# Fun Facts: Chaos and Fractals



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This is **not** a complete set of lecture notes for Math 320, Chaos and Fractals. Additional material will be covered in class and discussed in the textbook. Thanks to Paul Oldakowski, a student from the Math 320 - Spring 2000 course who typed some of the lecture notes below.

## Logic

In this section we give an informal overview of logic and proofs. For a more formal introduction see any logic textbook.

### Variables, Expressions, and Statements

**Definition** *A **set** is a collection of items called the **members** (or **elements**) of the set.*

**Remark** *An element is either in a set or it is not in a set, it cannot be in a set more than once.*

**Definition** *An **expression** is an arrangement of symbols which represents an element of a set called the **domain** (or **type**) of the expression.*

**Remark** *It is not necessary that we know specifically which element of the domain an expression represents, only that it represents some unspecified element in that set.*

**Definition** *The element of the domain that the expression represents is called a **value** of that expression.*

**Definition** *A **variable** is an expression consisting of a single symbol.*

**Definition** *A **constant** is an expression whose domain contains a single element.*

**Definition** *A **statement** (or **Boolean expression**) is an expression whose domain is* $\{true, false\}$.

**Remark** *We do not have to know if a statement is true or false, just that it is either true or false.*

**Definition** *The value of a statement is called its **truth value**.*

**Definition** *To **solve** a statement is to determine the set of all elements for which the statement is true.*

**Remark** *More precisely, if a statement contains* $\mathrm{n}$ *variables,* $\mathrm{x}_1, \ldots \mathrm{x}_\mathrm{n}$, *then to solve the statement is to find the set of all* $\mathrm{n}$*-tuples* $(\mathrm{a}_1, \ldots, \mathrm{a}_\mathrm{n})$ *such that each* $\mathrm{a}_\mathrm{i}$ *is an* *element of the domain of* $\mathrm{x_i}$ *and the statement becomes true when* $\mathrm{x}_1, \ldots, \mathrm{x_n}$ *are replaced by* $\mathrm{a}_1, \ldots, \mathrm{a_n}$ *respectively.*

**Definition** *An **equation** is a statement of the form* $A = B$ *where A and B are expressions.*

**Definition** *An **inequality** is a statement of the form* $A \star B$ *where A and B are expressions and* $\star$ *is one of* $\leq$, $\geq$, $>$, $<$, *or* $\neq$.

# Propositional Logic

## The Five Logical Operators

**Definition** *Let* $P, Q$ *be statements. Then the expressions*

1. $\sim P$
2. $P$ *and* $Q$
3. $P$ *or* $Q$
4. $P \Rightarrow Q$
5. $P \Leftrightarrow Q$

*are also statements whose truth values are completely determined by the truth values of P and Q as shown in the following table*

| $P$ | $Q$ | $\sim P$ | $P$ and $Q$ | $P$ or $Q$ | $P \Rightarrow Q$ | $P \Leftrightarrow Q$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $F$ | $T$ | $T$ |

## Rules of Inference and Proof

**Definition** *A **rule of inference** is a rule which takes zero or more statements (or other items) as input and returns one or more statements as output.*

**Notation** *An expression of the form*

$$\begin{array}{l} P_1 \\ \vdots \\ P_k \\ \hline Q_1 \\ \vdots \\ Q_n \end{array}$$

*represents a rule of inference whose inputs are* $P_1 \ldots P_k$ *and outputs are* $Q_1, \ldots, Q_n$.

**Notation** *The rule of inference shown above can also be expressed in **recipe notation** as*

$$\begin{array}{l} \textit{Show } P_1 \\ \vdots \\ \textit{Show } P_k \\ \textit{Conclude } Q_1 \\ \vdots \\ \textit{Conclude } Q_n \end{array}$$

*or equivalently,*

$$\begin{array}{l} \textit{To show } Q_1, \ldots, Q_n \\ \quad \textit{Show } P_1 \\ \quad \vdots \\ \quad \textit{Show } P_k \end{array}$$

**Definition** *A **formal logic system** consists of a set of statements and a set of rules of inference.*

**Definition** *A **proof** in a formal logic system consists of a finite sequence of statements (and other inputs to the rules of inference) such that each statement follows from the previous statements in the sequence by one or more of the rules of inference.*

## Natural Deduction

**Definition** *The symbol ← is an abbreviation for "end assumption".*

**Definition** *The rules of inference for propositional logic are shown in Table 1.*

| Table 1: Rules of inference for Propositional Logic | |
|---|---|
| **and +**<br>To show $W$ and $V$<br>1. Show $W$<br>2. Show $V$ | **and –**<br>To show $W$<br>1. Show $W$ and $V$<br><br>**and –**<br>To show $V$<br>1. Show $W$ and $V$ |
| **$\Rightarrow$ +**<br>To show $W \Rightarrow V$<br>1. Assume $W$<br>2. Show $V$<br>3. $\leftarrow$ | **$\Rightarrow$ – (modus ponens)**<br>To show $V$<br>1. Show $W$<br>2. Show $W \Rightarrow V$ |
| **$\Leftrightarrow$ +**<br>To show $W \Leftrightarrow V$<br>1. Show $W \Rightarrow V$<br>2. Show $V \Rightarrow W$ | **$\Leftrightarrow$ –**<br>To show $W \Rightarrow V$<br>1. Show $W \Leftrightarrow V$<br><br>**$\Leftrightarrow$ –**<br>To show $V \Rightarrow W$<br>1. Show $W \Leftrightarrow V$ |
| ***or* +**<br>To show $W$ or $V$<br>1. Show $W$<br><br>***or* +**<br>To show $W$ or $V$<br>1. Show $V$ | ***or* – (proof by cases)**<br>To show $U$<br>1. Show $W$ or $V$<br>2. Show $W \Rightarrow U$<br>3. Show $V \Rightarrow U$ |
| **$\sim$ + (proof by contradiction)**<br>To show $\sim W$<br>1. Assume $W$<br>2. Show $\rightarrow\leftarrow$<br>3. $\leftarrow$ | **$\sim$ – (proof by contradiction)**<br>To show $W$<br>1. Assume $\sim W$<br>2. Show $\rightarrow\leftarrow$<br>3. $\leftarrow$ |
| **$\rightarrow\leftarrow$ +**<br>To show $\rightarrow\leftarrow$<br>1. Show $W$<br>2. Show $\sim W$ | **$\rightarrow\leftarrow$ –**<br>To show $W$<br>1. Show $\rightarrow\leftarrow$ |

**Remark** *Note that the inputs "Assume -" and "←" are not themselves statements but rather inputs to rules of inference that may be inserted into a proof at any time. There is no reason however, to insert such statements unless you intend to use one* *of the rules of inference that requires them as inputs.*

**Example** *Prove* $P \Rightarrow (P \text{ or } Q)$ *and verify it with a truth table*

**Example** *Prove* $(P \text{ or } Q) \Rightarrow \sim(\sim P \text{ and } \sim Q)$ *and verify it with a truth table*

## Equality

**Definition** *The equality symbol, =, is defined by two rules of inference:*

| Reflexive = | Substitution |
|---|---|
| *To show* $x = x$ | *To show W with the* $n^{th}$ *free occurrence of x replaced by y*<br>*1. Show W*<br>*2. Show* $x = y$ |

**Remark** *Note that in the Reflexive rule there are no inputs, so you can insert a statement of the form* x = x *into your proof at any time.*

**Example** *Given* $x = y$ *and* $y = z$*, prove* $x = z$*.*

## Predicate Logic

### Quantifiers

**Definition** *The symbols ∀ and ∃ are* ***quantifiers****. The symbol ∀ is called "for all", "for every", or "for each". The symbol ∃ is called "for some" or "there exists".*

**Definition** *If W is a statement and x is any variable then* $\forall x, W$ *and* $\exists x, W$ *are both statements. The rules of inference for these quantifiers are given in Table 2.*

**Notation** *If x is a variable, t an expression, and* $W(x)$ *a statement then* $W(t)$ *is the statement obtained by replacing every free occurrence of x in* $W(x)$ *with* $(t)$*,*

**Table 2: Rules of Inference for Quantifiers**

| $\forall +$ | $\forall -$ |
|---|---|
| To show $\forall x, W(x)$<br>1. Let $s$ be arbitrary<br>2. Show $W(s)$ | To show $W(t)$<br>1. Show $\forall x, W(x)$ |
| $\exists +$ | $\exists -$ |
| To show $\exists x, W(x)$<br>1. Show $W(t)$ | To show $W(t)$ for some $t$<br>1. Show $\exists x, W(x)$ |

**Remark** *Note that there are restrictions on the rules of inference for quantifiers which are not listed in Table 2 (see the Proof Recipes sheet for details). In most situations they are not a concern.*

**Remark** *Precedence: Quantifiers have a lower precedence than* $\Leftrightarrow$. *Thus they quantify the largest statement to their right possible unless specifically limited by parentheses.*

**Example** *Prove* $(\exists x, P(x)) \Rightarrow (\sim\forall y, \sim P(y))$

# Sets, Functions, Numbers

## Some Definitions from Set theory

The symbol $\in$ is formally undefined, but it means “is an element of”. Many of the definitions below are informal definitions that are sufficient for our purposes.

**Set notation and operations**

| | |
|---|---|
| Finite set notation: | $x \in \{x_1, \ldots, x_n\} \Leftrightarrow x = x_1$ or $\cdots$ or $x = x_n$ |
| Set builder notation: | $x \in \{y : P(y)\} \Leftrightarrow P(x)$ |
| Cardinality: | $\#S$ = the number of elements in a finite set $S$ |
| Subset: | $A \subseteq B \Leftrightarrow \forall x, x \in A \Rightarrow x \in B$ |
| Set equality: | $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$ |
| Def. of $\notin$: | $x \notin A \Leftrightarrow \sim(x \in A)$ |
| Empty set: | $\exists \emptyset, \forall x, x \notin \emptyset$ |
| Relative Complement: | $x \in B - A \Leftrightarrow x \in B$ and $x \notin A$ |
| Intersection: | $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$ |
| Union: | $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$ |
| Indexed Intersection: | $x \in \bigcap_{i \in I} A_i \Leftrightarrow \forall i, i \in I \Rightarrow x \in A_i$ |
| Indexed Union: | $x \in \bigcup_{i \in I} A_i \Leftrightarrow \exists i, i \in I$ and $x \in A_i$ |
| Two convenient abbreviations: | $(\forall x \in A, P(x)) \Longleftrightarrow \forall x, x \in A \Rightarrow P(x)$<br>$(\exists x \in A, P(x)) \Longleftrightarrow \exists x, x \in A$ and $P(x)$ |

**Some Famous Sets**

| | |
|---|---|
| The Natural Numbers | $\mathbb{N} = \{0,1,2,3,4,\ldots\}$ |
| The Integers | $\mathbb{Z} = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$ |
| The Rational Numbers | $\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}, b \neq 0, \text{ and } \gcd(a,b) = 1\right\}$ |
| The Real Numbers | $\mathbb{R} = \left\{x : x \text{ can be expressed as a decimal number}\right\}$ |
| The Complex Numbers | $\mathbb{C} = \{x + yi : x,y \in \mathbb{R}\}$ where $i^2 = -1$ |
| The positive real numbers | $\mathbb{R}^+ = \left\{x : x \in \mathbb{R} \text{ and } x > 0\right\}$ |
| The negative real numbers | $\mathbb{R}^- = \left\{x : x \in \mathbb{R} \text{ and } x < 0\right\}$ |
| The positive reals in a set $A$ | $A^+ = A \cap \mathbb{R}^+$ |
| The negative reals in a set $A$ | $A^- = A \cap \mathbb{R}^-$ |
| The first $n$ positive integers | $\mathbb{I}_n = \{1,2,\ldots,n\}$ |
| The first $n+1$ natural numbers | $\mathbb{O}_n = \{0,1,2,\ldots,n\}$ |

**Cartesian products**

| | |
|---|---|
| Ordered Pairs: | $(x,y) = (u,v) \Leftrightarrow x = u$ and $y = v$ |
| Ordered $n$-tuple: | $(x_1,\ldots,x_n) = (y_1,\ldots,y_n) \Leftrightarrow x_1 = y_1$ and $\cdots$ and $x_n = y_n$ |
| Cartesian Product: | $A \times B = \{(x,y) : x \in A \text{ and } y \in B\}$ |
| Cartesian Product: | $A_1 \times \cdots \times A_n = \{(x_1,\ldots,x_n) : x_1 \in A_1 \text{ and } \cdots \text{ and } x_n \in A_n\}$ |
| Power of a Set | $A^n = A \times A \times \cdots \times A$ where there are $n$ "$A$'s" in the Cartesian product |

**Functions and Relations**

| | |
|---|---|
| Def of relation: | $R$ is a relation from $A$ to $B \Leftrightarrow R \subseteq A \times B$ |
| Def of function: | $f : A \to B \Leftrightarrow f \subseteq A \times B$ and<br>$(\forall x, \exists y, (x,y) \in f)$ and<br>$(\forall x, ((x,y) \in f$ and $(x,z) \in f) \Rightarrow y = z)$ |
| Alt function notation | $X \overset{f}{\to} Y \Leftrightarrow f : X \to Y$ |
| Def of $f(x)$: | $f(x) = y \Leftrightarrow f : A \to B$ and $(x,y) \in f$ |
| Domain: | $\text{Domain}(f) = A \Leftrightarrow f : A \to B$ |
| Codomain: | $\text{Codomain}(f) = B \Leftrightarrow f : A \to B$ |
| Image: | $f(S) = \{y : \exists x, x \in S$ and $y = f(x)\}$ |
| Range: | $\text{Range}(f) = f(\text{Domain}(f))$ |
| Identity Map: | $id_A : A \to A$ and $\forall x, id_A(x) = x$ |
| Composition: | $f : A \to B$ and $g : B \to C \Rightarrow (g \circ f) : A \to C$ and $\forall x, (g \circ f)(x) = g(f(x))$ |
| Injective (one-to-one): | $f$ is injective $\Leftrightarrow \forall x, \forall y, f(x) = f(y) \Rightarrow x = y$ |
| Surjective (onto): | $f$ is surjective $\Leftrightarrow f : A \to B$ and $(\forall y, y \in B \Rightarrow \exists x, y = f(x))$ |
| Bijective: | $f$ is bijective $\Leftrightarrow f$ is injective and $f$ is surjective |
| Inverse: | $f^{-1} : B \to A \Leftrightarrow f : A \to B$ and $f \circ f^{-1} = id_B$ and $f^{-1} \circ f = id_A$ |
| Inverse Image: | $f : A \to B$ and $S \subseteq B \Rightarrow f^{-1}(S) = \{x \in A : f(x) \in S\}$ |

**Example** *Prove that if $A \subseteq B$ then $A \cap B = A$.*

**Example** *(left cancellation for injective functions) Let $X, Y, Z$ be sets and $f : Y \to Z$. Show that if $f$ is injective then for any functions $g, h : X \to Y$*

$$(f \circ g = f \circ h) \Rightarrow g = h$$

## Sequences

**Definition** *A* ***finite sequence*** *is a function $t : \mathbb{I}_n \to A$ where $n$ is a natural number and $A$ is a set. An* ***infinite sequence*** *is a function $t : \mathbb{N}^+ \to A$ where $A$ is a set. In either case, $t(k)$ is called the $k^{th}$ term of the sequence.*

**Remark** *It is often convenient to say that* $\mathrm{t}$ *is a finite (resp infinite) sequence if* $\mathrm{t} : \mathbb{O}_\mathrm{n} \to \mathrm{A}$ *(resp.* $\mathrm{t} : \mathbb{N} \to \mathrm{A}$*). In this case we say that* $\mathrm{t(k)}$ *is the* $\mathrm{k} + 1^{st}$ *term of the sequence.*

**Notation** *If $t : \mathbb{I}_n \to A$ is a finite sequence we write*

$$t_1, t_2, t_3, \ldots, t_n$$

*as another notation for t, where $t_k = t(k)$ for all $k \in \mathbb{I}_n$. Similarly if $t : \mathbb{N}^+ \to A$ we write*

$$t_1, t_2, t_3, \ldots$$

*for t where $t_k = t(k)$ for all $k \in \mathbb{N}^+$.*

**Remark** *Sometimes for readability we might want to enclose a sequence in parenthesis. For example, we might write "Let* $t = (1,2,3,4)$*" instead of "Let* $t = 1,2,3,4$*". In this sense there is really no distinction between* n*-tuples and finite sequences.*

**Notation** *We use an overbar to indicate an infinite repeating sequence, i.e.*

$$t_0, t_1, \ldots, t_{k-1}, \overline{t_k, \ldots, t_{k+n-1}}$$

*denotes the sequence infinite sequence t such that* $t_i = t_{k+((i-k) \operatorname{Mod} n)}$ *for all* $i \geq k+n$.

**Example** *Write the first five terms of the sequence* $\mathbb{N} \xrightarrow{a} \mathbb{N}$ *by* $a_n = n^2 + 1$.

**Example** *What is the 1000*$^{th}$ *term in the sequence*

$$9,0,8,\overline{3,2,4,1,5,7,6}$$

**Example** *Write the first five terms of the sequence* $\mathbb{N} \xrightarrow{a} \mathbb{N}$ *given by*

$$a(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot a(n-1) & \text{otherwise} \end{cases}$$

## Some Facts from Number Theory

**Theorem (Math Induction)** *Let* $P(n)$ *be any statement about a natural number variable n. Then*

$$\big(P(0) \text{ and } \forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1)\big) \Rightarrow \forall n \in \mathbb{N}, P(n).$$

**Theorem (Division Algorithm)** *Let* $a, b \in \mathbb{Z}$, *and* $b > 0$ . *Then there exist unique integers* $q, r \in \mathbb{Z}$ *such that*

$$a = qb + r \text{ and } 0 \leq r < b.$$

**Remark** *In this theorem the number* q *is called the* ***quotient*** *and* r *is called the* ***remainder*** *when* a *is divided by* b.

**Definition** *Let* $a, b \in \mathbb{Z}$ *with* $b > 0$. *Then* $a \operatorname{Mod} b$ *is the remainder when a is divided by b. The quotient can be written as* $\lfloor \frac{a}{b} \rfloor$ *where* $\lfloor x \rfloor$ *is the greatest integer less than or equal to a real number x.*

**Definition** *Let* $a, b \in \mathbb{Z}$. *We say a* ***divides*** *b if* $ak = b$ *for some integer k. If a divides b we write* $a \mid b$.

**Definition** *Let* $a, b \in \mathbb{Z}$ *with* $a \neq 0$ *or* $b \neq 0$. *Then* $\gcd(a,b)$ *is the greatest positive integer which divides both a and b.*

**Example** *What is the quotient and remainder when* $2^{1000} + 1$ *is divided by* $32$*?*

**Example** *What is the quotient and remainder when* $-100$ *is divided by* $7$*?*

**Example** *True or False:*
*(a)* $14 \mid 7$ *(b)* $7 \mid 14$ *(c)* $7 \mid -14$ *(d)* $7 \mid 0$

**Example** *What is the* $\gcd(72,60)$*?* $\gcd(295927,304679)$*?*

# Iteration

## Discrete Dynamical Systems

**Definition** *Let X be any set. Any function* $f : X \to X$ *is called a* ***set theoretic discrete dynamical system*** *(or simply* ***discrete dynamical system****).*

**Definition** *Let X be a set and* $f : X \to X$*. Define* $f^0 = id_X$ *and for all* $k \geq 1$ *define*

$$f^k = f \circ f^{k-1}$$

**Example** *Let* $f : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = 2x + 1$*. Find a formula for* $f^k(x)$ *for* $k \geq 0$*.*

**Theorem (Power Theorem)** *Let* $f : X \to X$*. For any* $k, n \in \mathbb{N}$*,*

$$f^{k+n} = f^k \circ f^n$$

*and*

$$f^{kn} = \underbrace{f^n \circ f^n \circ \cdots \circ f^n}_{k \text{ terms}} = \left(f^n\right)^k$$

**Definition** *Let* $f : X \to X$ *and* $x \in X$*. The sequence*

$$x, f(x), f^2(x), f^3(x), \ldots$$

*is called the* ***f-orbit*** *of x. The first term, x, is called the* ***seed*** *of the orbit. The* $k+1^{st}$ *term is called the* $k^{th}$ ***f-iterate*** *of x (or* $k^{th}$ ***iterate*** *or* $k^{th}$ ***iteration****). We write* $\mathrm{Orb}_f(x)$ *for the f-orbit of x.*

**Remark** $\mathrm{Orb_f(x)} : \mathbb{N} \to \mathrm{X}$ *and* $\mathrm{Orb_f(x)(n)} = \mathrm{f^n(x)}$ *for all* $\mathrm{n} \in \mathbb{N}$*.*

**Example (1)** *Find the complete f-orbit of* $5$ *for* $\mathbb{C} - \{0,1\} \xrightarrow{f} \mathbb{C} - \{0,1\}$ *by* $f(z) = \frac{1}{1-z}$*. What is the f-orbit of* $3$*? How about a?*

**Definition** *Let X be a set,* $x \in X$*, and* $f : X \to X$*. Then the set of terms in the f-orbit of x is denoted* $\mathcal{O}_f(x)$*, i.e.*

$$\mathcal{O}_f(x) = \{f^k(x) : k \in \mathbb{N}\}$$

*We call* $\mathcal{O}_f(x)$ ***the set of f-iterates of*** *x (or simply* ***the set of terms in the f-orbit of*** *x).*

**Example** *What is* $\mathcal{O}_f(5)$ *in Example 1? How many elements are in* $\mathcal{O}_f(5)$*?*

## Types of Orbits

**Definition** *Let* $f : X \to X$ *and* $x \in X$*. The f-orbit of x is* ***cyclic*** *if* $f^n(x) = x$ *for some* $n \geq 1$*. In this situation we say that x is a* ***cyclic point*** *(or* ***periodic point****) for f.*

**Example** *Is* $\mathrm{Orb}_f(5)$ *cyclic in Example 1?*

**Definition** *Let* $f : X \to X$ *and* $x \in X$*. If* $f^n(x) = x$ *for some* $n \geq 1$*, we say x has* ***period*** *n. If in addition* $f^k(x) \neq x$ *for all* $1 \leq k < n$ *then we say x has* ***minimum period*** *n. If*

*$x$ has period 1 we say $x$ is a **fixed point** of $f$. If $x$ has period $n$ we also say that $\mathrm{Orb}_f(x)$ has **period** $n$ and if $x$ has minimum period $n$ we also say $\mathrm{Orb}_f(x)$ has **minimum period** $n$ as well.*

**Example** *What is the minimum period of 5 in Example 1?*

**Example** *Does $f$ have any fixed points in Example 1?*

**Lemma** *Let $f : X \to X$, $x \in X$, and $n \in \mathbb{N}^+$.*

$$x \text{ has minimum period } n \Rightarrow \#\mathcal{O}_f(x) = n$$

**Example** *Why isn't it if and only if?*

**Definition** *Let $f : X \to X$ and $x \in X$. The $f$-orbit of $x$ is **eventually cyclic** if $f^n(x) = f^m(x)$ for some $n, m$ with $n \neq m$. In this situation we also say that $x$ is an **eventually cyclic point** (or **eventually periodic point**) for $f$.*

**Definition** *Let $f : X \to X$ and $x \in X$ periodic point with period $n$. We say that $\mathcal{O}_f(x)$ is an $n$-**cycle** if and only if $\mathrm{Orb}_f(x)$ is cyclic with minimum period $n$.*

**Definition** *Let $f : X \to X$ and $x \in X$. The $f$-orbit of $x$ is **acyclic** if it is not eventually cyclic.*

**Example** *Can you come up with examples of each of these?*

## The Digraph

**Definition** *A **directed graph** (or **digraph**) is a pair $(V, E)$ where $V$ is a set of elements called the **nodes** and $E \subseteq V \times V$ is the set of **directed edges**.*

**Definition** *Let $X \xrightarrow{f} X$ be a discrete dynamical system. The **digraph** of $f$ is the directed graph $(X, S)$ where $S = \{ (a, f(a)) : a \in V \}$, i.e. the nodes are the elements of the domain and the directed edges connect each element $a$ in the domain to $f(a)$.*

## Examples of Iteration

### The Collatz Conjecture

**Definition** *Define $T : \mathbb{Z} \to \mathbb{Z}$ by $\forall x \in \mathbb{Z}$*

$$T(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{3x+1}{2} & \text{if } x \text{ is odd} \end{cases}.$$

**Conjecture (Collatz)** *For all $n \in \mathbb{N}^+$, $\exists k \geq 0, T^k(n) = 1$, i.e. the $T$-orbit of any positive integer contains one.*

**Remark** *Note that $\mathrm{Orb}_{\mathrm{T}}(1) = \overline{1,2}$ so that the conjecture is equivalent to saying that the* T*-orbit of any positive integer is eventually periodic and enters the* 2*-cycle* $\{1,2\}$.

### Sumerian Method for Computing Square Roots

**Claim** *Let* $a \in \mathbb{R}^+$ *and* $Root_a(x) = \frac{1}{2}(x + \frac{a}{x})$. *For any* $x \in \mathbb{R}^+$, *the* $Root_a$*-orbit of* $x$ *converges to* $\sqrt{a}$.

**Example** *Find a fraction and a decimal that are a good approximation to* $\sqrt{2}$ *and* $\sqrt{3}$ *by the Sumerian Method.*

### Multiple Inputs: The Euclidean Algorithm

**Claim** *Define* $Euc : \mathbb{N}^+ \times \mathbb{N} \to \mathbb{N}^+ \times \mathbb{N}$ *by*

$$Euc(n,m) = \begin{cases} (m,n) & \text{if } n < m \\ (n,0) & \text{if } m = 0 \\ (m, n \operatorname{Mod} m) & \text{otherwise} \end{cases}.$$

*for any* $(n,m) \in \mathbb{N}^+ \times \mathbb{N}$. *Then the Euc-orbit of any* $(n,m)$ *is eventually fixed and contains the fixed point* $(\gcd(n,m),0)$.

**Remark** *This method of computing* $\gcd(\mathrm{n},\mathrm{m})$ *is called the* ***Euclidean algorithm***.

**Remark** *If* $\mathrm{f} : \mathrm{A} \times \mathrm{B} \to \mathrm{C}$ *and* $\mathrm{a} \in \mathrm{A}, \mathrm{b} \in \mathrm{B}$, *we usually abbreviate* $\mathrm{f}((\mathrm{a},\mathrm{b}))$ *as* $\mathrm{f}(\mathrm{a},\mathrm{b})$.

**Example** *Reduce the fraction* $\frac{295927}{304679}$ *by hand.*

### Non-numeric Inputs: Post's Tag Problem

**Definition** *Let S be a set and let* $S^*$ *be the set of words (finite sequences) which can be made from the alphabet S, i.e.*

$$S^* = \bigcup_{n=0}^{\infty} \{f \mid f : \mathbb{I}_n \to S\}.$$

*If* $x \in S^*$ *then #x is the number of letters in the word x. If* $x,y \in S^*$ *then* $x \bullet y$ *is the word formed by concatenating the words x and y. If* $x = x_1 x_2 \cdots x_n \in S^*$ *then* $x[a\ldots b]$ *is the word* $x_a x_{a+1} x_{a+2} \cdots x_{b-1} x_b$ *and* $x[a] = x_a$.

**Example** *What is* $\{\bigstar\}^*$? $\{a,b\}^*$?

**Definition** *Define* $Tag : \{a,b\}^* \to \{a,b\}^*$ *as follows. If* $x \in \{a,b\}^*$ *and* $n = \#x$ *then*

$$Tag(x) = \begin{cases} x & \text{if } n < 3 \\ x[4..n] \bullet aa & \text{if } x[1] = a \\ x[4..n] \bullet bbab & \text{if } x[1] = b \end{cases}$$

*In other words, if a word is less than three letters long, Tag returns it unchanged, if it is 3 or more letters and begins with the letter a then Tag deletes the first three letters and appends aa on the right, and if it is 3 or more letters and begins with b then Tag deletes the first three letters and appends bbab.*

**Problem (TAG)** *(Emil Post 1921) Are there any Tag-orbits which are not eventually cyclic?*

**Example** *What is the Tag-orbit of a? baba? bbbaa?*

## Stick Figure Fractals

**Definition** *Let $A,B$ be any distinct points in the plane. Then $\overline{AB}$ denotes the line segment with endpoints A and B (i.e. the set of all points in the plane which are on the line containing A and B and are either between A and B or are equal to A and B). The* ***directed segment from*** *A* ***to*** *B is a pair $(\overline{AB},A)$, and is denoted $\overrightarrow{AB}$. In this case we say $\overline{AB}$ is the segment associated with $\overrightarrow{AB}$ (and can think of a directed segment as being a set of points in the plane in this sense).*

**Notation** *If $A = (a_1,a_2), B = (b_1,b_2)$ then $\overrightarrow{AB}$ can be written* $\mathrm{dseg}([a_1,a_2],[b_1,b_2])$ *and $\overline{AB}$ can be written* $\mathrm{seg}([a_1,a_2],[b_1,b_2])$.

**Remark** *A directed line segment can be thought of as a line segment with an arrow drawn on it in one of the two possible directions. Note that $\overline{\mathrm{AB}} = \overline{\mathrm{BA}}$ but $\overrightarrow{\mathrm{AB}} \neq \overrightarrow{\mathrm{BA}}$. A directed line segment can also be thought of as a set of points since the line segment associated with it is a set of points. So if we talk about a point being "on a directed segment" we mean that it is on the line segment associated with the directed segment and so on.*

**Definition** *A* ***stick*** *is either a line segment or a directed line segment. A* ***stick figure*** *is a finite set of sticks. Let $U_{sf}$ be the set of all stick figures.*

**Remark** *Note that we can also consider a stick figure to be a set of points in the plane by considering the union of the points in the line segments and (line segments associated with) directed segments.*

**Definition** *Let* $s = \mathrm{dseg}([a,b],[c,d])$. *Define $T_s : \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$T_s(x,y) = ((c-a)x + (b-d)y + a, (d-b)x + (c-a)y + b).$$

*$T_s$ is called the* ***affine map induced by*** *s.*

**Remark** *We will show how to derive this map later in the course. Intuitively, it is the map that sends the directed segment from* $(0,0)$ *to* $(1,0)$ *to the directed segment* $\mathrm{s} = \mathrm{dseg}([\mathrm{a},\mathrm{b}],[\mathrm{c},\mathrm{d}])$, *and the directed segment from* $(0,0)$ *to* $(0,1)$ *to the directed segment obtained by rotating* s *by* $90^\circ$ *CCW about* $(\mathrm{a},\mathrm{b})$.

**Example** *Find the affine map induced by the directed segment from* $(1,1)$ *to* $(2,2)$.

**Lemma** *Let s be a directed segment and t a line segment. Then $T_s(t)$ is a line segment.*

**Definition** *Let $s,t$ be directed segments with $t = \overrightarrow{AB}$. Then $T_s(t) = (T_s(\overline{AB}), T_s(A))$.*

**Definition** *If G is a stick figure and s a directed segment then $T_s(G)$ is the stick figure $\bigcup_{x \in G} \{T_s(x)\}$.*

**Definition** *For each stick figure G define a dynamical system $\gamma_G : U_{sf} \to U_{sf}$ as follows. Let $S \in U_{sf}$ be a stick figure. For each $x \in S$, define*

$$g(x) = \begin{cases} \{x\} & \text{if } x \text{ is a line segment} \\ T_x(G) & \text{if } x \text{ is a directed segment} \end{cases}.$$

*Then* $\gamma_G(S) = \bigcup_{x \in S} g(x)$. *The dynamical system* $\gamma_G$ *is called the* ***stick figure iterator associated with*** $G$. *The figure G is called the* ***generator*** *for the stick figure iterator.*

**Claim** *In many cases the* $\gamma_G$*-orbit of a seed S converges to a fractal shape.*

**Example** *See my Maple worksheet LectureExamples.mws for examples.*

### GeeBees (Grid Based Fractals)

**Algorithm** *Let* $n \in \mathbb{N}^+$ *and* $a_1, \ldots, a_k \in \mathbb{I}_{n^2}$. *Define a dynamical system* $GB(n; a_1, \ldots, a_k)$ *as follows. Let the seed be a set containing one uncolored square. The process is:*

1. *Subdivide each uncolored square in the input set into an* $n \times n$ *grid of congruent subsquares and number these subsquares from* $1$ *to* $n^2$ *from left to right and bottom to top, starting in the lower left corner.*
2. *Color the subsquares numbered* $a_1, \ldots, a_k$.
3. *Output the set of colored and uncolored subsquares.*

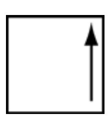
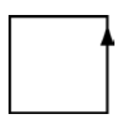
**Remark** *The background (uncolored subsquares of the original square) of a* GB *converges to a fractal shape.*

**Example** *Plot the first few iterations of* $GB(3; 2, 4)$.

### HeeBGB's

**Definition** *A* ***directed square*** *is a pair* $\left(S, \overrightarrow{AB}\right)$ *where S is a square in the plane and* $\overrightarrow{AB}$ *is a directed segment whose associated line segment is a side of S.*

**Notation** *When drawing a picture of a directed square we will draw the directed segment inside the square next to the edge instead of directly on top of the edge to avoid confusion when two directed squares share a common edge, i.e.*

 *instead of*  *. Note that the arrow is not part of the square S associated with the directed square.*

**Definition** *A* ***labeled square*** *is a member of the following 9 families:*

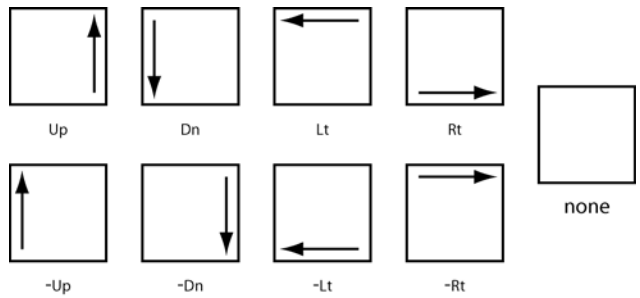



Figure 1: Labeled Squares

*Each labeled square must have one of the orientations shown above, but can have any position or size. Let $U_{LS}$ be the set of all labeled squares.*

**Remark** *Notice that every labeled square is a directed square except for the members of the family labeled* none. *Each labeled directed square is either positive or negative (the negative ones have the arrow on the left when viewed with the arrow pointing upwards).*

**Definition** *Define the* ***vertical mirror image*** *of each labeled square a to be $\overleftrightarrow{a}$ as follows:*

| $a$ | $Up$ | $-Up$ | $Dn$ | $-Dn$ | $Lt$ | $-Lt$ | $Rt$ | $-Rt$ | $none$ |
|---|---|---|---|---|---|---|---|---|---|
| $\overleftrightarrow{a}$ | $-Up$ | $Up$ | $-Dn$ | $Dn$ | $-Rt$ | $Rt$ | $-Lt$ | $Lt$ | $none$ |

*i.e. the sign always changes and left and right are interchanged.*

**Remark** *Note that this is what is obtained if each of the images in Figure 1 above are reflected about the vertical line through the center of the square.*

**Definition** *A* ***GB figure*** *is a finite set of labeled squares. Let $U_{GB}$ be the set of all GB figures.*

**Definition** *Let $n \in \mathbb{N}^+$ and $a_1, \ldots, a_{n^2} \in \{Up, Dn, Lt, Rt, -Up, -Dn, -Lt, -Rt, none\}$ (the label set). Define a dynamical system $HeeBGB(a_1, \ldots, a_{n^2}) : U_{GB} \to U_{GB}$ as follows. First define $g : U_{LS} \to U_{GB}$ as follows. For each $x \in U_{LS}$,*

*If $x$ is labeled none then $g(x) = \{x\}$.*

*If $x$ is a directed square then*

1. *Rotate $x$ so its arrow points upwards.*
2. *Subdivide $x$ into an $n \times n$ grid of congruent subsquares.*
    a. *if $x$ is positive, label these subsquares from $a_1$ to $a_{n^2}$ from left to right and bottom to top, starting in the lower left corner.*
    b. *if $x$ is negative, label these subsquares from $\overleftrightarrow{a_1}$ to $\overleftrightarrow{a_{n^2}}$ from right to left and bottom to top, starting in the lower right corner.*
3. *Undo the rotation from step number 1 to return the square (and all its new subsquares) to the original position and orientation. $g(x)$ is the set of these subsquares.*

*Now, let $S \in U_{GB}$. Define $HeeBGB(a_1, \ldots, a_{n^2})(S) = \bigcup_{x \in S} g(x)$.*

**Algorithm** *To draw a HeeBGB fractal start with a seed consisting of a set containing a single labeled square whose label is Up and a choice of labels $a_1, \ldots, a_{n^2}$. Computer the $HeeBGB(a_1, \ldots, a_{n^2})$-orbit of the seed, but color the squares labeled none as you iterate. The uncolored portion of the HeeBGB-orbit of this seed that is contained in the original square always converges to a fractal image (i.e. you are coloring the background, not the fractal).*

**Example** *Draw the first few iterations of $HeeBGB(Up, Up, Up, none)$.*

**Example** *Draw the first few iterations of $HeeBGB(Up, -Dn, Lt, none)$.*

**Example** *Draw the first few iterations of $HeeBGB(Dn, -Dn, Rt, none)$.*

### Newton's Method

**Definition** *Let $R \subseteq \mathbb{R}$, $f : R \to R$, and $r \in R$. We say $r$ is a* ***root*** *of $f$ if $f(r) = 0$.*

**Definition** *Let $R \subseteq \mathbb{R}$ and $f : R \to R$ a differentiable function. Define $Newt_f : \widetilde{R} \to \mathbb{R}$ by*

$$Newt_f(x) = x - \frac{f(x)}{f'(x)}$$

*for all $x \in \widetilde{R}$ where $\widetilde{R} = \left\{ x \in R : f'(x) \neq 0 \right\}$.*

**Remark** $\mathrm{Newt_f(a)}$ *is the* x *coordinate of point where the tangent line to the graph of* f *at* $(\mathrm{a}, \mathrm{f(a)})$ *meets the* x-*axis.*

**Theorem (Newton's Method)** *Let $R \subseteq \mathbb{R}$, $f : R \to R$ a differentiable function, and $r \in R$ a root of $f$. If $f'(r) \neq 0$ then there exists and interval $I \subseteq R$ such that $r \in I$, $Newt_f : I \to I$ and for all $x \in I$ the $Newt_f$-orbit of $x$ converges to $r$.*

**Example** *See my Maple worksheet LectureExamples.mws for an example.*

### Changing Integer Base

**Theorem (Base b representation)** *Let $b, n \in \mathbb{N}$, $b > 1$. There are unique integers $d_0, d_1, d_2, \ldots \in \mathbb{O}_{b-1}$ such that*

$$n = \sum_{i=0}^{\infty} d_i b^i.$$

**Definition** *The sequence $\ldots d_2 d_1 d_0$ is called the* ***base*** b ***representation*** *of n. If b is not clear from context we may write $\ldots d_2 d_1 d_{0\,(b)}$ to indicate the base.*

**Remark** *Since the sum is finite, there must exist* $\mathrm{k} \in \mathbb{N}$ *such that* $\mathrm{d_i} = 0$ *for all* $\mathrm{i} > \mathrm{k}$. *Thus we often abbreviate* $\mathrm{d_0 d_1 d_2 \ldots}$ *by* $\mathrm{d_0 d_1 \ldots d_k}$ *(i.e. drop the trailing zeros).*

**Definition** *Let $b \in \mathbb{N}$ and $b > 1$. Define $Base_b : \mathbb{N} \to \mathbb{N}$ by*

$$Base_b(x) = \frac{x - (x \text{ Mod } b)}{b}.$$

**Example** *If $b = 2$ then*

$$Base_2(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x-1}{2} & \text{if } x \text{ is odd} \end{cases}.$$

**Example** *If $b = 3$ then*

$$Base_3(x) = \begin{cases} \frac{x}{3} & \text{if } x \text{ Mod } 3 \text{ is } 0 \\ \frac{x-1}{3} & \text{if } x \text{ Mod } 3 \text{ is } 1 \\ \frac{x-2}{3} & \text{if } x \text{ Mod } 3 \text{ is } 2 \end{cases}.$$

**Theorem (Base Conversion)** *Let $n \in \mathbb{N}$. If each term in the $Base_b$-orbit of n is replaced by its value* Mod $b$, *the sequence produced will be the base b representation of n (with the digits listed from left to right from least significant to most significant.).*

**Example** *What happens if we apply this to a base ten number?*

**Example** *Convert* 314 *to base 2 by this method.*

**Conway's Fractran**

**Definition** *A **Fractran** program consists of a finite sequence of positive rational numbers*

$$F = [r_1, r_2, \ldots, r_k]$$

*with $r_k$ an integer. Each such sequence defines a dynamical system $f_F : \mathbb{Z}^+ \to \mathbb{Z}^+$ by*

$$f_F(n) = r_i n \quad \text{where } i = \min\{j : r_j n \in \mathbb{Z}\}$$

*i.e. $f_F$ multiplies n by the first rational number in the sequence for which the product is an integer.*

**Remark** *To "compute" with a Fractran program we simply compute the* $\mathrm{f_F}$*-orbit of some seed and look for certain terms in the orbit for the answers. For example, we might look at the exponents of the powers of two that appear in the orbit for the Fractan program's output.*

**Example:** (Conway) Let

$$PrimeGame = \left[ \frac{17}{91}, \frac{78}{85}, \frac{19}{51}, \frac{23}{38}, \frac{29}{33}, \frac{77}{29}, \frac{95}{23}, \frac{77}{19}, \frac{1}{17}, \frac{11}{13}, \frac{13}{11}, \frac{15}{2}, \frac{1}{7}, \frac{55}{1} \right]$$

and define $f = f_{PrimeGame}$. The powers of 2 which occur in the *f*-orbit of 2 are

$$2^2, 2^3, 2^5, 2^7, 2^{11}, 2^{13}, 2^{17}, 2^{19}, \ldots$$

in exactly that order, i.e. *PrimeGame* computes the prime numbers in order. See LectureExamples.mws to try it.

**Example: (**Monks) Let

$$CollatzGame = \left[ \frac{1}{11}, \frac{136}{15}, \frac{5}{17}, \frac{4}{5}, \frac{26}{21}, \frac{7}{13}, \frac{1}{7}, \frac{33}{4}, \frac{5}{2}, 7 \right]$$

and define $f = f_{CollatzGame}$. The powers of 2 which occur in the $f$-orbit of $2^n$ are

$$2^n, 2^{T(n)}, 2^{T^2(n)}, 2^{T^3(n)}, \ldots$$

in exactly that order, i.e. *CollatzGame* computes the Collatz orbits of natural numbers. See LectureExamples.mws to try it.

**Example: (**Conway) Let

$$PolyGame = \left[ \frac{583}{559}, \frac{629}{551}, \frac{437}{527}, \frac{82}{517}, \frac{615}{329}, \frac{371}{129}, \frac{1}{115}, \frac{53}{86}, \frac{43}{53}, \frac{23}{47}, \frac{341}{46}, \frac{41}{43}, \frac{47}{41}, \frac{29}{37}, \frac{37}{31}, \frac{299}{29}, \frac{47}{23}, \frac{161}{15}, \frac{527}{19}, \frac{159}{7}, \frac{1}{17}, \frac{1}{13}, \frac{1}{3} \right]$$

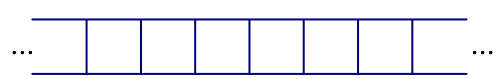
and define $f = f_{PolyGame}$. Define $f_c(n) = m$ if the $f_{PolyGame}$-orbit of $c2^{2^n}$ stops at $2^{2^m}$ and otherwise leave $f_c(n)$ undefined. Then every computable function appears among $f_0, f_1, f_2, \ldots$.

## Cellular Automata

**Definition** *An $n$-**dimensional** $k$-**state cellular automaton** is a discrete dynamical system $G \xrightarrow{c} G$ where $G$ is the set of all functions from $\mathbb{Z}^n$ to $\mathbb{I}_k$. Each element of $\mathbb{Z}^n$ is called a **cell**. Each element $f \in G$ is called a **state** and its value on a particular cell is called the **state** of that cell. The set $G$ is called the **state space**. To each cell we assign a finite **neighborhood** of cells such that the neighborhood of the translation of a cell is the translation of the neighborhood of the original cell. The map $c$ must be completely determined by a single rule that determines $c(f)(p)$ from the values of $f(q)$ for all $q$ in the neighborhood of $p$, i.e. the state of a cell after iterating is completely determined by the states of its neighbors before iterating.*

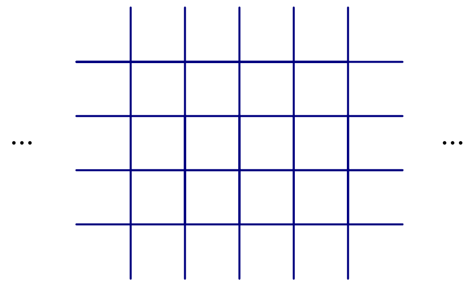
**Remark** *A cellular automaton (CA) is usually represented by a string or grid of squares, where each square in the grid represents a cell, and the states of each cell are represented by colors.*

**Example** *A one dimensional CA can be represented as a row of cells.*

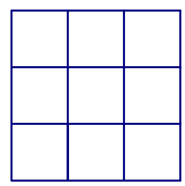


The states of the cells can be represented by coloring the cells different colors corresponding to the current state of that cell. The most common neighborhood to consider for a cell consists of the cell itself, the cell immediately to its left, and the cell immediately to its right (though others are possible).

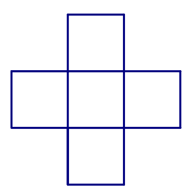
**Example** *A two dimensional CA can be represented as a grid of cells:*



**Definition** *There are two commonly used neighborhoods. The* ***Moore neighborhood*** *is a square shaped neighborhood centered at the cell. The most commonly used one consists of a cell all all of the cells that share a boundary point in common with that cell:*



*The* ***von Neumann neighborhood*** *consists of a diamond shaped neighborhood centered at the cell. The most commonly used one consists of the cell and its neighbors immediately to the left, right, above, and below the cell:*

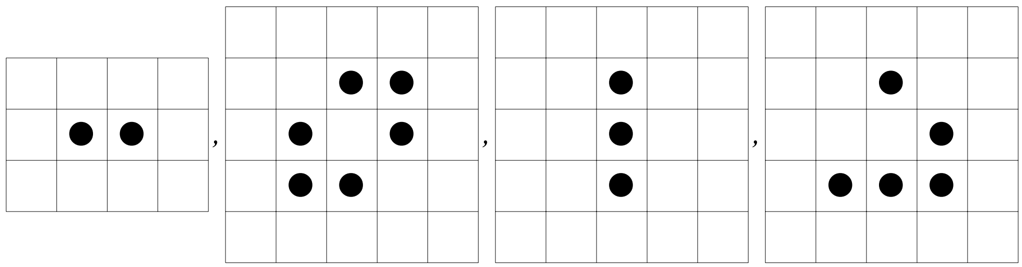


**Definition** *An* ***outer totalistic*** *(or simply* ***totalistic****) CA is one whose rule (map) is completely determined by the sum of the state values of the neighbors of each cell.*

**Definition** ***Binary cellular automata*** *are those with only two states for each cell. In this situation we say that a cell is* ***alive*** *if its state is 1 and* ***dead*** *if its state is 0.*

**Example** *The most famous CA is* ***Conway's Game of Life****. It is a 2 dimensional binary (=2 state) CA whose rule is given as follows. A dead cell becomes alive if exactly three of its Moore neighbors are alive, and a live cell stays alive if either two or three of its Moore neighbors (other than itself) are alive. Otherwise the cell becomes dead.*

**Example** *Compute the orbits of the following seed states in Conway's Game of Life (assuming that all cells other than the red ones shown are dead).*



**Definition** *A fixed point of the Game of Life cellular automaton is called a* ***Still Life****.*

**Example** *See Mirek's Cellebration or Life32 for interesting examples.*

**Example** *In 1999 Paul Rendell impelemented a Turing Machine in Life. In 2002, Paul Chapman extended this to construct a universal Turing Machine in Life.*

# Introduction to Maple

## Maple Basics

- Go through the *New User's Tour* in Maple. From the Maple help menu select *New User's Tour* and work through topic numbers 1,2 (briefly), 3, 4, 5, 10, 11.

## Maple Language

- Every Maple statement ends with either ';' or ':' . Statements that end with ';' will display their results, statement that end with ':' do not.
- **Assignment statement:**

  ```
  [variable]:=[expression]
  ```

  Examples: `x:=2; f:="Hello world"; a[0]:=x+2;`
- **Conditional Statement:**

  ```
  if [B_1] then
     [S_1]
  elif [B_2]
     [S_2]
    ⋮
  elif [B_n]
     [S_n]
  else
     [S_{n+1}]
  end if;
  ```

  where $B_1, B_2, \ldots, B_n$ are Boolean expressions and $S_1, S_2, \ldots, S_{n+1}$ are sequences of statements..
  
  Examples:

  ```
  if 0<x then
     print("It is positive.");
  end if;
  if x=0 then
  ```

```
    print("It is zero.");
  else
    print("It’s not zero.");
  end if;
  if 0<x then
    print("It is positive.");
  elif x=0 then
    print("It is zero.");
  else
    print("It is negative.");
  end if;
```

- **Loop statement**

  ```
  for [var] from [start] to [end] by [inc] while [B] do
  [S] end do;
  ```

  where `[var]` is a variable, `[start]`, `[end]`, `[inc]` are numbers, `[B]` a Boolean expression and `[S]` a sequences of statements. Note that any of the numbered pieces:

  $\underbrace{\textit{for [var]}}_{1}\ \underbrace{\textit{from [start]}}_{2}\ \underbrace{\textit{to [end]}}_{3}\ \underbrace{\textit{by [inc]}}_{4}\ \underbrace{\textit{while [B]}}_{5}$ `do`

  `[S] end do;`

  are optional. The default value for `[start]` and `[inc]`is 1. The default for `[end]` is infinity. The default for `[B]` is `true`.

  Examples:

  ```
  for i from 0 to 10 by 2 do
    print(i);
  end do;
  for n while n^2<100 do
    print(n^2);
  end do;
  ```

- **Procedures**

  ```
  proc([args])
    [S]
  end proc;
  ```

  where `[args]` is a sequence of zero or more variables and `[S]` is a sequence of statements. To return a value from a procedure use the `return [values]` command in the procedure. A procedure also returns the value of the last statements executed by default. See `?proc` for more details about procedures.

  Examples:

  ```
  Square:=proc(n)
    return n^2
  end proc;
  CollatzFunction:=proc(n)
  ```

```
    if n mod 2=0 then
       return n/2
    else
        return (3*x+1)/2
    end if;
  end proc;
```

To call any of the procedures above, simple use it like a function, e.g.

```
CollatzFunction(3);
Square(7);
```

## Maple Data types

- **Expressions**

  Maple has all the usual mathematical expressions built in.

  Examples:

  ```
  (x^2+3*x)/(x-3);
  sqrt(2);
  Pi*exp(x)-ln(3)+sin(x);
  x mod 6;
  ```

  etc.

- **Functions**

  While you can always use a proc to define a function, for simple functions it is easier to use the `x->` (think if this as "x maps to") notation as indicated in the following example.

  Ex:

  ```
  Square:=proc(x) RETURN(x^2) end proc;
  Square:=x->x^2;
  ```

  These statements both define the same function.

- **Sequences, Sets, and Lists**

  A Sequence: $a_0, \ldots, a_n$;

  A List: $[a_0, \ldots, a_n]$;

  A Set: $\{a_0, \ldots, a_n\}$;

  The `seq()` command can construct sequences. To convert a sequence to a list or set enclose it in `[]` or `{}`. To convert a list to a sequence use `op()`. The `nops()` command tells you how many elements are in a list or set. The operators `member`, `union`, and `intersect` work with sets as you would expect.

  Examples:

  ```
  S:=1,2,3,4;    # S is a sequence
  L:=[1,2,3,4]; # L is a list
  A:={1,2,3,4}; # A is a set
  T:=seq(i,i=1..4); # T is the same sequence as S
  M:=[T];  # M is the same list as L
  B:={T};  # B is the same set as A
  ```

```
Z:={op(M)};  # Z is the same set as A and B
n:=nops(L); # n is assigned 4
```

- **Subscripts (indexes)**

  The `i`th element in a list `L` is `L[i]`.

  Example:

  ```
  L:=[2,4,6,8];
  n:=L[3]; # n is assigned 6
  ```

  To access a sublist, use a range:

  Example:

  ```
  L:=[2,4,6,8,10];
  n:=L[2..4]; # n is assigned [4,6,8]
  ```

# Metric Spaces

**Definition** *A **metric space** is a pair $(X,d)$ where $X$ is a set and $d : X \times X \longrightarrow \mathbb{R}$ such that for all $x,y,z \in X$ :*

1. $d(x,y) \geq 0$
2. $d(x,y) = 0 \Leftrightarrow x = y$
3. $d(x,y) = d(y,x)$
4. $d(x,y) + d(y,z) \geq d(x,z)$

*In this situation, d is called a **metric** (or distance function) on X, and the elements of X are called the **points** in the metric space.*

## Examples of Metric Spaces

**Example** *$(\mathbb{R}, d_{\mathrm{Euc}})$ is a metric space where $d_{\mathrm{Euc}}(x,y) = |x-y|$ for all $x,y \in \mathbb{R}$.*

Notice this is just a special case of the more general theorem:

**Theorem** *$(\mathbb{R}^n, d_{\mathrm{Euc}})$ is a metric space where*

$$d_{\mathrm{Euc}}((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$$

*$d_{\mathrm{Euc}}$ is called the **Euclidean metric** on $\mathbb{R}^n$.*

**Definition** *Let $d_{\mathrm{Taxi}} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ by*

$$d_{\mathrm{Taxi}}((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \sum_{i=1}^{n}|x_i-y_i|$$

*The map $d_{\mathrm{Taxi}}$ is called the **lattice metric**, the **Manhattan metric**, or the **taxicab metric**.*

**Definition** *Let $d_{\max} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ by*

$$d_{\max}((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = max\{|x_i-y_i| : i \in \{1,\ldots,n\}\}$$

*The map $d_{\max}$ is called the **maximum metric**.*

**Definition** *The set of **2-adic integers,** denoted $\mathbb{Z}_2$, is the set of all infinite sequences of $0$'s and $1$'s, i.e.*

$$\mathbb{Z}_2 = \{(s_0, s_1, \ldots) : \forall i \in \mathbb{N}, s_i \in \{0,1\}\}$$

**Definition** *Let $d_2 : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{R}$ by*

$$d_2((s_0, s_1, \ldots), (t_0, t_1, \ldots)) = \frac{1}{2^k}$$

*where $k = \min\{i \mid s_i \neq t_i\}$ if $(s_0, s_{1,} \ldots) \neq (t_0, t_1, \ldots)$ and*

$$d_2((s_0, s_{1,} \ldots), (t_0, t_1, \ldots)) = 0$$

*if $(s_0, s_{1,} \ldots) = (t_0, t_1, \ldots)$. The map $d_2$ is called the* ***2-adic metric****.*

**Theorem** *$(\mathbb{R}^n, d_{\text{Taxi}})$, $(\mathbb{R}^n, d_{\max})$, and $(\mathbb{Z}_2, d_2)$ are metric spaces.*

**Remark** *It is a fact that $(\mathbb{Z}_2, \mathrm{d}_2)$ cannot be embedded in $(\mathbb{R}^{\mathrm{n}}, \mathrm{d}_{\mathrm{Euc}})$ for any n. The 2-adic metric is simple to compute and work with, but the geometry of $(\mathbb{Z}_2, \mathrm{d}_2)$ is very strange.*

## Properties of Metric Spaces

**Definition** *Let $(X,d)$ be a metric space, $\delta \in \mathbb{R}^+$, and $x \in X$. Then*

$$B(x;\delta) = \{y \in X \mid d(x,y) < \delta\} \text{ and}$$
$$\overline{B}(x;\delta) = \{y \in X \mid d(x,y) \leq \delta\}$$

*$B(x;\delta)$ is called the* ***open ball of radius*** *$\delta$ centered at $x$, and $\overline{B}(x;\delta)$ is called the* ***closed ball of radius*** *$\delta$ centered at $x$.*

**Definition** *Let $(X,d)$ be a metric space and $U \subseteq X$. Then $U$ is* ***open*** *if and only if*

$$\forall\ x \in U,\ \exists\ \delta \in \mathbb{R}^+ \text{ such that } B(x;\delta) \subseteq U$$

**Definition** *Let $(X,d)$ be a metric space and $U \subseteq X$. Then $U$ is* ***closed*** *if and only if $X - U$ is open.*

**Remark** *There are sets which are neither open nor closed.*

**Definition** *Let $(X,d)$ be a metric space and $U \subseteq X$. Then $U$ is* ***bounded*** *if and only if $\exists \delta \in \mathbb{R}^+, \exists x \in X$, such that $U \subseteq B(x;\delta)$.*

**Definition** *Let $x_0, x_1, x_2, \ldots \in X$ and $(X,d)$ a metric space. Let $x \in X$. Then*

$$\lim_{n\to\infty} x_n = x \Leftrightarrow \forall \varepsilon \in \mathbb{R}^+,\ \exists N \in \mathbb{N}^+ \text{ such that } \forall n \in \mathbb{N}, n > N \Rightarrow d(x_n, x) < \varepsilon$$

*In this case we say that the sequence $x_0, x_1, x_2, \ldots$* ***converges*** *to the limit $x$ in $(X,d)$.*

**Definition** *Let $x_0, x_1, x_2, \ldots \in X$ and $(X,d)$ a metric space. Then the sequence $x_0, x_1, x_2, \ldots$ is called a* ***Cauchy Sequence*** *if and only if*

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{N}^+ \text{ such that } \forall i > N, \forall j > N,\ d(x_i, x_j) < \varepsilon$$

*i.e., the terms of the sequence get arbitrarily close to each other.*

**Definition** *Let $(X,d)$ be a metric space. Then $(X,d)$ is a* ***complete*** *metric space if and only if every Cauchy sequence in $(X,d)$ converges to a limit $x \in X$.*

**Example** *$(\mathbb{R}, d_{\text{Euc}})$ is complete. In fact, this is one of the axioms that define the real* *numbers.*

**Example** $(\mathbb{Q}, d_{\text{Euc}})$ *is not complete.*

**Definition** *Let* $(X,d)$ *be a metric space and* $U \subseteq X$. $U$ *is* ***compact*** *if and only if every open cover has a finite subcover, i.e. whenever* $\{U_i\}_{i\in I}$ *satisfies* $U \subseteq \cup_{i\in I} U_i$ *and* $\forall i \in I$, $U_i$ *is open, then* $\exists\ i_1, i_2, \ldots, i_k \in I$ *for which* $U \subseteq U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_k}$.

**Theorem** *(Heine-Borel) Let* $A \subseteq (\mathbb{R}^n, d_{\text{Euc}})$. *Then A is compact if and only if A is closed and A is bounded.*

## Continuity

**Definition** *Let* $(X,d)$ *and* $(Y,d')$ *be metric spaces and* $f : X \to Y$. *Then f is* ***continuous*** *with respect to the metrics d and* $d'$ *if and only if*

$$\forall U \subseteq Y, U \text{ is open in } (Y,d') \Rightarrow f^{-1}(U) \text{ is open in } (X,d).$$

**Remark** *In other words a function between metric spaces is continuous if and only if the inverse image of every open set is open.*

**Theorem** *Let* $(X,d)$ *be a complete metric space,* $f : X \to X$ *a continuous map, and* $x_0, x_1, x_2, \ldots$ *a convergent sequence in X with* $\lim_{n\to\infty} x_n = x$. *Then* $f(x_0), f(x_1), f(x_2), \ldots$ *is a convergent sequence and* $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$, *i.e., limits commute with continuous maps.*

## The Metric Space of Shapes

**Definition** *Let* $n \in \mathbb{N}^+$. *Define*

$$K_n = \{A \subseteq \mathbb{R}^n \mid A \text{ is compact}\}.$$

**Example** $K_2$ *is the set of all compact subsets in the plane.*

**Definition** *Let* $(X,d)$ *be a metric space,* $S \subseteq X$, *and* $\delta \in \mathbb{R}^+$. *The* ***open collar of radius*** $\delta$ *about S is the set* $B(S;\delta) = \cup_{\alpha\in S} B(\alpha;\delta)$ *and the* ***closed collar of radius*** $\delta$ *about S is the set* $\overline{B}(S;\delta) = \cup_{\alpha\in S} \overline{B}(\alpha;\delta)$.

**Example** *Sketch* $B(MrFace;\delta)$ *for various values of* $\delta$.

**Definition** *Let* $S \subseteq \mathbb{R}$ *and* $t \in \mathbb{R} \cup \{\infty\}$. *Then* $t = \sup(S)$ *if and only if* $\forall x \in S, x \le t$ *and* $\forall u \in \mathbb{R}$, *if* $\forall x \in S, x \le u$ *then* $u \ge t$. $\sup(S)$ *is called the* ***supremum*** *of S.*

**Example** $\sup(0\ldots 1) = 1$ *but* $(0\ldots 1)$ *has no maximum value.*

**Definition** *Let* $S \subseteq \mathbb{R}$ *and* $t \in \mathbb{R} \cup \{-\infty\}$. *Then* $t = \inf(S)$ *if and only if* $\forall x \in S$, $x \ge t$ *and* $\forall u$, *if* $\forall x \in S, x \ge u$ *then* $u \le t$. $\inf(S)$ *is called the* ***infimum*** *of S.*

**Remark** *If a set is closed and bounded, then* $\sup(\mathrm{S}) = \max(\mathrm{S})$ *and* $\inf(\mathrm{S}) = \min(\mathrm{S})$.

**Definition** *Let* $d_H : K_n \times K_n \to \mathbb{R}$ *by* $d_H(S,T) = \inf\{\delta \mid S \subseteq \overline{B}(T;\delta)$ *and* $T \subseteq \overline{B}(S;\delta)\}$. $d_H$ *is called the* ***Hausdorff metric***.

**Remark** *Pages 267-269 of your textbook has a good explanation of this metric.*

**Example** *Let* $S = \{(x,x) : x \in [0\ldots 1]\}$ *and* $T = \{p : d_{Euc}(p,(1,0)) \leq 0.2\}$ *be elements of* $K_2$. *Compute* $d_H(S,T)$.

**Theorem** $(K_n, d_H)$ *(or the metric space where fractals live) is a complete metric space.*

# Chaos

## Dynamical Systems - Take 2

**Definition** *Let* $(X,d)$ *be a metric space (or a topological space). Any function* $f : X \to X$ *is called a* ***discrete dynamical system****. To indicate the metric we sometimes write* $f : (X,d) \to (X,d)$

**Definition** *Let* $f : (X,d) \to (X,d)$, $g : \left(Y,d'\right) \to \left(Y,d'\right)$. *We say the dynamical systems* $f,g$ *are* ***conjugate*** *if and only if there exists* $h : X \to Y$ *such that*
*1.* $h$ *is a* ***homeomorphism*** *(a continuous bijection with continuous inverse) and*
*2.* $h \circ f = g \circ h$
*In this situation* $h$ *is called a* ***topological conjugacy*** *(or simply* ***conjugacy****) between* $f$ *and* $g$.

**Remark** *The study of discrete dynamical systems is the study of those properties which are preserved by conjugacy.*

**Definition** *Let* $(X,d)$ *be a metric space,* $f : X \to X$, *and let* $q$ *be a fixed point of* $f$. *Then* $q$ *is an* ***attracting fixed point*** *if and only if*

$$\exists \delta \in \mathbb{R}^+, \forall x \in B(q;\delta), \lim_{n\to\infty} f^n(x) = q$$

*i.e. if the* $f$*-orbit of every point in some ball centered at* $q$ *converges to* $q$.

**Definition** *Let* $(X,d)$ *be a metric space,* $f : X \to X$, *and let* $q$ *be a fixed point of* $f$. *Then* $q$ *is a* ***repelling fixed point*** *if and only if*

$$\exists \delta \in \mathbb{R}^+, \forall x \in B(q;\delta) - \{q\}, \exists N \in \mathbb{N}, f^N(x) \notin B(q;\delta)$$

*i.e. if the* $f$*-orbit of every point other than* $q$ *in some ball centered at* $q$ *contains a point outside the ball.*

**Definition** *Let* $(X,d)$ *be a metric space,* $f : X \to X$, *and let* $q$ *be a periodic point of* $f$ *with period* $n$. *We say the* $n$*-cycle containing* $q$ *is an* ***attracting cycle*** *(resp.* ***repelling cycle****) if and only if* $q$ *is an attracting (resp. repelling) fixed point of* $f^n$.

**Example** *Classify the fixed points of* $f : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = 3x$.

**Example** *Classify the fixed points of* $f : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = \frac{1}{2}x$.

**Theorem** *Attracting and repelling fixed points are presevered by topological conjugacy.*

### Graphical Analysis and Time Series Plots

**Definition** *Let* $f : \mathbb{R} \to \mathbb{R}$, *and* $x \in X$. *Then the* ***time series plot*** *of the orbit of* $x$ *is the* *graph of the points*

$$(0,x),(1,f(x)),(2,f^2(x)),\ldots,(k,f^k(x)),\ldots$$

**Remark** *Sometimes we connect the points with line segments to make them more visible.*

**Definition** *Let* $f:\mathbb{R}\to\mathbb{R}$, *and* $x\in X$. *Then the* ***graphical analysis*** *of the orbit of x is the graph consisting of:*

*(a) the graph of f*
*(b) the graph of* $y=x$
*(c) a line segment connecting* $(x_k,x_{k+1})$ *to* $(x_{k+1},x_{k+1})$ *for each* $k\in\mathbb{N}$
*(d) a line segment connecting to* $(x_{k+1},x_{k+1})$ *to* $(x_{k+1},x_{k+2})$ *for each* $k\in\mathbb{N}$

*where* $x_k=f^k(x)$.

**Remark** *Usually we connect these line segments in order starting from* $\mathrm{k}=0$, *drawing the segment in part (c) before part (d). It is often customary to add the segment from* $(\mathrm{x},0)$ *to* $(\mathrm{x},\mathrm{x}_1)$ *as an initial segment.*

**Example** *Draw the graphical analysis for the f-orbit of seeds* $0.23$ *and* $0.230001$ *for*

*(a)* $f(x)=\frac{1-x}{2}$
*(b)* $g(x)=2x$
*(c)* $h(x)=x^2-2$

## Devaney's Definition

**Definition** *A dynamical system* $f:(X,d)\to(X,d)$ *is said to be* ***transitive*** *if and only if*

$$\forall x,y\in X,\forall\varepsilon\in\mathbb{R}^+,\exists z\in B(x;\varepsilon),\exists k\in\mathbb{N},f^k(z)\in B(y;\varepsilon)$$

*i.e. for any* $\varepsilon\in\mathbb{R}^+$ *and for any two points in X there is a third point whose orbit passes within* $\varepsilon$ *of both points.*

**Remark** *Sometimes this property is called* ***mixing***.

**Definition** *A dynamical system* $f:(X,d)\to(X,d)$ *is said to have* ***sensitive dependence on initial conditions*** *if and only if*

$$\exists\delta\in\mathbb{R}^+,\forall x\in X,\forall\varepsilon\in\mathbb{R}^+,\exists y\in B(x;\varepsilon)-\{x\},\exists k\in\mathbb{N},d(f^k(x),f^k(y))>\delta$$

*i.e. there is a positive constant so that for any x there is a point y arbitrarily close to x such that the orbits of x and y will eventually be separated by at least the constant.*

**Definition** *Let* $(X,d)$ *be a metric space and* $A\subseteq X$. *Then A is* ***dense*** *in X if and only if*

$$\forall x\in X,\forall\varepsilon\in\mathbb{R}^+,B(x;\varepsilon)\cap A\neq\emptyset$$

*i.e. A is dense if every open ball contains a point of A.*

**Definition (Devaney)** *A discrete dynamical system is* ***chaotic*** *if and only if*

*1. it has dense periodic points,*
*2. it is transitive and*

*3. it has sensitive dependence on initial conditions.*

**Remark** *Chaotic maps give us a model for unpredicatable deterministic systems.*

## Touhey's Definition

**Theorem (Touhey 1997)** *A discrete dynamical system (on an infinite set) is chaotic if and only if every finite collection of open sets shares infinitely many periodic orbits.*

## Chaotic Maps

The following are examples of chaotic maps:

1. Quadratic maps

   For each $c \in \mathbb{C}$ define $Q_c(x) = x^2 + c$.

   a. $Q_{-2}(x) = x^2 - 2$ is chaotic on $[-2 \ldots 2]$.

   b. $Q_0(z) = z^2$ is chaotic on the unit circle.

   c. $Q_c$ is chaotic on a fractal set called $J_c$ (more later).

   d. The Logistic Map $Q(x) = 4x(1-x)$ is chaotic on $[0 \ldots 1]$.

2. The Doubling Map

$$D(x) = \begin{cases} 2x & \text{if } x \in [0 \ldots 1/2) \\ 2x - 1 & \text{if } x \in [1/2 \ldots 1] \end{cases}$$

   is chaotic on $[0 \ldots 1]$

3. The Tent Map

$$T(x) = 1 - |2x - 1|$$

   is chaotic on $[0 \ldots 1]$.

4. (J. Joseph) The Extended Collatz Map

$$T(z) = \begin{cases} \frac{1}{2}z & \text{if } z \underset{2}{\equiv} 0 \\ \frac{3z+1}{2} & \text{if } z \underset{2}{\equiv} 1 \\ \frac{3z+i}{2} & \text{if } z \underset{2}{\equiv} i \\ \frac{3z+1+i}{2} & \text{if } z \underset{2}{\equiv} 1+i \end{cases}$$

   is chaotic on $\mathbb{Z}_2[i]$.

# The Contraction Mapping Theorem

**Definition** *Let $(X,d)$ be a metric space and $f : X \to X$. Then $f$ is called a **contraction mapping** if and only if $\exists s \in (0 \ldots 1), \forall x,y \in X, d(f(x),f(y)) \le s\,d(x,y)$. In this situation $s$ is called a **contraction factor** of $f$.*

**Theorem** *Every contraction mapping is continuous.*

**Theorem (The Derivative Test)** *Let $I = (a \ldots b) \subseteq \mathbb{R}$ and $f : I \to I$ differentiable* *on I. If* $\exists s \in (0\ldots1)$ *such that* $\forall x \in I,\ \left|f'(x)\right| \leq s < 1$, *then f is a contraction mapping with contraction factor* $s$.

**Theorem (The Contraction Mapping Theorem)** *Let* $f : X \to X$ *be a contraction mapping on a complete metric space* $(X,d)$ *with contraction factor* $s$.

*(1) f has a unique fixed point, q.*

*(2) The f-orbit of every element of X converges to q*
*(i.e.,* $\forall x \in X,\ \lim_{n\to\infty} f^n(x) = q$*).*

*(3) If* $x_0, x_1, x_2, \ldots$ *is the f-orbit of* $x_0 \in X$ *then*

$$d(x_n,q) \leq \frac{s^n}{1-s} d(x_0,x_1)$$

*for all* $n \in \mathbb{N}$.

**Remark** *Every contraction map has an attracting fixed point.*

## Hutchinson Operators

**Definition** *Let* $w_0, w_1, \ldots, w_k$ *be contraction mappings on* $\mathbb{R}^n$ *with contraction factors* $c_0, c_1, \ldots, c_k$ *respectively and define* $W : K_n \to K_n$ *by*

$$W(A) = w_0(A) \cup w_1(A) \cup \cdots \cup w_k(A).$$

*W is called the* ***Hutchinson operator*** *associated with* $w_0, w_1, \ldots, w_k$ *and we write* $W = Hutch(w_0, w_1, \ldots, w_k)$.

**Theorem (Hutchinson)** *W is a contraction mapping on* $(K_n, d_H)$ *with contraction factor* $c = \max\{c_0, c_1, \ldots, c_k\}$.

**Definition** *If W is a Hutchinson operator then the unique fixed point of W is called the* ***attractor of*** *W and is denoted* $F_W$.

**Remark** *Applying the three parts of the contraction mapping theorem to* W *gives us a lot of information about producing fractals with Hutchinson operators.*

# Iterated Function Systems

## Complex Numbers

**Definition** *Let* $\mathbb{C} = \mathbb{R}^2$. *For each* $(x,y) \in \mathbb{C}$ *we formally write* $(x,y) = x + yi$. *This form,* $x + yi$, *is called the* ***standard form*** *of the complex number* $(x,y)$.

**Definition** *Let* $x + yi, a + bi \in \mathbb{C}$, *then:*

*1.* $\overline{x+yi} = x - yi$. *(This is called the* ***complex conjugate****.)*
*2.* $|x+yi| = \sqrt{x^2+y^2}$ . *(This is called the* ***complex norm****.)*
*3.* $\operatorname{Arg}(x+yi) =$ *the angle in* $[0\ldots2\pi)$ *of* $(x,y)$ *in polar form (not defined for* $x = y = 0$*) . (This is called the* ***Argument*** *of* $x + yi$.*)*
*4.* $\operatorname{Re}(x+yi) = x$. *(This is called the* ***real part*** *of* $x + yi$.*)*
*5.* $\operatorname{Im}(x+yi) = y$. *(This is called the* ***imaginary part*** *of* $x + yi$.*)*
*6.* $(x+yi) + (a+bi) = (x+a) + (y+b)i$. *(This is the definition of* ***addition*** *in* $\mathbb{C}$.*)*
*7.* $(x+yi)(a+bi) = (xa-yb) + (ya+xb)i$. *(This is the definition of* ***multiplication*** *in* $\mathbb{C}$.*)*

**Notation** *We can abbreviate* $0+yi$ *as* $yi$, $x+0i$ *as* $x$, $x+1i$ *as* $x+i$*, and* $x-1i$ *as* $x-i$ *with no ambiguity in the above definitions. With this notation* $i=(0,1)$ *and* $i^2=-1$*. It is easy to verify that the usual laws of addition and multiplication (associative, commutative, distributive, identity, etc.) hold for the complex numbers as well.*

**Definition** *Let* $\theta \in \mathbb{R}$*. Then* $e^{i\theta} = \cos\theta + i\sin\theta$

**Definition** *Let* $x+yi \in \mathbb{C}-\{0\}$*. The **standard polar form** of* $x+yi$ *is* $re^{i\theta}$ *where* $r=|x+yi|$ *and* $\theta = \operatorname{Arg}(x+yi)$.

**Theorem** $e^{i\pi}+1=0$ *(The most beautiful theorem in mathematics?)*

**Theorem** *Let* $\theta,\gamma \in \mathbb{R}$

1. $e^{i\theta}e^{i\gamma} = e^{i(\theta+\gamma)}$.
2. $|e^{i\theta}| = 1$.
3. $\overline{e^{i\theta}} = e^{i(-\theta)}$.

**Theorem** *Let* $z, z_1, z_2 \in \mathbb{C}$*. Then:*

1. $|z_1 z_2| = |z_1||z_2|$
2. $d_{\text{Euc}}(z_1,z_2) = |z_2 - z_1|$
3. $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$ *i.e. the conjugate of a product is the product of conjugates.*
4. $\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$ *i.e. the conjugate of a sum is the sum of the conjugates.*
5. $z\,\overline{z} = |z|^2$
6. $|z| = |\overline{z}|$
7. *If* $z=re^{i\theta}$ *in polar form, then* $\overline{z} = re^{i(-\theta)}$

## Affine Maps

**Definition** *Define* $M_{m,n}(\mathbb{R})$ *to be the set of all* $m\times n$ *matrices with real number entries. Let* $A\in M_{m,n}(\mathbb{R})$ *and* $i\in \mathbb{I}_m$, $j \in \mathbb{I}_n$*. Then* $A\langle i,j\rangle$ *is the entry in the* $i^{th}$ *row and* $j^{th}$ *column of* $A$*. If* $c\in\mathbb{R}$ *then* $cA \in M_{m,n}(\mathbb{R})$ *and*

$$(cA)\langle i,j\rangle = c(A\langle i,j\rangle)$$

*If* $B\in M_{m,n}(\mathbb{R})$ *then* $A+B \in M_{m,n}(\mathbb{R})$ *and*

$$(A+B)\langle i,j\rangle = A\langle i,j\rangle + B\langle i,j\rangle.$$

*If* $B \in M_{n\times p}(\mathbb{R})$ *then* $AB\in M_{m,p}(\mathbb{R})$ *and for all* $i\in\mathbb{I}_m$, $j\in\mathbb{I}_p$

$$AB\langle i,j\rangle = \sum_{k=1}^{n} A\langle i,k\rangle B\langle k,j\rangle.$$

**Remark** *We often identify elements of* $\mathbb{R}^n$ *with elements in* $M_{1,n}(\mathbb{R})$ *and* $M_{n,1}(\mathbb{R})$ *when appropriate.*

**Definition** $T:\mathbb{R}^n \to \mathbb{R}^n$ *is called an **affine map** or **affine transformation** if and only if* $T(x) = Mx+B$ *for some* $n\times n$ *matrix* $M$ *and* $B\in\mathbb{R}^n$.

**Remark** *We will mostly restrict our attention to affine maps on* $\mathbb{R}^2$ *in this course.*

**Theorem** *An affine transformation on* $\mathbb{R}^2$ *is completely determined by where it maps*

*3 non-collinear points.*

**Representations of Affine Maps on $\mathbb{R}^2$**

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an affine map, and let $p \in \mathbb{R}^2$. Let $x,y \in \mathbb{R}$ such that $p = (x,y)$ and let $z = x + yi$. Then there exist a $2 \times 2$ real matrix $M$, $B \in \mathbb{R}^2$, $\alpha,\beta,\gamma \in \mathbb{C}$, and $a,b,c,d,e,f,r,s,\theta,\phi \in \mathbb{R}$ such that:

| Form Name | Math Notation |
|---|---|
| Matrix | $T(p) = Mp + B$ |
| Standard | $T(x,y) = (ax + by + e,\ cx + dy + f)$ |
| Geometric | $T(x,y) = (r\cos(\theta)x - s\sin(\phi)y + e, r\sin(\theta)x + s\cos(\phi)y + f)$ |
| Complex | $T(z) = \alpha z + \beta\overline{z} + \gamma$ |

These are expressed in the *chaos* Maple package in the following notation:

| Form Name | Maple Notation |
|---|---|
| Matrix | *affineM*$(M,B)$ |
| Standard | *affine*$(a,b,c,d,e,f)$ |
| Geometric | *Affine*$(r,s,\theta,\phi,e,f)$ |
| Complex | *affineC*$(\alpha,\beta,\gamma)$ |

An IFS (see below) in Maple, is a Maple list of one or more of these affine maps. Note that in the Geometric Form, the Maple program assumes that $\theta,\phi$ are in degrees, not radians.

Each form has its own advantages. We can convert from any form to any other form. It suffices to give the formulas for converting between any form and standard form. Thus, if you are given Matrix form and want to convert to Complex form, first convert to standard and then to Complex.

**Theorem** *In the above notation:*

***Converting Standard form to Matrix form and vice-versa:***

$affine(a,b,c,d,e,f) = affineM(M,B) \Leftrightarrow$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} e \\ f \end{pmatrix}$$

***Converting Standard form to Complex form and vice-versa:***

$affine(a,b,c,d,e,f) = affineC(A + Bi,\ C + Di,\ E + Fi) \Leftrightarrow$

$$\begin{aligned} a &= A + C \quad \textit{and} \quad & A &= \tfrac{1}{2}(a + d) \\ b &= D - B & B &= \tfrac{1}{2}(c - b) \\ c &= B + D & C &= \tfrac{1}{2}(a - d) \\ d &= A - C & D &= \tfrac{1}{2}(c + b) \\ e &= E & E &= e \\ f &= F & F &= f \end{aligned}$$

***Converting Standard form to Geometric form and vice-versa:***
$\mathit{affine}(a,b,c,d,e,f) = \mathit{Affine}(r,s,\theta,\phi,E,F) \Leftrightarrow$

$$\begin{aligned} a &= r\cos(\theta) \quad \textit{and} \quad & r &= \sqrt{a^2 + c^2} \\ b &= -s\sin(\phi) & s &= \sqrt{b^2 + d^2} \\ c &= r\sin(\theta) & \theta &= \arctan(\tfrac{c}{a}) \\ d &= s\cos(\phi) & \phi &= \arctan(\tfrac{d}{b}) - 90^\circ \\ e &= E & E &= e \\ f &= F & F &= f \end{aligned}$$

**Remark** *The effect of the affine map* $\mathrm{Affine}(r, s, \theta, \phi, E, F)$ *on a geometric figure is as follows:*

| | |
|---|---|
| r | *scales the figure horizontally by a factor of* $\lvert r\rvert$ <br> *(if* r *is negative, it also reflects the figure across the* y*-axis)* |
| s | *scales the figure vertically by a factor of* $\lvert s\rvert$ <br> *(if* s *is negative, it also reflects the figure across the* x*-axis)* |
| θ | *rotates horizontal lines by* θ *degrees CCW about the point where they intersect the* y*-axis* |
| ϕ | *rotates vertical lines by* ϕ *degrees CCW about the point where they intersect the* x*-axis* |
| e | *translates the figure horizontally by an amount* e |
| f | *translates the figure vertically by an amount* f |

*Note that if* $\theta = \phi$, *then the effect of both numbers combined is to rotate the entire figure about the origin by an angle* $\theta$ *counterclockwise (CCW). Negative angles rotate clockwise (CW) instead of counterclockwise. Also note that* $\mathrm{Affine}(r, s, \theta, \phi, e, f)$ *always sends the origin,* $(0, 0)$, *to the point* $(e, f)$.

**Contraction Factor for Affine Maps**

**Theorem** *Let* $\alpha, \beta, \gamma \in \mathbb{C}$ *and* $c = |\alpha| + |\beta|$. *Then the map* $T = \mathit{affineC}(\alpha, \beta, \gamma)$ *is a* *contraction mapping if and only if $c < 1$. Further if T is a contraction mapping then c is a contraction factor for T.*

## IFS's

**Definition** *A Hutchinson operator $W = Hutch(w_0, w_1, \ldots, w_k)$ such that $w_0, w_1, \ldots, w_k$ are all affine maps is called an* ***iterated function system*** *or IFS. We write $W = [w_0, w_1, \ldots, w_k]$ in this case.*

**Remark** *$\{$contraction affine mappings$\} \subseteq \{$affine mappings$\}$. But there are many contraction mappings that aren't affine. For example, $\mathrm{f(x)} = \frac{1}{2}\cos(\mathrm{x})$ on $\mathbb{R}$, or $\mathrm{f(x)} = \sqrt{\mathrm{x}}$ on $[1\ldots\infty)$.*

**Remark** *A stick figure generator that contains finitely many directed segments whose associated affine maps are contraction maps (and no ordinary segments) is an example of a IFS. The attractor of the IFS can be obtained by iterating the stick figure dynamical system starting with a single directed segment as the seed.*

**Remark** *Both HeeBGB's and GB's are IFS's which map the original square onto the appropriate subsquares in the given manner. Coloring a HeeBGB is just producing the attractor of the IFS by the Deterministic Method (see below) where we color the background, not the image itself.*

## The Deterministic Method

**Algorithm** *To draw the attractor of an IFS, W, simply compute the terms in the W-orbit of any seed in $K_n$ until the image is as close to the attractor as you desire.*

**Remark** *By the contraction mapping theorem, the IFS $\mathrm{W}$ has an attractor, no matter what shape we start with for a seed, its orbit will converge to the attractor, and we can compute the number of iterations required to obtain an image that is within any desired accuracy of the attractor.*

## Guess My IFS

**Remark** *Let $\mathrm{W} = [\mathrm{w}_0, \mathrm{w}_1, \ldots, \mathrm{w}_k]$ be an IFS. By the contraction mapping theorem, $\mathrm{F_W}$ is the unique fixed point of $\mathrm{W}$, i.e. $\mathrm{F_W}$ is the only element of $\mathrm{K_n}$ such that*

$$\mathrm{W}(\mathrm{F_W}) = \mathrm{F_W}.$$

*By the definition of $\mathrm{W}$ this means that $\mathrm{F_W}$ is the unique solution $\mathrm{A}$ in $\mathrm{K_n}$ to the equation:*

$$\mathrm{A} = \mathrm{w}_0(\mathrm{A}) \cup \mathrm{w}_1(\mathrm{A}) \cup \cdots \cup \mathrm{w}_k(\mathrm{A}).$$

*In particular,*

$$\mathrm{F_w} = \mathrm{w}_0(\mathrm{F_w}) \cup \mathrm{w}_1(\mathrm{F_w}) \cup \cdots \cup \mathrm{w}_k(\mathrm{F_w})$$

*so that the attractor is a union of finitely many affine images of itself (each of which must therefore be a finite union of strictly smaller affine images of itself, and so on ad infinitum).*

*Thus, given the attractor of an IFS, we can determine an IFS that produces it (not unique!) by identifying a finite number of strictly smaller affine images if the* *attractor whose union is the entire shape.*

# Address My IFS

**Definition** *Let $n \geq 2$. Define $\Sigma_n$ to be the set of all infinite sequences whose terms are in $\{0,1,2,\ldots,n-1\}$, i.e.*

$$\sum_n = \{s_0, s_1, s_2, \ldots | s_i \in \{0,1,2,\ldots,n-1\}\}$$

*$\Sigma_n$ is called the **sequence space on** n **letters**.*

**Example** *$\Sigma_2$ is the set of 2-adic integers $\mathbb{Z}_2$.*

**Definition** *Let $W = [w_0, w_1, \ldots, w_{n-1}]$ be an IFS and let $F_w$ be the attractor of $W$. Define the **address map** $\Phi : \Sigma_n \to F_w$ by*

$$\Phi(s_0 s_1 s_2 \ldots) = \bigcap_{i=0}^{\infty} w_{s_0} \circ w_{s_1} \circ \cdots \circ w_{s_i}(F_w)$$

**Theorem** *Let $W = [w_0, w_1, \ldots, w_{n-1}]$ be an IFS.*
*1. $\Phi(s)$ is a single point in $F_w$ for any $s = s_0 s_1 s_2 \ldots \in \sum_n$.*
*2. $\Phi$ is onto.*
*3. $\lim_{i\to\infty} w_{s_0} \circ w_{s_1} \circ \cdots \circ w_{s_i}(F_w) = \{\Phi(s)\}$ in $(K_n, d_H)$*
*4. $\lim_{i\to\infty} w_{s_0} \circ w_{s_1} \circ \cdots \circ w_{s_i}(x) = \Phi(s)$ for any $x \in \mathbb{R}^m$ in $(\mathbb{R}^m, d_{\mathrm{Euc}})$*

**Definition** *$s$ is called an **address** of the point $\Phi(s)$ in the attractor.*

**Definition** *Let $W$ be an IFS and $F_w$ its attractor. Then $W$ is said to be **totally disconnected** $\Leftrightarrow$ every point in $F_w$ has a unique address, i.e. $\Phi$ is bijective.*

**How big is the Attractor?**

**Definition** *Let $S \in K_2$ be a compact set. Then the **diameter** of $S$ is the real number*

$$\mathrm{diam}(S) = \sup\{d_{\mathrm{Euc}}(x,y) \,|\, x,y \in S\}.$$

**Example** *The diameter of a circle is a special case of this definition.*

**Remark** *You will prove the following for homework: Let $\mathrm{W} = [\mathrm{w}_0, \ldots, \mathrm{w}_{n-1}]$ be an IFS and $\mathrm{c}_0, \ldots, \mathrm{c}_{n-1}$ the contraction factors of $\mathrm{w}_0, \ldots, \mathrm{w}_{n-1}$ respectively. Let $\mathrm{a}$, $\mathrm{b} \in \mathbb{R}^k$ and $\mathrm{c} = \max\{\mathrm{c}_0, \ldots, \mathrm{c}_{n-1}\}$. Let $\mathrm{a} = \Phi(\mathrm{t}_1, \ldots, \mathrm{t}_m, \mathrm{t}_{m+1}, \ldots)$ and $\mathrm{b} = \Phi(\mathrm{t}_1, \ldots, \mathrm{t}_m, \mathrm{t}'_{m+1}, \ldots)$ then*

$$\mathrm{d}_{\mathrm{Euc}}(\mathrm{a}, \mathrm{b}) \leq \mathrm{c}^{\mathrm{m}} \,\mathrm{diam}(\mathrm{F}_{\mathrm{W}}).$$

*i.e. if two points have addresses that agree in the first $\mathrm{m}$ digits, then they will be no further than $\mathrm{c}^{\mathrm{m}}\,\mathrm{diam}(\mathrm{F}_{\mathrm{W}})$ apart.*

**Theorem (Monks)** *Let $W = [w_0, \ldots, w_n]$ be an IFS, $c_0, \ldots, c_n$ the contraction factors of $w_0, \ldots, w_n$ respectively, and $q_0, \ldots, q_n$ the fixed points of $w_0, \ldots, w_n$ respectively. Define $c = \max\{c_0, \ldots, c_n\}$ and $r = \max\left\{ d(q_i, q_j) : i,j \in \mathbb{O}_n \right\}$. Then for any $a \in F_W$ and any $i \in \mathbb{O}_n$,*

$$d_{Euc}(a, q_i) \leq \frac{1}{1-c} r.$$

**Corollary** $F_W \subseteq \bigcap_{i=0}^{n} \overline{B}\left(q_i; \frac{r}{1-c}\right)$

## The Shift Map

**Definition** *Let* $W = [w_0, \ldots, w_{n-1}]$ *be a totally disconnected IFS. Define* $\sigma : F_W \to F_W$ *by* $\sigma(\Phi(s_0 s_1 s_2 \ldots)) = \Phi(s_1 s_2 s_3 \ldots)$*. Then* $\sigma$ *is called the* ***shift map*** *on* $F_W$*.*

**Theorem** *A shift map is chaotic!*

## The Address Method

**Algorithm** *To draw the attractor of an IFS,* $W = [w_0, \ldots, w_n]$*, start with a fixed point, q, of one of the maps* $w_0, \ldots, w_n$*. Use the preceeding theorem and remark to determine m such that any two points with addresses that agree on the first m digits will be less than a pixel width apart. For each finite sequence* $t_1, \ldots, t_m$ *in* $\mathbb{O}_n^m$ *plot the point*

$$w_{t_1} \circ w_{t_2} \circ \cdots \circ w_{t_m}(q).$$

**Remark** *There are* $(n+1)^m$ *sequences in* $\mathbb{O}_n^m$ *so that sometimes this method may be limited by the number of points you can compute and plot.*

## The Random Iteration Method

**Remark** *If we choose* $t_1, t_2, t_3, \ldots$ *at random from* $\mathbb{O}_n$*, then it is very likely that every finite sequence of any given length will eventually occur as a subsequence of our choices.*

**Algorithm** *To draw the attractor of an IFS,* $W = [w_0, \ldots, w_n]$*. Start with a fixed point, q, of one of the maps* $w_0, \ldots, w_n$*. This is your current point.*
*1. Choose a random number i from* $\mathbb{O}_n$ *and plot* $w_i$ *of the current point. This becomes the new current point.*
*2. Iterate!*

**Remark** *You can actually start with any point you like, not necessarily a fixed point, but starting with the fixed point guarentees that all points you plot will be in the attractor, not just near to it after a sufficient number of iterations.*

# Applications

## Fractal Randomness Testing

**Algorithm** *Given a sequence of values s whose terms are in* $\mathbb{O}_3$*, draw the attractor of* $HeeBGB(Up, Up, Up, Up)$ *by the random iteration method, using s as the source of the "random" numbers.*

**Remark** *Since the attractor of* $\mathrm{HeeBGB(Up, Up, Up, Up)}$ *is the filled-in unit square, if the sequence is missing any addresses there will be holes in the attractor where the points having those addresses would be.*

## Fractal Curves

**Definition** *Let b be an integer greater than* 1. *Then the* ***Base*** $b$ ***Ruler IFS*** *is* $W = [w_0, w_1, \ldots, w_{b-1}]$ *where* $w_i : \mathbb{R} \to \mathbb{R}$ *by*

$$w_i(x) = \frac{1}{b}x + \frac{i}{b}$$

*for* $i \in \mathbb{O}_{b-1}$.

**Theorem** *The attractor of any Base b Ruler IFS is* $[0\ldots 1]$. *For each* $t \in [0\ldots 1]$, $t = 0.t_1t_2t_3\cdots_{(b)}$ *if and only if* $t = \Phi(t_1t_2t_3\cdots)$, *i.e. the digits in the base b representation of t and an address of t are the same sequence.*

**Example** *Determine the points in the Middle Thirds Cantor Set.*

**Example** *Determine the coordinates of all points in the right Sierpinski triangle.*

**Theorem** *Let* $W = [w_0, \ldots, w_{n-1}]$ *be an IFS and define*

$$f_\Phi(0.t_1t_2t_3\cdots_{(n)}) = \Phi(t_1t_2t_3\cdots)$$

*Then* $f_\Phi$ *will be a function from* $[0..1] \to F_w$ *if and only if* $\Phi(s_1s_2\cdots) = \Phi(t_1t_2\cdots)$ *whenever* $0.s_1s_2s_3\cdots_{(n)} = 0.t_1t_2t_3\cdots_{(n)}$.

**Theorem** *If* $f_\Phi$ *is a function, then it is continuous.*

**Theorem** ***(Barnsley):*** *Let* $W = [w_0, \ldots, w_{n-1}]$ *be an IFS and* $F_w$ *its attractor. If there exist distinct points* $\{(s_i, t_i) \in F_w \mid i \in \{0\ldots n\}\}$ *such that for* $i \in \mathbb{O}_{n-1}$

1. $w_i(s_0, t_0) = (s_i, t_i)$

*and*

2. $w_i(s_n, t_n) = (s_{i+1}, t_{i+1})$

*then the map* $f_\Phi$ *is a continuous map from* $f : [0..1] \to F_w$.

**Example** *See Lecture Examples Maple worksheet.*

## Fractal Interpolation

**Definition** *A set of* ***data*** *is a collection* $\{(x_i, y_i) \in \mathbb{R}^2 \mid i \in \{0, 1, \ldots, n\}$ *and* $x_0 < x_1 < \cdots < x_n\}$. *An* ***interpolation function*** *for a given set of data is a continuous map* $f : [x_0 \ldots x_n] \to \mathbb{R}$ *such that* $f(x_i) = y_i$ *for all* $i \in \{0, 1, \ldots n\}$, *i.e its graph passes through all of the data points.*

**Example** *Linear interpolation, cubic spline, polynomial, etc.*

**Definition** *Let* $(x_0, y_0), \ldots, (x_n, y_n)$ *be a set of data with* $n > 2$. *Then a* ***Barnsley Interpolation Function*** *is an IFS* $W = [w_0, w_2, \ldots, w_{n-1}]$ *such that for all* $i \in \mathbb{O}_{n-1}$,

1. $w_i\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ c_i & d_i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix}$

2. $w_i\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$ *and* $w_i\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix}$

**Remark** *Simply put, we connect the data points with the chins of Mr Face, keeping* *the sides of his head vertical. The seed is the cousin of Mr Face whose chin connects the first and last data points. We choose the* $d_i$*'s with* $|d_i| < 1$ *to vary the fractal dimension (ruggedness) of the interpolation graph. Choosing* $|d_i| < 1$ *guarentees that our function is a contraction mapping.*

**Theorem** *Let W be the IFS in the previous definition. Then for each* $i \in \mathbb{O}_{n-1}$,

$$w_i = \text{affine}\left(\frac{x_{i+1} - x_i}{x_n - x_0}, 0, \frac{y_{i+1} - y_i}{x_n - x_0} - d_i \frac{y_n - y_0}{x_n - x_0}, d_i, \frac{x_n x_i - x_0 x_{i+1}}{x_n - x_0}, \frac{x_n y_i - x_0 y_{i+1}}{x_n - x_0} - d_i \frac{x_n y_0 - x_0 y_n}{x_n - x_0}\right)$$

*Furthermore,* $F_W$ *is the graph of an interpolation function.*

# Dimension

## Topological Dimension

### Topological Background

**Definition** *Let* $(X,d)$ *and* $(Y,d')$ *be metric spaces. Then* $(X,d)$ *is said to be* ***homeomorphic*** *to* $(Y,d')$ *if* $\exists : f : X \to Y$ *such that f is bijective, continuous, and its inverse is continuous. In this case, f is said to be a* ***homeomorphism***.

**Definition** *Any property of a metric space which is preserved by homeomorphisms is called a* ***topological invariant****, i.e. if P is a topological invariant, and if* $(X,d)$ *and* $(Y,d')$ *are homeomorphic, then* $P(X,d) \Leftrightarrow P(Y,d')$.

**Definition** *Let* $(X,d)$ *be a metric space and* $U \subseteq X$. *Then the* ***interior*** *of U is* $U^\circ = \{ x \in U \mid \exists \delta > 0, B(x;\delta) \subseteq U \}$. *Also, the* ***boundary*** *of U is* $\partial U = X - U^\circ - (X-U)^\circ$, *i.e. we take away the interior of the set and the interior of the complement to get the boundary.*

**Theorem** $U^\circ$ *is open for any set.*

**Theorem** *A set S is open* $\Leftrightarrow S = S^\circ$.

### Topological Dimension

**Definition** *Let* $U \subseteq \mathbb{R}^m$. *Define the* ***topological dimension*** *of* $U$, *denoted* $\dim_T(U)$, *to be the integer given by*
*1.* $\dim_T(\phi) = -1$ *and* $\phi$ *is the only subset A of* $\mathbb{R}^m$ *for which* $\dim_T(A) = -1$.
*2.* $\dim_T(U) \leq n$ *if and only if for any* $x \in U$ *and any open set* $W \subseteq U$ *containing x, there exists an open set V with* $x \in V \subseteq W$ *such that the topological dimension* $\dim_T(\partial V) \leq n-1$. *[Note: V, W must be open in U as a metric subspace of* $\mathbb{R}^m$*, but not necessarily open as a subset of* $\mathbb{R}^m$*. Similary,* $\partial V$ *refers to the boundary of V in U, not in* $\mathbb{R}^m$*.]*
*Then* $\dim_T(U) = n$ *if and only if* $\dim_T(U) \leq n$ *but* $\dim_T(U) \nleq n-1$.

**Remark** $\dim_T(U)$ *is always an integer by definition.*

## Hausdorff Dimension

**Example** *Consider the following*

| *Figure:* | *a point* | *unit segment* | *Sierpinski triangle* | *unit square* | *unit cube* |
|---|---|---|---|---|---|
| *# of pts:* | 1 | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| *Length:* | 0 | 1 | $\infty$ | $\infty$ | $\infty$ |
| *Area:* | 0 | 0 | 0 | 1 | $\infty$ |
| *Volume:* | 0 | 0 | 0 | 0 | 1 |

*Notice that none of these standard measures give a nontrivial value for the Sierpinski triangle.*

**Definition** *Let* $\varepsilon \in \mathbb{R}^+, 0 \le p < \infty,$ *and* $A \subseteq \mathbb{R}^2$ *a bounded subset of the plane. Then*

$$M_p(A;\varepsilon) = \inf\left\{\sum_{i=1}^{\infty} diam(A_i)^p \,:\, A = \cup_{i=1}^{\infty} A_i \text{ and } \forall i, diam(A_i) < \varepsilon\right\}$$

*and*

$$M_p(A) = \sup\{M_p(A;\varepsilon) \,:\, \varepsilon \in \mathbb{R}^+\}.$$

$M_p(A)$ *is called the Hausdorff p-measure of A.*

**Remark** $M_p(A)$ *can equal* $\infty$. $M_p(A;\varepsilon)$ *is obviously a nonincreasing function of* $\varepsilon$, *so* $M_p(A)$ *is the same as* $\lim_{\varepsilon \to 0} M_p(A;\varepsilon)$.

**Theorem** *For each A,* $\exists d \in \mathbb{R}$ *such that* $M_p(A) = \infty$ *for* $p < d$ *and* $M_p(A) = 0$ *for* $p > d$,

**Definition** *The number d in the previous theorem is called the* ***Hausdorff dimension of*** *A, and written* $d = \dim_H(A)$.

**Similarity and Congruence**

**Definition** *Let* $(X,d)$ *be a metric space. A* ***similitude*** *(or* ***similarity map****) is a surjective map* $f : X \to X$ *such that*

$$\exists c \in \mathbb{R}^+, \forall x,y \in X, d(f(x),f(y)) = cd(x,y).$$

*In this case c is called the* ***similarity factor*** *(or* ***scaling factor*** *or* ***ratio of similarity****).*

**Definition** *Let* $(X,d)$ *be a metric space and* $A,B \subseteq X$. *Then A is* ***similar*** *to B if* $B = f(A)$ *for some similitude f.*

**Theorem** $f : \mathbb{R}^2 \to \mathbb{R}^2$ *is a similitude with scaling factor c if and only if* $f = \text{Affine}(r,s,\theta,\phi,e,f)$ *with* $|r| = |s| = c$ *and* $\theta = \phi + \pi k$ *for some* $k \in \mathbb{Z}$.

**Theorem** *If f is a similitude with scaling factor c, then f is bijective and* $f^{-1}$ *is a similitude with scaling factor* $1/c$.

**Definition** *Let* $(X,d)$ *be a metric space. Then* $f : X \to X$ *is called an* ***isometry*** *if and only if f is a similitude with scaling factor equal to 1.*

**Remark** *In other words* f *is an isometry if*
*1.* f *is surjective and*
*2.* $\forall x,y \in X,\ d(f(x),f(y)) = d(x,y)$, *i.e.* f *preserves all distances.*

**Definition** *Let $(X,d)$ be a metric space and $A,B \subseteq X$. Then $A$ is* ***congruent*** *to $B$ if $B = f(A)$ for some isometry $f$.*

**Corollary** *If $f$ is an isometry, then $f$ is bijective and $f^{-1}$ is an isometry.*

**Theorem** $f : R^2 \to R^2$ *is an isometry iff* $f = \mathrm{Affine}(r,s,\theta,\phi,e,f)$ *with* $|r| = |s| = 1$ *and* $\theta = \phi + \pi k$ *for some* $k \in \mathbb{Z}$.

**Self-similarity and Dimension**

**Definition** *Let $W = [w_0, w_1, \ldots, w_{n-1}]$ be an IFS and $F_w$ its attractor. Then $W$ is said to be* ***just touching*** *$\Leftrightarrow$ it is not totally disconnected and there exists and open set $U \in \mathbb{R}^2$ such that*
*1. $W(U) \subseteq U$ and*
*2. $w_i(U) \cap w_j(U) = \phi$ for all $i,j \in [0,1,\ldots,n-1]$ for $i \neq j$.*

**Definition** *Let $W$ be an IFS. Then $W$ is said to be* ***overlapping*** *$\Leftrightarrow$ it is not totally disconnected and not just touching.*

**Example** *The middle thirds Cantor set is totally disconnected.*

**Example** *The Sierpinski triangle is just touching.*

**Example** *A trivial example of an overlapping IFS is an IFS containing the same map twice.*

**Definition** *Let $W = [w_0, w_1, \ldots, w_{n-1}]$ be an IFS. $F_w$ is said to be* ***self-similar*** *$\Leftrightarrow$ $F_w$ is not overlapping and each $w_i$ is a similarity. $F_w$ is* ***strictly self-similar*** *if and only if the similarity factors are all equal.*

**Definition** *If $F_w$ is a strictly self-similar attractor of a non-overlapping IFS $W = [w_0, w_1, \ldots, w_N]$, then the* ***similarity dimension*** *of $F_w$ is defined to be the unique number d such that $N = (\frac{1}{r})^d$ where r is the similarity factor (= contraction factor for the affine maps). We write $\dim_S(F_w) = d$ in this case.*

**Remark** *If we solve the defining equation for* d *we obtain*

$$\dim_S(F_w) = \frac{\ln N}{\ln(\frac{1}{r})}$$

**Theorem** *For attractors of non-overlapping IFS's,* $\dim_S(F_w) = \dim_H(F_w)$.

**Theorem** *(Moran) Let $F_w$ be the attractor of a non-overlapping IFS, $W = [w_0, w_1, \ldots, w_{n-1}]$ such that each $w_i$ is a similitude with similarity factor $c_i$ respectively. Then $\dim_H(F_w)$ is the unique number $d$ such that*

$$c_0^d + c_1^d + \cdots + c_n^d = 1.$$

*If $W$ is overlapping then $\dim_H(F_w) \leq d$.*

**Theorem** *Let $W$ be the Barnsley interpolation function IFS given in the definition. If $\sum_{k=0}^{n-1} |d_k| > 1$ and the interpolation points do not lie on a straight line, then the fractal dimension of $F_W$ is the unique real number $D$ such that*

$$\sum_{k=0}^{n-1} |d_k| a_k^{D-1} = 1.$$

**Approximating the Hausdorff Dimension**

*Grid Dimension*

**Algorithm** *To estimate the Grid dimension of a shape:*
*1. Cover the shape with grids of size* $l_1, l_2, \ldots, l_k$
*2. For each grid, count the number* $N_i$ *of grid boxes whose interior intersects the shape*
*3. Plot* $\ln(N_i)$ *vs.* $\ln(\frac{1}{l_i})$ *and compute the least squares linear regression line through the points* $\{(\ln(\frac{1}{l_1}), \ln(N_1)), \ldots, (\ln(\frac{1}{l_k}), \ln(N_k))\}$
*4. The slope is an estimate of the Grid dimension (and is also an estimate of the Hausdorff dimension).*

**Remark** *The slope is independent of the units used.*

# Complex Fractals

## Julia Sets

**Definition** *If* $z \in \mathbb{C}$ *and* $z = re^{i\theta}$ *where* $r = |z|$ *and* $\theta \in [0 \ldots 2\pi]$ *then* $\sqrt{z} = z^{\frac{1}{2}}$ *is* $(re^{i\theta})^{\frac{1}{2}} = r^{\frac{1}{2}} e^{\frac{i\theta}{2}}$. $\sqrt{z}$ *is the* ***principal square root*** *of a complex number.*

**Example** *Let* $c \in \mathbb{C}$. *Let* $w_0(z) = \sqrt{z-c}$, $w_1(z) = -\sqrt{z-c}$, *and* $W = [w_0, w_1]$. *Since* $w_0, w_1$ *are not affine, W is not an IFS.*

Is $W$ a Hutchinson operator? Not always. But it behaves like a Hutchinson Operator in the sense that $\exists \gamma \subseteq \mathbb{R}^2$ such that for any $A \in K_2$, $W^n(A \cap (\mathbb{R}^2 - \gamma))$ converges to a unique set $F_w$ such that $W(F_w) = F_w$.

**Definition** *That unique set* $F_w$ *is called the* ***Julia Set*** *associated with c and is denoted* $J_c$.

**Definition** *Let* $c \in \mathbb{C}$. *Let* $Q_c(z) = z^2 + c$. *The* ***filled in Julia Set***, $K_c$, *is*

$$K_c = \left\{ z : \text{ the } Q_c \text{ orbit of } z \text{ is bounded} \right\}$$

Facts about Julia Sets:
1. $J_c = \partial K_c$
2. $J_c = K_c$ if $J_c$ is totally disconnected
3. All Julia Sets fit into a closed ball of radius two centered at the origin, i.e. if $|Q_c^n(z)| > 2$ for any $n$ then $z \notin K_c$.

**Remark** *For each* $\mathrm{c} \in \mathbb{C}$, *there is a Julia Set* $\mathrm{J_c}$.

**Theorem** $J_c$ *is connected if and only if the* $Q_c$ *orbit of* $0$ *is bounded, i.e. if and only if* $0$ *is in the filled in Julia Set* $K_c$.

## The Mandelbrot Set

**Definition** *The* ***Mandelbrot Set***, *M, is the set of all* $c \in \mathbb{C}$ *such that* $J_c$ *is connected.*

*Because of the previous theorem, we write*

$$M = \{c \in \mathbb{C} : \text{the } Q_c\text{-orbit of } 0 \text{ is bounded}\}$$

**Remark** *While there are infinitely many Julia Sets, there is only one Mandelbrot Set.*

Facts about the Mandelbrot Set:
1. It is symmetric with respect to the x-axis
2. It is connected
3. Every open set containing any point on the boundary of $M$ contains infinitely many "baby $M$'s". Note: The babies are not similiar to $M$.
4. $K_c$ "looks like" $M$ near $c$.
5. Every bulb, $B$, has the property that $\exists n \in \mathbb{N}^+$ such that $\forall c \in B$, the $Q_c$ orbit of 0 converges to an $n$-cycle. The integer $n$ is called the **period of the bulb**.

## The Escape Time Algorithm

**Algorithm** *Assign to each screen pixel a representative complex number and color the pixel based on the number of iterations it required for a term in the $Q_c$-orbit to have absolute value greater than 2 (or some particular color if no such term is obtained after a predetermined number of iterations).*

# Proofs

### The Power Theorem

**Theorem** *Let $f : X \to X$. For any $k, n \in \mathbb{N}$,*

$$f^{k+n} = f^k \circ f^n.$$

*and*

$$f^{kn} = \underbrace{f^n \circ f^n \circ \cdots \circ f^n}_{k \text{ terms}}.$$

*(where "0 terms" means the identity map).*

**Pf.**

1. ⋆*We proceed by induction on $k$* ⋆
2. Let $f : X \to X$
3. Let $n \in \mathbb{N}$

   ⋆*Basis step* ⋆
4. $f^{0+n} = f^n$ — arithmetic
5. $\quad = id_X \circ f^n$ — (homework)
6. $\quad = f^0 \circ f^n$ — def $f^0$
7. $\forall n \in \mathbb{N}, f^{0+n} = f^0 \circ f^n$ — $\forall +$

   ⋆ inductive hypothesis ⋆
8. Let $k \in \mathbb{N}$
9. $\quad$ Assume $f^{k+n} = f^k \circ f^n$
10. $\quad f^{k+1+n} = f^{1+k+n}$ — arithmetic
11. $\quad\quad = f \circ f^{k+n}$ — def $f^k$

**12.** $= f \circ \left(f^k \circ f^n\right)$ substitution
**13.** $= \left(f \circ f^k\right) \circ f^n$ $\circ$ is associative (homework)
**14.** $= f^{1+k} \circ f^n$ def $f^k$
**15.** $\leftarrow$
**16.** $f^{k+n} = f^k \circ f^n \Rightarrow f^{(k+1)+n} = f^{k+1} \circ f^n$ $\Rightarrow +$
**17.** $\forall k \in \mathbb{N}, \forall n \in \mathbb{N}, f^{k+n} = f^k \circ f^n \Rightarrow f^{(k+1)+n} = f^{k+1} \circ f^n$ $\forall$ + (twice)
**18.** $\forall k \in \mathbb{N}, \forall n \in \mathbb{N}, f^{k+n} = f^k \circ f^n$ induction

⋆ *the proof of the second equation is similar* ⋆

QED

**Lemma** *Let $f : X \to X$, $x \in X$, and $n \in \mathbb{N}^+$.*

*$x$ has minimum period $n \Rightarrow \#\mathcal{O}_f(x) = n$*

**Pf:**

**1.** Let $f : X \to X$, $x \in X$, and $n \in \mathbb{N}^+$.
($\Rightarrow$)
**2.** Assume $x$ has minimum period $n$
**3.** $f^n(x) = x$ and $\forall k \in \mathbb{I}_{n-1}, f^k(x) \neq x$ def min period
**4.** Define $S = \{x, f(x), \ldots, f^{n-1}(x)\}$

⋆*Let's first show that $\mathcal{O}_f(x) = S$* ⋆

⋆ *First we show $\mathcal{O}_F(x) \subseteq S$* ⋆

**5.** Let $y \in \mathcal{O}_f(x)$
**6.** $y = f^j(x)$ for some $j \in \mathbb{N}$ def $\mathcal{O}$
**7.** $j = nq + r$ and $0 \leq r < n$ for some $q, r \in \mathbb{N}$ division algorithm
**8.** $y = f^j(x)$ line 6
**9.** $= f^{nq+r}(x)$ substitution
**10.** $= f^r(f^{nq}(x))$ Power Thm
**11.** $= f^r(f^n(f^n(\cdots f^n(x))))$ (with $q$ $f^n$'s) Power Thm
**12.** $= f^r(x)$ substitution ($q$ times)
**13.** $\in S$ def $S$
**14.** $\mathcal{O}_f(x) \subseteq S$ def subset

⋆ *Now we show $S \subseteq \mathcal{O}_F(x)$* ⋆

**15.** Let $z \in S$
**16.** $z = f^i(x)$ for some $i \in \mathbb{O}_{n-1}$ def $S$
**17.** $\in \mathcal{O}_f(x)$ def $\mathcal{O}$
**18.** $S \subseteq \mathcal{O}_f(x)$ def subset
**19.** $\mathcal{O}_f(x) = S$ def set =

⋆ *Now let's show that the n elements of S are distinct. Assume two powers are equal and show that only happens if they are the same power* ⋆

**20.** Let $a, b \in \mathbb{O}_{n-1}$ and $a \geq b$
**21.** Assume $f^a(x) = f^b(x)$

**22.** $f^{a-b}(x) = f^{a-b}(f^n(x))$ subst
**23.** $= f^{n+a-b}(x)$ Power Thm
**24.** $= f^{n-b}(f^a(x))$ Power Thm
**25.** $= f^{n-b}(f^b(x))$ subst
**26.** $= f^n(x)$ Power Thm
**27.** $= x$ subst
**28.** $a - b \notin \mathbb{I}_{n-1}$ by lines 3,22,27
**29.** $a - b \in \mathbb{O}_{n-1}$ since $a, b \in \mathbb{O}_{n-1}$
**30.** $a - b \in \mathbb{O}_{n-1} - \mathbb{I}_{n-1}$ def set –
**31.** $= \{0\}$ def $\mathbb{O}_{n-1}$ and $\mathbb{I}_{n-1}$
**32.** $a - b = 0$ def set notation
**33.** $a = b$ arithmetic
**34.** $\leftarrow$
**35.** the elements of $S$ are distinct def distinct
**36.** $\#S = n$ def #
**37.** $\#\mathcal{O}_f(x) = n$ subst
**38.** $\leftarrow$
**39.** $x$ has minimum period $n \Rightarrow \#\mathcal{O}_f(x) = n$ $\Rightarrow +$

QED

**Change of Basis**

**Theorem** *Let $n \in \mathbb{N}$. If each term in the $Base_b$-orbit of n is replaced by its value* $\operatorname{Mod} b$, *the sequence produced will be the base b representation of n (with the least significant digit on the left).*

**Pf:**

**1.** $\star$ We proceed by induction on $n$ $\star$
**2.** Let $b \in \mathbb{N}, b > 1$.
**3.** Define Orb $= \text{Orb}_{Base_b}$

$\star$ basis step $\star$

**4.** $Base_b(1) = \frac{1-(1 \operatorname{Mod} b)}{b} = \frac{1-1}{b} = 0$ by def of $Base_b$
**5.** $Base_b(1) = \frac{0-(0 \operatorname{Mod} b)}{b} = \frac{0-0}{b} = 0$ by def of $Base_b$
**6.** $\text{Orb}(1) = 1, 0, 0, 0, \ldots$ by definition of orbit
**7.** For any sequence $s$, define $s \operatorname{Mod} b$ to be the sequence whose $i^{\text{th}}$ term is $s_i \operatorname{Mod} b$.
**8.** $1 = 100000\ldots_{(b)} = \text{Orb}(1) \operatorname{Mod} b$

$\star$ inductive step

**9.** Let $n \in \mathbb{N}^+$
**10.** Assume $\text{Orb}(m) \operatorname{Mod} b = m_0 m_1 m_2 \ldots_{(b)} = m$ for all $m < n$
**11.** $n = (n_0 n_1 \ldots n_k)_{(b)}$ for some $n_i \in \mathbb{O}_{b-1}$ and some $k \in \mathbb{N}$ by the representation theorem

$\star$ Let's calculate $Base_b(n)$

**12.** $Base_b(n) = \frac{n-(n \text{ Mod } b)}{b}$

**13.** $= \frac{(n_0 n_1 \ldots n_k)_{(b)} - n_0}{b}$

**14.** $= \frac{(n_0 n_1 \ldots n_k)_{(b)} - n_0}{b}$

**15.** $= \frac{(0 n_1 n_2 \ldots n_k)_{(b)}}{b}$

**16.** $= (n_1 n_2 \ldots n_k)_{(b)}$

**17.** $\leq n$

$\star$ Since its less than $n$ the assumption holds for $Base_b(n)$ $\star$

**18.** $\text{Orb}(Base_b(n)) \text{ Mod } b = n_1 n_2 \ldots n_k \overline{0}$

**19.** $\text{Orb}(n) = n, \text{Orb}(Base_b(n))$ by definition of orbit

**20.** $\text{Orb}(n) \text{ Mod } b = n \text{ Mod } b, \text{Orb}(Base_b(n)) \text{ Mod } b$

**21.** $= n_0, n_1 n_2 \ldots n_k \overline{0}$

**22.** $= n_0, n_1 n_2 \ldots n_k \overline{0}_{(b)}$

**23.** $= n$

**24.** $\leftarrow$

**25.** $\forall n \in \mathbb{N}, \text{Orb}(n) \text{ Mod } b = n_0 n_1 n_2 \ldots_{(b)} = n$

QED

**Triangle Inequality and the Euclidean Metric**

**Lemma (Euclidean Triangle Inequality)** *Let $x, y, z \in \mathbb{R}^n$. Then*

$$d_{\text{Euc}}(x,z) \leq d_{\text{Euc}}(x,y) + d_{\text{Euc}}(y,z).$$

**Pf:**

**1.** Let $x, y, z \in \mathbb{R}^n$.

**2.** $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n), z = (z_1, \ldots, z_n)$ for some $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n \in \mathbb{R}$ by def of $\mathbb{R}^n$

**3.** Define $a = d_{\text{Euc}}(x,z), b = d_{\text{Euc}}(x,y), c = d_{\text{Euc}}(y,z)$

$\star$ in this notation we are trying to show that $a \leq b + c$

$\star$ we will prove the case where $b \neq 0$ and $c \neq 0$. The other cases are for homework.

**4.** For $i \in \mathbb{I}_n$ define $a_i = x_i - z_i, b_i = x_i - y_i, c_i = y_i - z_i$

**5.** $\forall i \in \mathbb{I}_n, a_i = b_i + c_i$

**6.** $a, b, c \geq 0$

**7.** $a^2 = (d_{\text{Euc}}(x,z))^2$

**8.** $= \left(\sqrt{\sum_{i=1}^n (x_i - z_i)^2}\right)^2$

**9.** $= \sum_{i=1}^n (x_i - z_i)^2$

**10.** $= \sum_{i=1}^n a_i^2$

($\star$ Note that a similar argument shows $b^2 = \sum_{i=1}^n b_i^2$ and $c^2 = \sum_{i=1}^n c_i^2$)

**11.** $= \sum_{i=1}^n (b_i + c_i)^2$

**12.** $= \sum_{i=1}^n (b_i^2 + 2b_i c_i + c_i^2)$

**13.** $= \sum_{i=1}^n b_i^2 + \sum_{i=1}^n (2b_i c_i) + \sum_{i=1}^n c_i^2$

**14.** $= b^2 + c^2 + \sum_{i=1}^{n}(2b_ic_i)$
**15.** $= b^2 + 2bc + c^2 + \sum_{i=1}^{n}(2b_ic_i) - 2bc$
**16.** $= (b+c)^2 + \sum_{i=1}^{n}(2b_ic_i) - 2bc$
**17.** $= (b+c)^2 + \sum_{i=1}^{n}(2b_ic_i) - (1+1)bc$
**18.** $= (b+c)^2 + \sum_{i=1}^{n}(2b_ic_i) - \left(\frac{b^2}{b^2} + \frac{c^2}{c^2}\right)bc$
**19.** $= (b+c)^2 + \frac{bc}{bc}\sum_{i=1}^{n}(2b_ic_i) - \left(\frac{1}{b^2}\sum_{i=1}^{n} b_i^2 + \frac{1}{c^2}\sum_{i=1}^{n} c_i^2\right)bc$
**20.** $= (b+c)^2 + \left(\sum_{i=1}^{n}\frac{2b_ic_i}{bc} - \sum_{i=1}^{n}\frac{b_i^2}{b^2} - \sum_{i=1}^{n}\frac{c_i^2}{c^2}\right)bc$
**21.** $= (b+c)^2 - \left(-\sum_{i=1}^{n}\frac{2b_ic_i}{bc} + \sum_{i=1}^{n}\frac{b_i^2}{b^2} + \sum_{i=1}^{n}\frac{c_i^2}{c^2}\right)bc$
**22.** $= (b+c)^2 - \sum_{i=1}^{n}\left(\frac{b_i^2}{b^2} - \frac{2b_ic_i}{bc} + \frac{c_i^2}{c^2}\right)bc$
**23.** $= (b+c)^2 - \left(\sum_{i=1}^{n}\left(\frac{b_i}{b} - \frac{c_i}{c}\right)^2\right)bc$
**24.** $\leq (b+c)^2$
**25.** $a^2 \leq (b+c)^2$
**26.** $a \leq b+c$

QED

**Theorem** *$(\mathbb{R}^n, d_{\text{Euc}})$ is a metric space.*

**Pf:**

**1.** $d_{Euc} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$
**2.** Let $x,y,z \in \mathbb{R}^n$
**3.** $x = (x_1,\ldots,x_n), y = (y_1,\ldots,y_n), z = (z_1,\ldots,z_n)$ for some $x_1,\ldots,x_n,y_1,\ldots,y_n,z_1,\ldots,z_n \in \mathbb{R}$
**4.** $d_{Euc}(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$
**5.** $\geq 0$
**6.** $d_{Euc}(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$
**7.** $= \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2}$
**8.** $= d_{Euc}(y,x)$
**9.** $d_{Euc}(x,z) \leq d_{Euc}(x,y) + d_{Euc}(y,z)$
**10.** $d_{Euc}(x,x) = \sqrt{\sum_{i=1}^{n}(x_i - x_i)^2}$
**11.** $= \sqrt{\sum_{i=1}^{n} 0^2}$
**12.** $= 0$
**13.** Assume $d_{Euc}(x,y) = 0$
**14.** $\sum_{i=1}^{n}(x_i - y_i)^2 = \left(\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}\right)^2$
**15.** $= (d_{Euc}(x,y))^2$
**16.** $= 0^2$
**17.** $= 0$

**18.** $(x_i - y_i)^2 = 0$ for all $i \in \mathbb{I}_n$
**19.** $x_i - y_i = 0$ for all $i \in \mathbb{I}_n$
**20.** $x_i = y_i$ for all $i \in \mathbb{I}_n$
**21.** $x = (x_1, \ldots, x_n)$
**22.** $= (y_1, \ldots, y_n)$
**23.** $= y$
**24.** $(\mathbb{R}^n, d_{Euc})$ is a metric space
QED

**Contraction Mappings are Continuous**

**Theorem** *Every contraction mapping is continuous.*

**Pf:**

**1.** Let $(X,d)$ be a metric space and $f : X \to X$ a contraction mapping.
**2.** $f$ has contraction factor $s$ for some $s \in (0\ldots 1)$
**3.** Let $U \in X$ be an open set
**4.** Let $x \in f^{-1}(U)$
**5.** $f(x) \in U$
**6.** $B(f(x);\delta) \subseteq U$ for some $\delta \in \mathbb{R}^+$
**7.** Let $y \in B(x;\delta)$
**8.** $d(x,y) < \delta$
**9.** $d(f(x),f(y)) \leq sd(x,y)$
**10.** $< s\delta$
**11.** $< \delta$
**12.** $f(y) \in B(f(x);\delta)$
**13.** $f(y) \in U$
**14.** $y \in f^{-1}(U)$
**15.** $\forall y \in B(x;\delta),\ y \in f^{-1}(U)$
**16.** $B(x;\delta) \subseteq f^{-1}(U)$
**17.** $\forall x \in f^{-1}(U), \exists \delta \in \mathbb{R}^+, B(x;\delta) \subseteq f^{-1}(U)$
**18.** $f^{-1}(U)$ is open
**19.** $f$ is continuous
QED

**The Derivative and Contraction Maps of $\mathbb{R}$**

**Theorem** *Let $I = (a\ldots b) \subseteq \mathbb{R}$ and $f : I \to I$ differentiable on $I$. If $\exists s \in (0\ldots 1)$ such that $\forall x \in I$, $|f'(x)| \leq s < 1$, then $f$ is a contraction mapping with contraction factor $s$.*

**Pf:**

**1.** Let $I = (a\ldots b) \subseteq \mathbb{R}$ and $f : I \to I$ differentiable on $I$, and $d = d_{Euc}$.
**2.** Assume $\exists s \in (0\ldots 1), \forall x \in I,\ |f'(x)| \leq s < 1$
**3.** Let $x,y \in I$
**4.** $x = y$ or $x \neq y$

⋆ Case 1 ⋆

**5.** Assume $x = y$

**6.** $d(f(x),f(y)) = d(f(x),f(x))$

**7.** $= 0$

**8.** $\leq s \cdot 0$

**9.** $= s \cdot d(x,x)$

**10.** $= s \cdot d(x,y)$

**11.** $\leftarrow$

⋆ Case 2 ⋆

**12.** Assume $x \neq y$

**13.** $f'(c) = \frac{f(x)-f(y)}{x-y}$ for some $c$ between $x$ and $y$

**14.** $\left|f'(c)\right| \leq s$

**15.** $d(f(x),f(y)) = |f(x) - f(y)|$

**16.** $= |f(x) - f(y)|\frac{|x-y|}{|x-y|}$

**17.** $= \left|\frac{f(x)-f(y)}{x-y}\right||x-y|$

**18.** $= \left|f'(c)\right||x-y|$

**19.** $\leq s|x-y|$

**20.** $= s \cdot d(x,y)$

**21.** $\leftarrow$

**22.** $\exists s \in (0\ldots 1), \forall x,y \in I, d(f(x),f(y)) \leq s \cdot d(x,y)$

**23.** $f$ is a contraction map with contraction factor $s$

**24.** $\leftarrow$

**25.** If $\exists s \in (0\ldots 1), \forall x \in I,\ |f'(x)| \leq s < 1$, then $f$ is a contraction mapping with contraction factor $s$.

QED

**Contraction Mapping Theorem**

**Theorem (The Contraction Mapping Theorem)** *Let $f : X \to X$ be a contraction mapping on a complete metric space $(X,d)$ with contraction factor $s$.*

*(1) $f$ has a unique fixed point, $q$.*

*(2) The $f$-orbit of every element of $X$ converges to $q$*
*(i.e., $\forall x \in X,\ \lim_{n\to\infty} f^n(x) = q$).*

*(3) If $x_0,x_1,x_2,\ldots$ is the $f$-orbit of $x_0 \in X$ then*

$$d(x_n,q) \leq \frac{s^n}{1-s}d(x_0,x_1)$$

*for all $n \in \mathbb{N}$.*

**Pf:**

**1.** Let $f : X \to X$ be a contraction mapping on a complete metric space $(X,d)$ with contraction factor $s \in (0\ldots 1)$.

**2.** Let $x_0 \in X$

**3.** Define $x_i = f^i(x_0)$ for $i \in \mathbb{N}^+$, i.e. $x_0,x_1,x_2,\ldots$ is the $f$-orbit of $x_0$

$\star$ *Goal: We want to prove* $x_0, x_1, x_2, \ldots$ *is a Cauchy sequence* $\star$

**4.** Define $\alpha = d(x_0, x_1)$

$\star$ *Let's first prove that* $d(x_i, x_{i+1}) \leq s^i\alpha$ *for all* $i$ *by induction on* $i$ $\star$

$\star$ *Base case* $\star$

**5.** $d(x_0, x_1) = \alpha \leq s^0\alpha$

$\star$ *inductive step* $\star$

**6.** Let $i \in \mathbb{N}^+$

**7.** Assume $d(x_i, x_{i+1}) \leq s^i\alpha$

**8.** $d(x_{i+1}, x_{i+2}) = d(f(x_i), f(x_{i+1}))$

**9.** $\leq s \cdot d(x_i, x_{i+1})$

**10.** $\leq s \cdot s^i\alpha$

**11.** $= s^{i+1}\alpha$

**12.** $\leftarrow$

**13.** $\forall i \in \mathbb{N}, d(x_i, x_{i+1}) \leq s^i\alpha$

**14.** Let $m, n \in \mathbb{N}$ with $m \leq n$

**15.** $m = n$ or $m < n$

$\star$ *Case 1* $\star$

**16.** Assume $m = n$

**17.** $d(x_m, x_n) = d(x_m, x_m)$

**18.** $= 0$

**19.** $\leq \frac{s^m}{1-s}\alpha$

**20.** $\leftarrow$

$\star$ *Case 2* $\star$

**21.** Assume $m < n$

**22.** $d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n)$

**23.** $\leq s^m\alpha + s^{m+1}\alpha + \cdots + s^{n-1}\alpha$

**24.** $= s^m\alpha(1 + s + s^2 + s^3 + \cdots + s^{n-1-m})$

**25.** $\leq s^m\alpha(1 + s + s^2 + s^3 + \cdots)$

**26.** $\leq s^m\alpha\frac{1}{1-s}$

**27.** $= \frac{s^m}{1-s}\alpha$

**28.** $\leftarrow$

**29.** $\forall m, n \in \mathbb{N}, d(x_m, x_n) \leq \frac{s^m}{1-s}\alpha$

**30.** Let $\varepsilon \in \mathbb{R}^+$

**31.** $\lim_{m\to\infty} \frac{s^m}{1-s}\alpha = 0$

**32.** $\exists N \in \mathbb{N}, \forall m \geq N, \frac{s^m}{1-s}\alpha < \varepsilon$

**33.** Assume $m, n \geq N$

**34.** $d(x_m, x_n) \leq \frac{s^m}{1-s}\alpha$

**35.** $< \varepsilon$

**36.** $\leftarrow$

**37.** $\exists N \in \mathbb{N}, \forall m,n \geq N, d(x_m,x_n) < \varepsilon$
**38.** $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m,n \geq N, d(x_m,x_n) < \varepsilon$
**39.** $x_0,x_1,x_2,\ldots$ is a Cauchy sequence
$\star$ *we use completeness to get q* $\star$
**40.** $\lim_{i\to\infty} x_i = q$ for some $q \in X$ (since $(X,d)$ is complete)
$\star$ *Now let's show that q is a fixed point* $\star$
**41.** $f$ is continuous
**42.** $f(q) = f\left(\lim_{i\to\infty} x_i\right)$
**43.** $= \lim_{i\to\infty} f(x_i)$
**44.** $= \lim_{i\to\infty} f(f^i(x_0))$
**45.** $= \lim_{i\to\infty} f^{i+1}(x_0)$
**46.** $= \lim_{i\to\infty} x_{i+1}$
**47.** $= \lim_{i\to\infty} x_i$
**48.** $= q$
**49.** $q$ is a fixed point of $f$
**50.** the $f$-orbit of every element of $X$ converges to a fixed point
$\star$ *Let's show it is unique* $\star$
**51.** Let $p \in X$
**52.** Assume $p$ is a fixed point of $f$
**53.** $f(p) = p$
**54.** $d(p,q) = d(f(p),f(q))$
**55.** $\leq s \cdot d(p,q)$
**56.** Assume $d(p,q) \neq 0$
**57.** $1 \leq s$
**58.** $\rightarrow\leftarrow$
**59.** $\leftarrow$
**60.** $d(p,q) = 0$
**61.** $p = q$
**62.** $\leftarrow$
**63.** $q$ is a unique fixed point of $f$
$\star$ *Part (2) now follows immediatly from lines 49 and 63* $\star$
**64.** $\forall x \in X, \lim_{i\to\infty} f^i(x) = q$
$\star$ *Now for the final part, part (3), we will prove it by contradiction* $\star$
**65.** Assume $d(x_n,q) > \frac{s^n}{1-s} d(x_0,x_1)$ for some $x_0 \in X$ and some $n \in \mathbb{N}$
**66.** $d(x_n,q) - \frac{s^n}{1-s} d(x_0,x_1) > 0$
**67.** Define $\varepsilon_1 = d(x_n,q) - \frac{s^n}{1-s} d(x_0,x_1)$
**68.** $\varepsilon_1 > 0$

**69.** $\exists N_1 \in \mathbb{N}, \forall m > N_1, d(x_m, q) < \varepsilon_1$
**70.** $\forall m \in \mathbb{N}, d(x_n, x_m) \leq \frac{s^n}{1-s} d(x_0, x_1)$
**71.** Let $m > N_1$
**72.** $d(x_m, q) < \varepsilon_1$ and $d(x_n, x_m) \leq \frac{s^n}{1-s} d(x_0, x_1)$
**73.** $d(x_n, q) \leq d(x_n, x_m) + d(x_m, q)$
**74.** $< \frac{s^n}{1-s} d(x_0, x_1) + \varepsilon_1$
**75.** $= \frac{s^n}{1-s} d(x_0, x_1) + d(x_n, q) - \frac{s^n}{1-s} d(x_0, x_1)$
**76.** $= d(x_n, q)$
**77.** $\rightarrow\leftarrow$
**78.** $\leftarrow$
**79.** $\forall x_0 \in X, \forall n \in \mathbb{N}, d(x_n, q) \leq \frac{s^n}{1-s} d(x_0, x_1)$

QED

**Hutchinson Operators are Contraction Maps**

**Theorem (Hutchinson)** *Let $w_0, w_1, \ldots, w_k$ be contraction mappings on $\mathbb{R}^n$ with contraction factors $c_0, c_1, \ldots, c_k$ respectively and define $W : K_n \to K_n$ by*

$$W(A) = w_0(A) \cup w_1(A) \cup \cdots \cup w_k(A).$$

*$W$ is a contraction mapping on $(K_n, d_H)$ with contraction factor $c = \max\{c_0, c_1, \ldots, c_k\}$.*

**Pf:**

**1.** Let $w_0, w_1, \ldots, w_k$ be contraction mappings on $\mathbb{R}^n$ with contraction factors $c_0, c_1, \ldots, c_k$ respectively and $W : K_n \to K_n$ by

$$W(A) = w_0(A) \cup w_1(A) \cup \cdots \cup w_k(A).$$

**2.** Define $c = \max\{c_0, c_1, \ldots, c_k\}$.
**3.** Let $X, Y \in K_n$
**4.** Define $d = d_H$
**5.** Define $r = d(X, Y)$
**6.** $X \subseteq \overline{B}(Y; r)$ and $Y \subseteq \overline{B}(X; r)$

$\star$we want to show that $d(W(x), W(Y)) \leq cd(X, Y)$, so let's compute $d(W(X), W(Y))$ $\star$

**7.** Let $x \in W(X)$
**8.** $x \in \bigcup_{i=0}^{k} w_i(X)$
**9.** $x \in w_j(X)$ for some $j \in \mathbb{O}_k$
**10.** $x = w_j(a)$ for some $a \in X$
**11.** $a \in \overline{B}(Y; r)$
**12.** $a \in \bigcup_{z \in Y} \overline{B}(z; r)$
**13.** $a \in \overline{B}(z; r)$ for some $z \in Y$
**14.** $d_{Euc}(a, z) \leq r$
**15.** $d_{Euc}(x, w_j(z)) = d_{Euc}(w_j(a), w_j(z))$

**16.** $\leq c_j d_{Euc}(a,z)$
**17.** $\leq c_j r$
**18.** $\leq cr$
**19.** $w_j(z) \in w_j(Y)$
**20.** $\subseteq \bigcup_{i=0}^{k} w_i(Y)$
**21.** $= W(Y)$
**22.** $x \in \overline{B}(w_j(z); cr)$
**23.** $\subseteq \bigcup_{\alpha \in W(Y)} \overline{B}(\alpha; cr)$
**24.** $= \overline{B}(W(Y); cr)$
**25.** $W(X) \subseteq \overline{B}(W(Y); cr)$
**26.** $W(Y) \subseteq \overline{B}(W(X); cr)$ (by a similar argument)
**27.** $d(W(X), W(Y)) \leq cr$
**28.** $= cd(X,Y)$
**29.** $W$ is a contraction mapping on $(K_n, d_H)$with contraction factor $c$.
QED

**Planar Affine Maps are Determined by 3 Points**

**Theorem** *An affine transformation on* $\mathbb{R}^2$ *is completely determined by where it maps 3 non-collinear points.*

**Pf:**
**1.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an affine map
**2.** Let $p_1, p_2, p_3 \in \mathbb{R}^2$ be noncollinear
**3.** $p_1 = (x_1, y_1), p_2 = (x_2, y_2), p_3 = (x_3, y_3)$ for some $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$
**4.** $\forall x \in \mathbb{R}^2, T(x) = Mx + B$ for some $M \in M_{2,2}(\mathbb{R})$ and some $B \in \mathbb{R}^2$
**5.** Define $u = p_2 - p_1$ and $v = p_3 - p_1$
**6.** $u = (x_2 - x_1, y_2 - y_1)$ and $v = (x_3 - x_1, y_3 - y_1)$
**7.** Define $u_1 = x_2 - x_1,\ u_2 = y_2 - y_1,\ v_1 = x_3 - x_1,\ v_2 = y_3 - y_1$
**8.** Assume $u_1 v_2 - u_2 v_1 = 0$
**9.** $u_1 v_2 = u_2 v_1$
**10.** Assume $v_1 = v_2 = 0$
**11.** $x_3 - x_1 = 0$ and $y_3 - y_1 = 0$
**12.** $x_3 = x_1 = 0$ and $y_3 = y_1$
**13.** $(x_1, y_1) = (x_3, y_3)$
**14.** $p_1 = p_3$
**15.** $p_1, p_2, p_3$ are collinear
**16.** $\to\leftarrow$
**17.** $\leftarrow$
**18.** $v_1 \neq 0$ or $v_2 \neq 0$
$\star$ So we have two cases. We will prove the case where $v_1 \neq 0$ since the other case is

similar ⋆

**19.** Assume $v_1 \neq 0$

**20.** Define $c = \frac{u_1}{v_1}$

**21.** $u_1 = \frac{u_1}{v_1} v_1$

**22.** $= cv_1$

**23.** $u_2 = u_2 \frac{v_1}{v_1}$

**24.** $= \frac{u_2 v_1}{v_1}$

**25.** $= \frac{u_1 v_2}{v_1}$

**26.** $= \frac{u_1}{v_1} v_2$

**27.** $= cv_2$

**28.** $u = (u_1, u_2)$

**29.** $= (cv_1, cv_2)$

**30.** $= c(v_1, v_2)$

**31.** $= cv$

**32.** $p_1, p_2, p_3$ are collinear

**33.** $\rightarrow\leftarrow$

**34.** $\leftarrow$

**35.** $v_1 \neq 0$

**36.** $v_2 \neq 0$ by a similar argument

**37.** $\rightarrow\leftarrow$

**38.** $\leftarrow$

**39.** $u_1 v_2 - u_2 v_1 \neq 0$

**40.** Let $z \in \mathbb{R}^2$

**41.** $z = (z_1, z_2)$ for some $z_1, z_2 \in \mathbb{R}$

⋆Note we should write $a_z$ and $b_z$ below but omit where possible to avoid clutter ⋆

**42.** Define $a = \frac{z_1 v_2 - z_2 v_1}{u_1 v_2 - u_2 v_1}, b = \frac{u_1 z_2 - u_2 z_1}{u_1 v_2 - u_2 v_1}$

**43.** $z = (z_1, z_2)$

**44.** $= \left(\frac{z_1 v_2 - z_2 v_1}{u_1 v_2 - u_2 v_1} u_1 + \frac{u_1 z_2 - u_2 z_1}{u_1 v_2 - u_2 v_1} v_1, \frac{z_1 v_2 - z_2 v_1}{u_1 v_2 - u_2 v_1} u_2 + \frac{u_1 z_2 - u_2 z_1}{u_1 v_2 - u_2 v_1} v_2\right)$

**45.** $= (au_1 + bv_1, au_2 + bv_2)$

**46.** $= (au_1 + bv_1, au_2 + bv_2)$

**47.** $= au + bv$

**48.** $Mz = M(au + bv)$

**49.** $= aMu + bMv$

**50.** $= aM(p_2 - p_1) + bM(p_3 - p_1)$

**51.** $= a(Mp_2 - Mp_1) + b(Mp_3 - Mp_1)$

**52.** $= a(Mp_2 + B - Mp_1 - B) + b(Mp_3 + B - Mp_1 - B)$

**53.** $= a((Mp_2 + B) - (Mp_1 + B)) + b((Mp_3 + B) - (Mp_1 + B))$

**54.** $= a(T(p_2) - T(p_1)) + b(T(p_3) - T(p_1))$

**55.** $\forall z \in \mathbb{R}^2, Mz = a_z(T(p_2) - T(p_1)) + b_z(T(p_3) - T(p_1))$
**56.** $M\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \frac{v_2}{u_1v_2 - u_2v_1}(T(p_2) - T(p_1)) + \frac{-u_2}{u_1v_2 - u_2v_1}(T(p_3) - T(p_1))$
**57.** $M(\begin{smallmatrix}0\\1\end{smallmatrix}) = \frac{-v_1}{u_1v_2 - u_2v_1}(T(p_2) - T(p_1)) + \frac{u_1}{u_1v_2 - u_2v_1}(T(p_3) - T(p_1))$
**58.** $M\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ is the first column of $M$ and $M(\begin{smallmatrix}0\\1\end{smallmatrix})$ is the second column of $M$
**59.** $M$ is completely determined by $p_1, p_2, p_3, T(p_1), T(p_2), T(p_3)$
**60.** $T(p_1) = Mp_1 + B$
**61.** $B = T(p_1) - Mp_1$
**62.** $B$ is completely determined by $p_1, p_2, p_3, T(p_1), T(p_2), T(p_3)$
**63.** $T$ is completely determined by $p_1, p_2, p_3, T(p_1), T(p_2), T(p_3)$
**64.** $T$ is completely determined by where it sends any three noncollinear points
QED

**Contraction Factor for Affine Maps**

**Theorem** *Let $\alpha, \beta, \gamma \in \mathbb{C}$ and $c = |\alpha| + |\beta|$. Then the map $T = affineC(\alpha, \beta, \gamma)$ is a contraction mapping if and only if $c < 1$. Further, if $T$ is a contraction mapping then $c$ is a contraction factor for $T$.*

**Pf**:

**1.** Let $\alpha, \beta, \gamma \in \mathbb{C}$, $c = |\alpha| + |\beta|$, and $T = affineC(\alpha, \beta, \gamma)$
**2.** Define $d = d_{\text{Euc}}$.
**3.** Let $z, w \in \mathbb{C}$
**4.** Define $q = z - w$.
**5.** $|q| = |z - w|$ — substitution
**6.** $\quad = d(z, w)$ — def of $d_{\text{Euc}}$
**7.** Define $r = |q|$, $r_1 = |\alpha|$, $r_2 = |\beta|, \theta = \text{Arg}(q), \theta_1 = \text{Arg}(\alpha), \theta_2 = \text{Arg}(\beta)$
**8.** $q = re^{i\theta}$, $\alpha = r_1e^{i\theta_1}$, and $\beta = r_2e^{i\theta_2}$ — def polar form
**9.** $d(T(z), T(w)) = d(\alpha z + \beta\overline{z} + \gamma, \alpha w + \beta\overline{w} + \gamma)$ — def of *affineC*
**10.** $\quad = |(\alpha z + \beta\overline{z} + \gamma) - (\alpha w + \beta\overline{w} + \gamma)|$ — definition of $d_{\text{Euc}}$
**11.** $\quad = |\alpha(z - w) + \beta(\overline{z} - \overline{w})|$ — arithmetic
**12.** $\quad = |\alpha(z - w) + \beta(\overline{z - w})|$ — property of conjugates
**13.** $\quad = |\alpha q + B\overline{q}|$ — definition of $q$
**14.** $\quad \leq |\alpha q| + |B\overline{q}|$ — by the triangle inequality
**15.** $\quad = |\alpha||q| + |B||\overline{q}|$ — property of absolute value
**16.** $\quad = |\alpha||q| + |B||q|$ — property of conjugates
**17.** $\quad = |q|(|\alpha| + |\beta|)$ — arithmetic
**18.** $\quad = cd(z, w)$ — definitions of $c, q$
**19.** $d(T(z), T(w)) \leq cd(z, w)$ — transitivity

$(\Leftarrow)$

**20.** Assume $c < 1$
**21.** $c > 0$ — def of $c$ and property of $|\ \ |$
**22.** $0 < c < 1$ — by the two last two lines

**23.** $T$ is a contraction mapping — definition of contraction mapping
**24.** $\leftarrow$
**25.** $c < 1 \Rightarrow T$ is a contraction mapping

$(\Rightarrow)$

**26.** Assume $T$ is a contraction mapping
**27.** $T$ has contraction factor $s$ for some $s \in (0\ldots 1)$ — def of contraction mapping
**28.** Define $u = 0,\ v = e^{i(\frac{\theta_2-\theta_1}{2})}$
**29.** $d(T(v),T(u)) \ = |\alpha v + \beta\overline{v} + \gamma - (\alpha u + \beta\overline{u} + \gamma)|$ — complex notation
**30.** $= |\alpha v + \beta\overline{v}|$ — subst $w = 0$ and arithmetic
**31.** $= \left|\alpha e^{i(\frac{\theta_2-\theta_1}{2})} + \beta e^{-i(\frac{\theta_2-\theta_1}{2})}\right|$ — substitution
**32.** $= \left|r_1 e^{i\theta_1} e^{i(\frac{\theta_2-\theta_1}{2})} + r_2 e^{i\theta_2} e^{-i(\frac{\theta_2-\theta_1}{2})}\right|$ — substitution
**33.** $= \left|r_1 e^{i\left(\theta_1+\frac{\theta_2-\theta_1}{2}\right)} + r_2 e^{i\left(\theta_2-\frac{\theta_2-\theta_1}{2}\right)}\right|$ — property of exponentials
**34.** $= \left|r_1 e^{i\left(\frac{\theta_1+\theta_2}{2}\right)} + r_2 e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}\right|$ — arithmetic
**35.** $= \left|(r_1 + r_2) e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}\right|$ — distributive law
**36.** $= |r_1 + r_2|\left|e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}\right|$ — property of $|\ \ |$
**37.** $= |r_1 + r_2|$ — property of exponentials
**38.** $= r_1 + r_2$ — def of $|\ \ |$ (since $r_1, r_2 \geq 0$)
**39.** $= |\alpha| + |\beta|$ — substitution
**40.** $= c$ — substitution
**41.** $d(T(v),T(u)) \ = c$ — transitivity
**42.** $c = d(T(v),T(u))$ — substitution
**43.** $\leq sd(z,w)$ — def of contraction map
**44.** $= sd\left(e^{i(\frac{\theta_2-\theta_1}{2})}, 0\right)$ — substitution
**45.** $= s\left|e^{i(\frac{\theta_2-\theta_1}{2})} - 0\right|$ — def of $d_{\text{Euc}}$
**46.** $= s\left|e^{i(\frac{\theta_2-\theta_1}{2})}\right|$ — arithmetic
**47.** $= s$ — property of exponentials
**48.** $c$ is a contraction factor for $T$ — substitution
**49.** $c < 1$ — def of contraction factor
**50.** $\leftarrow$
**51.** $T$ is a contraction mapping $\Rightarrow c < 1$ and $c$ is a contraction factor for $T$

QED

**Attractor Size Theorem**

**Theorem** *Let $W = [w_0, \ldots, w_n]$ be an IFS, $c_0, \ldots, c_n$ the contraction factors of $w_0, \ldots, w_n$ respectively, and $q_0, \ldots, q_n$ the fixed points of $w_0, \ldots, w_n$ respectively.*

*Define* $c = \max\{c_0,\ldots,c_n\}$ *and* $r = \max\left\{ d(q_i,q_j) \,:\, i,j \in \mathbb{O}_n \right\}$. *Then for any* $a \in F_W$ *and any* $i \in \mathbb{O}_n$,

$$d_{Euc}(a,q_i) \le \frac{1}{1-c}r.$$

**Pf:**

1. Let $W = [w_0,\ldots,w_n]$ be an IFS on $\mathbb{R}^k$, $c_0,\ldots,c_n$ the contraction factors of $w_0,\ldots,w_n$ respectively, and $q_0,\ldots,q_n$ the fixed points of $w_0,\ldots,w_n$ respectively.
2. Define $c = \max\{c_0,\ldots,c_n\}$, $r = \max\left\{ d(q_i,q_j) \,:\, i,j \in \mathbb{O}_n \right\}$, and $d = d_{Euc}$.
3. Let $i,j \in \mathbb{O}_n$.
4. Let $z \in \mathbb{R}^k$.
5. $d(w_j(z),q_i) \le d(w_j(z),w_j(q_j)) + d(w_j(q_j),q_i)$
6. $\quad\le c_j d(z,q_j) + d(q_j,q_i)$
7. $\quad\le cd(z,q_j) + r$
8. $\forall i,j \in \mathbb{O}_n, \forall z \in \mathbb{R}^k, d(w_j(z),q_i) \le cd(z,q_j) + r$
9. Let $a \in F_w$.
10. $a = \Phi(t_1t_2\ldots)$ for some $t_1t_2\ldots \in \Sigma_{n+1}$
11. $\quad= \lim_{n\to\infty} w_{t_1}w_{t_2}\ldots w_{t_n}(q_i)$
12. Let $n \in \mathbb{N}$.
13. $d(w_{t_n}(q_i),q_i) \le cd(q_i,q_{t_n}) + r \le cr + r = (1+c)r$
14. $d(w_{t_{n-1}}w_{t_n}(q_i),q_i) \le cd(w_{t_n}(q_i),q_{t_{n-1}}) + r \le c(1+c)r + r = (1+c+c^2)r$
15. $\quad\vdots$
16. $d(w_{t_1}w_{t_2}\circ\cdots\circ w_{t_{n-1}}w_{t_n}(q_i),q_i) \le (1+c+c^2+\cdots+c^n)r$
17. $\quad\le (1+c+c^2+\cdots)r$
18. $\quad\le \frac{1}{1-c}r$
19. Let $\varepsilon > 0$
20. $\exists N > 0, \forall n \ge N, d(a,w_{t_1}\circ\cdots\circ w_{t_n}(q_i)) < \varepsilon$
21. Let $n > N$.
22. $d(a,q_i) \le d(a,w_{t_1}\circ\cdots\circ w_{t_n}(q_i)) + d(w_{t_1}\circ\cdots\circ w_{t_n}(q_i),q_i)$
23. $\quad\le d(a,w_{t_1}\circ\cdots\circ w_{t_n}(q_i)) + \frac{1}{1-c}r$
24. $\quad< \varepsilon + \frac{1}{1-c}r$
25. $\forall \varepsilon > 0, d(a,q_i) < \frac{1}{1-c}r + \varepsilon$
26. $d(a,q_i) \le \frac{1}{1-c}r$

QED

**Corollary** $F_W \subseteq \bigcap_{i=0}^{n} \overline{B}\left(q_i; \frac{r}{1-c}\right)$

**Pf:**

1. Let $W = [w_0,\ldots,w_n]$ be an IFS on $\mathbb{R}^k$, $c_0,\ldots,c_n$ the contraction factors of $w_0,\ldots,w_n$ respectively, and $q_0,\ldots,q_n$ the fixed points of $w_0,\ldots,w_n$ respectively.
2. Let $a \in F_W$, $i \in \mathbb{O}_n$.

**3.** $d(a, q_i) \leq \frac{1}{1-c} r$

**4.** $a \in \overline{B}\left(q_i; \frac{r}{1-c}\right)$

**5.** $\forall i \in \mathbb{O}_n, a \in \overline{B}\left(q_i; \frac{r}{1-c}\right)$

**6.** $a \in \bigcap_{i=0}^{n} \overline{B}\left(q_i; \frac{r}{1-c}\right)$

**7.** $F_W \subseteq \bigcap_{i=0}^{n} \overline{B}\left(q_i; \frac{r}{1-c}\right)$

QED