x, y))^2$
16. $= 0^2$
17. $= 0$

18. $(x_i - y_i)^2 = 0$ for all $i \in \mathbb{I}_n$
19. $x_i - y_i = 0$ for all $i \in \mathbb{I}_n$
20. $x_i = y_i$ for all $i \in \mathbb{I}_n$
21. $x = (x_1, \dots, x_n)$
22. $= (y_1, \dots, y_n)$
23. $= y$
24. $(\mathbb{R}^n, d_{\text{Euc}})$ is a metric space

QED

Contraction Mappings are Continuous

Theorem Every contraction mapping is continuous.

Pf:

1. Let (X, d) be a metric space and $f : X \rightarrow X$ a contraction mapping.
2. f has contraction factor s for some $s \in (0 \dots 1)$
3. Let $U \subseteq X$ be an open set
4. Let $x \in f^{-1}(U)$
5. $f(x) \in U$
6. $B(f(x); \delta) \subseteq U$ for some $\delta \in \mathbb{R}^+$
7. Let $y \in B(x; \delta)$
8. $d(x, y) < \delta$
9. $d(f(x), f(y)) \leq sd(x, y)$
10. $< s\delta$
11. $< \delta$
12. $f(y) \in B(f(x); \delta)$
13. $f(y) \in U$
14. $y \in f^{-1}(U)$
15. $\forall y \in B(x; \delta), y \in f^{-1}(U)$
16. $B(x; \delta) \subseteq f^{-1}(U)$
17. $\forall x \in f^{-1}(U), \exists \delta \in \mathbb{R}^+, B(x; \delta) \subseteq f^{-1}(U)$
18. $f^{-1}(U)$ is open
19. f is continuous

QED

The Derivative and Contraction Maps of \mathbb{R}

Theorem Let $I = (a \dots b) \subseteq \mathbb{R}$ and $f : I \rightarrow \mathbb{R}$ differentiable on I . If $\exists s \in (0 \dots 1)$ such that $\forall x \in I, |f'(x)| \leq s < 1$, then f is a contraction mapping with contraction factor s .

Pf:

1. Let $I = (a \dots b) \subseteq \mathbb{R}$ and $f : I \rightarrow \mathbb{R}$ differentiable on I , and $d = d_{\text{Euc}}$.
2. Assume $\exists s \in (0 \dots 1), \forall x \in I, |f'(x)| \leq s < 1$
3. Let $x, y \in I$
4. $x = y$ or $x \neq y$

★ Case 1 ★

5. Assume $x = y$
 6. $d(f(x), f(y)) = d(f(x), f(x))$
 7. $= 0$
 8. $\leq s \cdot 0$
 9. $= s \cdot d(x, x)$
 10. $= s \cdot d(x, y)$
 11. ←

★ Case 2 ★

12. Assume $x \neq y$
 13. $f'(c) = \frac{f(x)-f(y)}{x-y}$ for some c between x and y
 14. $|f'(c)| \leq s$
 15. $d(f(x), f(y)) = |f(x) - f(y)|$
 16. $= |f(x) - f(y)| \frac{|x-y|}{|x-y|}$
 17. $= \left| \frac{f(x)-f(y)}{x-y} \right| |x-y|$
 18. $= |f'(c)| |x-y|$
 19. $\leq s|x-y|$
 20. $= s \cdot d(x, y)$
 21. ←
 22. $\exists s \in (0 \dots 1), \forall x, y \in I, d(f(x), f(y)) \leq s \cdot d(x, y)$
 23. f is a contraction map with contraction factor s
 24. ←

25. If $\exists s \in (0 \dots 1), \forall x \in I, |f'(x)| \leq s < 1$, then f is a contraction mapping with contraction factor s .

QED

Contraction Mapping Theorem

Theorem (The Contraction Mapping Theorem) Let $f : X \rightarrow X$ be a contraction mapping on a complete metric space (X, d) with contraction factor s .

- (1) f has a unique fixed point, q .
- (2) The f -orbit of every element of X converges to q
 (i.e., $\forall x \in X, \lim_{n \rightarrow \infty} f^n(x) = q$).
- (3) If x_0, x_1, x_2, \dots is the f -orbit of $x_0 \in X$ then

$$d(x_n, q) \leq \frac{s^n}{1-s} d(x_0, x_1)$$

for all $n \in \mathbb{N}$.

Pf:

1. Let $f : X \rightarrow X$ be a contraction mapping on a complete metric space (X, d) with contraction factor $s \in (0 \dots 1)$.
2. Let $x_0 \in X$
3. Define $x_i = f^i(x_0)$ for $i \in \mathbb{N}^+$, i.e. x_0, x_1, x_2, \dots is the f -orbit of x_0

- ★ Goal: We want to prove x_0, x_1, x_2, \dots is a Cauchy sequence ★
4. Define $\alpha = d(x_0, x_1)$
 - ★ Let's first prove that $d(x_i, x_{i+1}) \leq s^i \alpha$ for all i by induction on i ★
 - ★ Base case ★
 5. $d(x_0, x_1) = \alpha \leq s^0 \alpha$
 - ★ inductive step ★
 6. Let $i \in \mathbb{N}^+$
 7. Assume $d(x_i, x_{i+1}) \leq s^i \alpha$
 8. $d(x_{i+1}, x_{i+2}) = d(f(x_i), f(x_{i+1}))$
 9. $\leq s \cdot d(x_i, x_{i+1})$
 10. $\leq s \cdot s^i \alpha$
 11. $= s^{i+1} \alpha$
 12. ←
 13. $\forall i \in \mathbb{N}, d(x_i, x_{i+1}) \leq s^i \alpha$
 14. Let $m, n \in \mathbb{N}$ with $m \leq n$
 15. $m = n$ or $m < n$
 - ★ Case 1 ★
 16. Assume $m = n$
 17. $d(x_m, x_n) = d(x_m, x_m)$
 18. $= 0$
 19. $\leq \frac{s^m}{1-s} \alpha$
 20. ←
 - ★ Case 2 ★
 21. Assume $m < n$
 22. $d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \dots + d(x_{n-1}, x_n)$
 23. $\leq s^m \alpha + s^{m+1} \alpha + \dots + s^{n-1} \alpha$
 24. $= s^m \alpha (1 + s + s^2 + s^3 + \dots + s^{n-1-m})$
 25. $\leq s^m \alpha (1 + s + s^2 + s^3 + \dots)$
 26. $\leq s^m \alpha \frac{1}{1-s}$
 27. $= \frac{s^m}{1-s} \alpha$
 28. ←
 29. $\forall m, n \in \mathbb{N}, d(x_m, x_n) \leq \frac{s^m}{1-s} \alpha$
 30. Let $\varepsilon \in \mathbb{R}^+$
 31. $\lim_{m \rightarrow \infty} \frac{s^m}{1-s} \alpha = 0$
 32. $\exists N \in \mathbb{N}, \forall m \geq N, \frac{s^m}{1-s} \alpha < \varepsilon$
 33. Assume $m, n \geq N$
 34. $d(x_m, x_n) \leq \frac{s^m}{1-s} \alpha$
 35. $< \varepsilon$
 36. ←

37. $\exists N \in \mathbb{N}, \forall m, n \geq N, d(x_m, x_n) < \varepsilon$
38. $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N, d(x_m, x_n) < \varepsilon$
39. x_0, x_1, x_2, \dots is a Cauchy sequence
 ★ we use completeness to get q ★
40. $\lim_{i \rightarrow \infty} x_i = q$ for some $q \in X$ (since (X, d) is complete)
 ★ Now let's show that q is a fixed point ★
41. f is continuous
42. $f(q) = f\left(\lim_{i \rightarrow \infty} x_i\right)$
43. $= \lim_{i \rightarrow \infty} f(x_i)$
44. $= \lim_{i \rightarrow \infty} f(f^i(x_0))$
45. $= \lim_{i \rightarrow \infty} f^{i+1}(x_0)$
46. $= \lim_{i \rightarrow \infty} x_{i+1}$
47. $= \lim_{i \rightarrow \infty} x_i$
48. $= q$
49. q is a fixed point of f
50. the f -orbit of every element of X converges to a fixed point
 ★ Let's show it is unique ★
51. Let $p \in X$
52. Assume p is a fixed point of f
53. $f(p) = p$
54. $d(p, q) = d(f(p), f(q))$
55. $\leq s \cdot d(p, q)$
56. Assume $d(p, q) \neq 0$
57. $1 \leq s$
58. $\rightarrow \leftarrow$
59. \leftarrow
60. $d(p, q) = 0$
61. $p = q$
62. \leftarrow
63. q is a unique fixed point of f
 ★ Part (2) now follows immediately from lines 49 and 63 ★
64. $\forall x \in X, \lim_{i \rightarrow \infty} f^i(x) = q$
 ★ Now for the final part, part (3), we will prove it by contradiction ★
65. Assume $d(x_n, q) > \frac{s^n}{1-s} d(x_0, x_1)$ for some $x_0 \in X$ and some $n \in \mathbb{N}$
66. $d(x_n, q) - \frac{s^n}{1-s} d(x_0, x_1) > 0$
67. Define $\varepsilon_1 = d(x_n, q) - \frac{s^n}{1-s} d(x_0, x_1)$
68. $\varepsilon_1 > 0$

69. $\exists N_1 \in \mathbb{N}, \forall m > N_1, d(x_m, q) < \varepsilon_1$
 70. $\forall m \in \mathbb{N}, d(x_n, x_m) \leq \frac{s^n}{1-s} d(x_0, x_1)$
 71. Let $m > N_1$
 72. $d(x_m, q) < \varepsilon_1$ and $d(x_n, x_m) \leq \frac{s^n}{1-s} d(x_0, x_1)$
 73. $d(x_n, q) \leq d(x_n, x_m) + d(x_m, q)$
 74. $< \frac{s^n}{1-s} d(x_0, x_1) + \varepsilon_1$
 75. $= \frac{s^n}{1-s} d(x_0, x_1) + d(x_n, q) - \frac{s^n}{1-s} d(x_0, x_1)$
 76. $= d(x_n, q)$
 77. $\rightarrow \leftarrow$
 78. \leftarrow
 79. $\forall x_0 \in X, \forall n \in \mathbb{N}, d(x_n, q) \leq \frac{s^n}{1-s} d(x_0, x_1)$

QED

Hutchinson Operators are Contraction Maps

Theorem (Hutchinson) Let w_0, w_1, \dots, w_k be contraction mappings on \mathbb{R}^n with contraction factors c_0, c_1, \dots, c_k respectively and define $W : K_n \rightarrow K_n$ by

$$W(A) = w_0(A) \cup w_1(A) \cup \dots \cup w_k(A).$$

W is a contraction mapping on (K_n, d_H) with contraction factor $c = \max\{c_0, c_1, \dots, c_k\}$.

Pf:

1. Let w_0, w_1, \dots, w_k be contraction mappings on \mathbb{R}^n with contraction factors c_0, c_1, \dots, c_k respectively and $W : K_n \rightarrow K_n$ by

$$W(A) = w_0(A) \cup w_1(A) \cup \dots \cup w_k(A).$$

2. Define $c = \max\{c_0, c_1, \dots, c_k\}$.
 3. Let $X, Y \in K_n$
 4. Define $d = d_H$
 5. Define $r = d(X, Y)$
 6. $X \subseteq \bar{B}(Y; r)$ and $Y \subseteq \bar{B}(X; r)$
 ★we want to show that $d(W(X), W(Y)) \leq cd(X, Y)$, so let's compute $d(W(X), W(Y))$ ★
 7. Let $x \in W(X)$
 8. $x \in \bigcup_{i=0}^k w_i(X)$
 9. $x \in w_j(X)$ for some $j \in \mathbb{O}_k$
 10. $x = w_j(a)$ for some $a \in X$
 11. $a \in \bar{B}(Y; r)$
 12. $a \in \bigcup_{z \in Y} \bar{B}(z; r)$
 13. $a \in \bar{B}(z; r)$ for some $z \in Y$
 14. $d_{Euc}(a, z) \leq r$
 15. $d_{Euc}(x, w_j(z)) = d_{Euc}(w_j(a), w_j(z))$

16. $\leq c_j d_{\text{Euc}}(a, z)$
17. $\leq c_j r$
18. $\leq cr$
19. $w_j(z) \in w_j(Y)$
20. $\subseteq \bigcup_{i=0}^k w_i(Y)$
21. $= W(Y)$
22. $x \in \bar{B}(w_j(z); cr)$
23. $\subseteq \bigcup_{\alpha \in W(Y)} \bar{B}(\alpha; cr)$
24. $= \bar{B}(W(Y); cr)$
25. $W(X) \subseteq \bar{B}(W(Y); cr)$
26. $W(Y) \subseteq \bar{B}(W(X); cr)$ (by a similar argument)
27. $d(W(X), W(Y)) \leq cr$
28. $= cd(X, Y)$
29. W is a contraction mapping on (K_n, d_H) with contraction factor c .

QED

Planar Affine Maps are Determined by 3 Points

Theorem An affine transformation on \mathbb{R}^2 is completely determined by where it maps 3 non-collinear points.

Pf:

1. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an affine map
 2. Let $p_1, p_2, p_3 \in \mathbb{R}^2$ be noncollinear
 3. $p_1 = (x_1, y_1), p_2 = (x_2, y_2), p_3 = (x_3, y_3)$ for some $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$
 4. $\forall x \in \mathbb{R}^2, T(x) = Mx + B$ for some $M \in M_{2,2}(\mathbb{R})$ and some $B \in \mathbb{R}^2$
 5. Define $u = p_2 - p_1$ and $v = p_3 - p_1$
 6. $u = (x_2 - x_1, y_2 - y_1)$ and $v = (x_3 - x_1, y_3 - y_1)$
 7. Define $u_1 = x_2 - x_1, u_2 = y_2 - y_1, v_1 = x_3 - x_1, v_2 = y_3 - y_1$
 8. Assume $u_1 v_2 - u_2 v_1 = 0$
 9. $u_1 v_2 = u_2 v_1$
 10. Assume $v_1 = v_2 = 0$
 11. $x_3 - x_1 = 0$ and $y_3 - y_1 = 0$
 12. $x_3 = x_1 = 0$ and $y_3 = y_1$
 13. $(x_1, y_1) = (x_3, y_3)$
 14. $p_1 = p_3$
 15. p_1, p_2, p_3 are collinear
 16. \leftrightarrow
 17. \leftarrow
 18. $v_1 \neq 0$ or $v_2 \neq 0$
- ★ So we have two cases. We will prove the case where $v_1 \neq 0$ since the other case is

similar ★

19. Assume $v_1 \neq 0$
 20. Define $c = \frac{u_1}{v_1}$
 21. $u_1 = \frac{u_1}{v_1} v_1$
 22. $= cv_1$
 23. $u_2 = u_2 \frac{v_1}{v_1}$
 24. $= \frac{u_2 v_1}{v_1}$
 25. $= \frac{u_1 v_2}{v_1}$
 26. $= \frac{u_1}{v_1} v_2$
 27. $= cv_2$
 28. $u = (u_1, u_2)$
 29. $= (cv_1, cv_2)$
 30. $= c(v_1, v_2)$
 31. $= cv$
 32. p_1, p_2, p_3 are collinear
 33. $\rightarrow \leftarrow$
 34. \leftarrow
 35. $v_1 \neq 0$
 36. $v_2 \neq 0$ by a similar argument
 37. $\rightarrow \leftarrow$
 38. \leftarrow
 39. $u_1 v_2 - u_2 v_1 \neq 0$
 40. Let $z \in \mathbb{R}^2$
 41. $z = (z_1, z_2)$ for some $z_1, z_2 \in \mathbb{R}$

★Note we should write a_z and b_z below but omit where possible to avoid clutter ★

42. Define $a = \frac{z_1 v_2 - z_2 v_1}{u_1 v_2 - u_2 v_1}$, $b = \frac{u_1 z_2 - u_2 z_1}{u_1 v_2 - u_2 v_1}$
 43. $z = (z_1, z_2)$
 44. $= \left(\frac{z_1 v_2 - z_2 v_1}{u_1 v_2 - u_2 v_1} u_1 + \frac{u_1 z_2 - u_2 z_1}{u_1 v_2 - u_2 v_1} v_1, \frac{z_1 v_2 - z_2 v_1}{u_1 v_2 - u_2 v_1} u_2 + \frac{u_1 z_2 - u_2 z_1}{u_1 v_2 - u_2 v_1} v_2 \right)$
 45. $= (au_1 + bv_1, au_2 + bv_2)$
 46. $= (au_1 + bv_1, au_2 + bv_2)$
 47. $= au + bv$
 48. $Mz = M(au + bv)$
 49. $= aMu + bMv$
 50. $= aM(p_2 - p_1) + bM(p_3 - p_1)$
 51. $= a(Mp_2 - Mp_1) + b(Mp_3 - Mp_1)$
 52. $= a(Mp_2 + B - Mp_1 - B) + b(Mp_3 + B - Mp_1 - B)$
 53. $= a((Mp_2 + B) - (Mp_1 + B)) + b((Mp_3 + B) - (Mp_1 + B))$
 54. $= a(T(p_2) - T(p_1)) + b(T(p_3) - T(p_1))$

55. $\forall z \in \mathbb{R}^2, Mz = a_z(T(p_2) - T(p_1)) + b_z(T(p_3) - T(p_1))$
 56. $M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{v_2}{u_1v_2 - u_2v_1} (T(p_2) - T(p_1)) + \frac{-u_2}{u_1v_2 - u_2v_1} (T(p_3) - T(p_1))$
 57. $M \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{-v_1}{u_1v_2 - u_2v_1} (T(p_2) - T(p_1)) + \frac{u_1}{u_1v_2 - u_2v_1} (T(p_3) - T(p_1))$
 58. $M \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the first column of M and $M \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the second column of M
 59. M is completely determined by $p_1, p_2, p_3, T(p_1), T(p_2), T(p_3)$
 60. $T(p_1) = Mp_1 + B$
 61. $B = T(p_1) - Mp_1$
 62. B is completely determined by $p_1, p_2, p_3, T(p_1), T(p_2), T(p_3)$
 63. T is completely determined by $p_1, p_2, p_3, T(p_1), T(p_2), T(p_3)$
 64. T is completely determined by where it sends any three noncollinear points
 QED

Contraction Factor for Affine Maps

Theorem Let $\alpha, \beta, \gamma \in \mathbb{C}$ and $c = |\alpha| + |\beta|$. Then the map $T = \text{affineC}(\alpha, \beta, \gamma)$ is a contraction mapping if and only if $c < 1$. Further, if T is a contraction mapping then c is a contraction factor for T .

Pf:

1. Let $\alpha, \beta, \gamma \in \mathbb{C}$, $c = |\alpha| + |\beta|$, and $T = \text{affineC}(\alpha, \beta, \gamma)$
 2. Define $d = d_{\text{Euc}}$.
 3. Let $z, w \in \mathbb{C}$
 4. Define $q = z - w$.
 5. $|q| = |z - w|$ substitution
 6. $= d(z, w)$ def of d_{Euc}
 7. Define $r = |q|$, $r_1 = |\alpha|$, $r_2 = |\beta|$, $\theta = \text{Arg}(q)$, $\theta_1 = \text{Arg}(\alpha)$, $\theta_2 = \text{Arg}(\beta)$
 8. $q = re^{i\theta}$, $\alpha = r_1e^{i\theta_1}$, and $\beta = r_2e^{i\theta_2}$ def polar form
 9. $d(T(z), T(w)) = d(\alpha z + \beta \bar{z} + \gamma, \alpha w + \beta \bar{w} + \gamma)$ def of *affineC*
 10. $= |(\alpha z + \beta \bar{z} + \gamma) - (\alpha w + \beta \bar{w} + \gamma)|$ definition of d_{Euc}
 11. $= |\alpha(z - w) + \beta(\bar{z} - \bar{w})|$ arithmetic
 12. $= |\alpha(z - w) + \beta(\bar{z} - \bar{w})|$ property of conjugates
 13. $= |\alpha q + B\bar{q}|$ definition of q
 14. $\leq |\alpha q| + |B\bar{q}|$ by the triangle inequality
 15. $= |\alpha||q| + |B||\bar{q}|$ property of absolute value
 16. $= |\alpha||q| + |B||q|$ property of conjugates
 17. $= |q|(|\alpha| + |\beta|)$ arithmetic
 18. $= cd(z, w)$ definitions of c, q
 19. $d(T(z), T(w)) \leq cd(z, w)$ transitivity
- (\Leftarrow)
20. Assume $c < 1$
 21. $c > 0$ def of c and property of $| \cdot |$
 22. $0 < c < 1$ by the two last two lines

23. T is a contraction mapping definition of contraction mapping
24. \leftarrow
25. $c < 1 \Rightarrow T$ is a contraction mapping
- (\Rightarrow)
26. Assume T is a contraction mapping
27. T has contraction factor s for some $s \in (0 \dots 1)$ def of contraction mapping
28. Define $u = 0$, $v = e^{i(\frac{\theta_2 - \theta_1}{2})}$
29. $d(T(v), T(u)) = |\alpha v + \beta \bar{v} + \gamma - (\alpha u + \beta \bar{u} + \gamma)|$ complex notation
30. $= |\alpha v + \beta \bar{v}|$ subst $w = 0$ and arithmetic
31. $= \left| \alpha e^{i(\frac{\theta_2 - \theta_1}{2})} + \beta e^{-i(\frac{\theta_2 - \theta_1}{2})} \right|$ substitution
32. $= \left| r_1 e^{i\theta_1} e^{i(\frac{\theta_2 - \theta_1}{2})} + r_2 e^{i\theta_2} e^{-i(\frac{\theta_2 - \theta_1}{2})} \right|$ substitution
33. $= \left| r_1 e^{i(\theta_1 + \frac{\theta_2 - \theta_1}{2})} + r_2 e^{i(\theta_2 - \frac{\theta_2 - \theta_1}{2})} \right|$ property of exponentials
34. $= \left| r_1 e^{i(\frac{\theta_1 + \theta_2}{2})} + r_2 e^{i(\frac{\theta_1 + \theta_2}{2})} \right|$ arithmetic
35. $= \left| (r_1 + r_2) e^{i(\frac{\theta_1 + \theta_2}{2})} \right|$ distributive law
36. $= |r_1 + r_2| \left| e^{i(\frac{\theta_1 + \theta_2}{2})} \right|$ property of $| \cdot |$
37. $= |r_1 + r_2|$ property of exponentials
38. $= r_1 + r_2$ def of $| \cdot |$ (since $r_1, r_2 \geq 0$)
39. $= |\alpha| + |\beta|$ substitution
40. $= c$ substitution
41. $d(T(v), T(u)) = c$ transitivity
42. $c = d(T(v), T(u))$ substitution
43. $\leq sd(z, w)$ def of contraction map
44. $= sd\left(e^{i(\frac{\theta_2 - \theta_1}{2})}, 0\right)$ substitution
45. $= s \left| e^{i(\frac{\theta_2 - \theta_1}{2})} - 0 \right|$ def of d_{Euc}
46. $= s \left| e^{i(\frac{\theta_2 - \theta_1}{2})} \right|$ arithmetic
47. $= s$ property of exponentials
48. c is a contraction factor for T substitution
49. $c < 1$ def of contraction factor
50. \leftarrow
51. T is a contraction mapping $\Rightarrow c < 1$ and c is a contraction factor for T

QED

Attractor Size Theorem

Theorem Let $W = [w_0, \dots, w_n]$ be an IFS, c_0, \dots, c_n the contraction factors of w_0, \dots, w_n respectively, and q_0, \dots, q_n the fixed points of w_0, \dots, w_n respectively.

Define $c = \max\{c_0, \dots, c_n\}$ and $r = \max\{d(q_i, q_j) : i, j \in \mathbb{O}_n\}$. Then for any $a \in F_W$ and any $i \in \mathbb{O}_n$,

$$d_{\text{Euc}}(a, q_i) \leq \frac{1}{1-c}r.$$

Pf:

1. Let $W = [w_0, \dots, w_n]$ be an IFS on \mathbb{R}^k , c_0, \dots, c_n the contraction factors of w_0, \dots, w_n respectively, and q_0, \dots, q_n the fixed points of w_0, \dots, w_n respectively.
2. Define $c = \max\{c_0, \dots, c_n\}$, $r = \max\{d(q_i, q_j) : i, j \in \mathbb{O}_n\}$, and $d = d_{\text{Euc}}$.
3. Let $i, j \in \mathbb{O}_n$.
4. Let $z \in \mathbb{R}^k$.
5. $d(w_j(z), q_i) \leq d(w_j(z), w_j(q_j)) + d(w_j(q_j), q_i)$
6. $\leq c_j d(z, q_j) + d(q_j, q_i)$
7. $\leq cd(z, q_j) + r$
8. $\forall i, j \in \mathbb{O}_n, \forall z \in \mathbb{R}^k, d(w_j(z), q_i) \leq cd(z, q_j) + r$
9. Let $a \in F_W$.
10. $a = \Phi(t_1 t_2 \dots)$ for some $t_1 t_2 \dots \in \Sigma_{n+1}$
11. $= \lim_{n \rightarrow \infty} w_{t_1} w_{t_2} \dots w_{t_n}(q_i)$
12. Let $n \in \mathbb{N}$.
13. $d(w_{t_n}(q_i), q_i) \leq cd(q_i, q_{t_n}) + r \leq cr + r = (1+c)r$
14. $d(w_{t_{n-1}} w_{t_n}(q_i), q_i) \leq cd(w_{t_n}(q_i), q_{t_{n-1}}) + r \leq c(1+c)r + r = (1+c+c^2)r$
15. \vdots
16. $d(w_{t_1} w_{t_2} \circ \dots \circ w_{t_{n-1}} w_{t_n}(q_i), q_i) \leq (1+c+c^2+\dots+c^n)r$
17. $\leq (1+c+c^2+\dots)r$
18. $\leq \frac{1}{1-c}r$
19. Let $\varepsilon > 0$
20. $\exists N > 0, \forall n \geq N, d(a, w_{t_1} \circ \dots \circ w_{t_n}(q_i)) < \varepsilon$
21. Let $n > N$.
22. $d(a, q_i) \leq d(a, w_{t_1} \circ \dots \circ w_{t_n}(q_i)) + d(w_{t_1} \circ \dots \circ w_{t_n}(q_i), q_i)$
23. $\leq d(a, w_{t_1} \circ \dots \circ w_{t_n}(q_i)) + \frac{1}{1-c}r$
24. $< \varepsilon + \frac{1}{1-c}r$
25. $\forall \varepsilon > 0, d(a, q_i) < \frac{1}{1-c}r + \varepsilon$
26. $d(a, q_i) \leq \frac{1}{1-c}r$

QED

Corollary $F_W \subseteq \bigcap_{i=0}^n \bar{B}(q_i; \frac{r}{1-c})$

Pf:

1. Let $W = [w_0, \dots, w_n]$ be an IFS on \mathbb{R}^k , c_0, \dots, c_n the contraction factors of w_0, \dots, w_n respectively, and q_0, \dots, q_n the fixed points of w_0, \dots, w_n respectively.
2. Let $a \in F_W, i \in \mathbb{O}_n$.

3. $d(a, q_i) \leq \frac{1}{1-c}r$
4. $a \in \bar{B}(q_i; \frac{r}{1-c})$
5. $\forall i \in \mathbb{O}_n, a \in \bar{B}(q_i; \frac{r}{1-c})$
6. $a \in \bigcap_{i=0}^n \bar{B}(q_i; \frac{r}{1-c})$
7. $F_W \subseteq \bigcap_{i=0}^n \bar{B}(q_i; \frac{r}{1-c})$

QED

* SPECIAL THANKS TO PAUL OLDAKOWSKI, A STUDENT FROM THE MATH 320 - SPRING 2000 COURSE WHO TYPED SOME OF THESE LECTURE NOTES.