Homework Problems - Math 320

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Course: Math 320 - Chaos and Fractals
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Instructions: Unless stated otherwise in a particular problem, you may do any computations in these problems either by hand, by writing your own Maple program, or by using one of my Maple libraries available online or in the math lab. The choice is yours.

Logic and Proofs

1. Math Zoology 101: (1 point each) Let x, y be real numbers, f, g functions from the set of real numbers to the set of real numbers, and A, B sets. Classify each of the following expressions as either a number, statement, function, or set (assuming the expression is defined).
   a. $x^2 > 0$
   b. $x^2 + y^2$
   c. $A \cup B$
   d. $x \in A$
   e. $\{x\}$
   f. $f \circ g$
   g. 3
   h. $f(x) = 2x$
   i. g' (the derivative of g)
   j. $3 < 2$
   k. $A \subseteq B$
   l. $f(y)$
   m. $\sqrt{\frac{x+y}{2}}$
2. (1 point each) Let P, Q be statements. Use a truth table to show that each of the following is a tautology.
   a. $P \Rightarrow P$
   b. $(P \text{ and } {\sim}P) \Rightarrow Q$
   c. $P \Rightarrow (Q \Rightarrow P)$
   d. $\big((P \text{ or } Q) \text{ and } {\sim}Q\big) \Rightarrow P$
   e. $P \text{ or } {\sim}P$
   f. ${\sim}({\sim}P) \Leftrightarrow P$
3. (2 points each) Use the rules of natural deduction to prove each of the tautologies in exercise 2.

4. (2 points each) Let $P(x)$ be a statement containing x and $Q(x,y)$ a statement containing x,y. Use the rules of natural deduction to prove the following.
   a. $(\exists x, P(x)) \Rightarrow (\exists y, P(y))$
   b. $(\forall x, P(x)) \Rightarrow (\exists y, P(y))$
   c. $(\exists y, \forall x, Q(x,y)) \Rightarrow (\forall x, \exists y, Q(x,y))$
   d. $\sim(\forall x, P(x)) \Rightarrow \exists x, \sim P(x)$
   e. $x = y \Leftrightarrow y = x$
5. (1 point each) Give examples of particular statements $P(x)$ and $Q(x,y)$, and then translate each of the statements in exercise 4 into English statements using your examples.

Sets, Functions, and Sequences

In the following problems, let A, B, C, X, Y, Z be sets.

1. (2 points) Prove that $A \cap B \subseteq A \cup B$.
2. (2 points) Prove that if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.
3. (2 points) Let $A \xrightarrow{f} B$. Prove that $f^{-1}(B) = A$.
4. (3 points) Let $A \xrightarrow{f} B$. Prove $f(A) = B$ if and only if f is surjective.
5. (2 points) Prove id_A is bijective.
6. (2 points) Let $f : X \to X$. Prove that $id_X \circ f = f \circ id_X = f$.
7. (2 points) Prove that composition of functions is associative, i.e. if $f : Z \to W$, $g : Y \to Z$, and $h : X \to Y$ then $f \circ (g \circ h) = (f \circ g) \circ h$.
8. (3 points) Let $A \xrightarrow{f} A \times A$ by $f(x) = (x,x)$ for all $x \in A$. Prove f is injective.
9. (2 points) Let t be the infinite sequence

$$1, 2, 4, 7, 11, 16, 22, 29, 37, 46, 56, 67, 79, 92, 106, 121, \ldots$$

   whose sequence of consecutive differences is arithmetic. What is the 1,000,000$^{\text{th}}$ term of t? What is t_{t_4} (if the first term is t_1)?
10. (2 points) (*left cancellation law for injective functions*) Let $Y \xrightarrow{f} Z$. Prove that f is injective if and only if for all functions $g, h : X \to Y$

$$(f \circ g = f \circ h) \Rightarrow g = h$$

11. (4 points) (*right cancellation law for surjective functions*) Let $X \xrightarrow{f} Y$. Prove that f is surjective if and only if for all functions $g, h : Y \to Z$

$$(g \circ f = h \circ f) \Rightarrow g = h$$

12. Math Zoology: (1 point each) Let $x \in \mathbb{R}$, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, and $A, B \subseteq \mathbb{R}$. Classify each of the following expressions as either a number, statement, function, ordered pair, or set (assuming the expression is defined).
    a. $f(A \times A)$
    b. $Range(f)$
    c. f^{-1}
    d. $f^{-1}(x,x)$
    e. $f(\{(x,x)\})$

f. $f(x,x)$
g. id_A
h. $Domain(f) = \mathbb{R} \times \mathbb{R}$
i. $A \times B$
j. $f(\mathbb{I}_4 \times \mathbb{I}_4)$

Number Fun

1. (1 point) What is the quotient and remainder when -371 is divided by 17?
2. Consider the function $\mathbb{N} \times \mathbb{N} \xrightarrow{A} \mathbb{N}$ given by

$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ A(m-1,1) & \text{if } n = 0 \text{ and } m > 0 \\ A(m-1, A(m,n-1)) & \text{otherwise} \end{cases}$$

   a. (1 point) Compute $A(2,2)$
   b. (2 points) Find an explicit formula for $A(1,n)$ ("explicit" means that the formula does not contain the symbol A).
   c. (3 points) Find an explicit formula for $A(2,n)$ in terms of n
   d. (4 points) Find an explicit formula for $A(3,n)$
   e. (4 points) Compute $A(4,2)$. Can you come up with an explicit formula for $A(4,n)$?
   f. (2 points) Write an essay discussing just how big $A(5,1)$ is!

Discrete Dynamical Systems

1. (3 points) Let $A \xrightarrow{P} A$, $s_0 \in A$, and $s = (s_0, s_1, s_2, \ldots)$ the P-orbit of s_0. Show that if s is cyclic with period n then it is also cyclic with period kn for any positive integer k. [Hint: Use induction on k.]
2. (3 points) Let $A \xrightarrow{P} A$, $s_0 \in A$, and $s = (s_0, s_1, s_2, \ldots)$ the P-orbit of s_0. Show that if $s_n = s_m$ then $s_{n+k} = s_{m+k}$ for any positive integer k.
3. (4 points) Let $A \xrightarrow{P} A$, $s_0 \in A$, and $s = (s_0, s_1, s_2, \ldots)$ the P-orbit of s_0. Show that if s is both a cyclic with period n and cyclic with period m, then s is cyclic with period $\gcd(n,m)$. [Hint: Use the fact that $\gcd(n,m) = sn + tm$ for some integers s,t.]
4. (3 points) Fun with composition! Let $A \xrightarrow{f} A$ and $A \xrightarrow{g} A$. Show that if $f \circ g \circ f = g$ and $g \circ f \circ f = f$ then $g = f$.

The Directed Graph

1. (1 point each) Draw the directed graph of the following discrete dynamical systems.
   a. $\mathbb{O}_{11} \xrightarrow{f} \mathbb{O}_{11}$ by $f(n) = n + 3 \operatorname{Mod} 12$.
   b. $\mathbb{O}_{11} \xrightarrow{f} \mathbb{O}_{11}$ by $f(n) = n + 7 \operatorname{Mod} 12$.
   c. $\mathbb{O}_{11} \xrightarrow{f} \mathbb{O}_{11}$ by $f(n) = n^6 \operatorname{Mod} 12$.
   d. $\mathbb{O}_6 \xrightarrow{f} \mathbb{O}_6$ by $f(n) = n^6 \operatorname{Mod} 7$.
2. (1 point each) Give an example of, and draw the directed graph of a dynamical system

$X \xrightarrow{f} X$ which has the following properties.

 a. f is neither injective nor surjective
 b. f is bijective
 c. f is injective and not surjective
 d. f is surjective and not injective

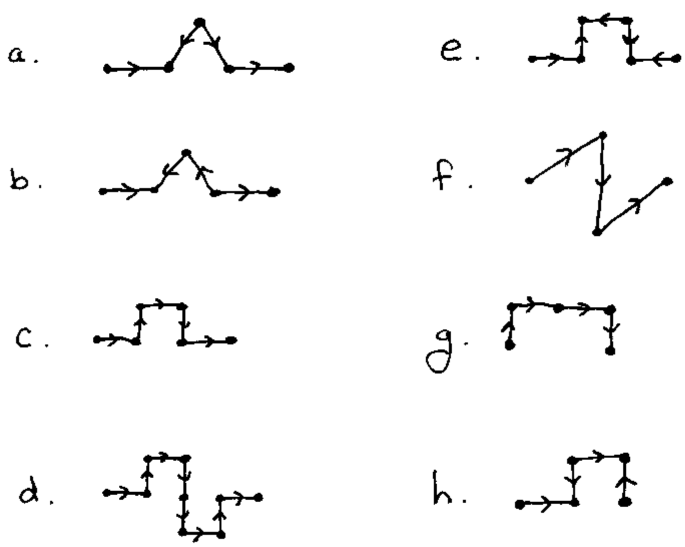
Iteration

1. (3 points for each part)
   a. Verify the Collatz conjecture for the first 100 positive integers by computing the T-orbit of n for $1 \le n \le 100.$ Note that you should compute the complete orbit of each $n,$ indicating any repeating parts with an overbar or other appropriate notation. You can do this by hand or by computer or calculator, its your choice.
   b. Define the **total stopping time** of n to be the number of iterations of T required for the orbit of n to reach $1.$ For example, the total stopping time of 1 is $0,$ the total stopping time of 2 is 1, and the total stopping time of 3 is 5. Compute the total stopping time of the integers from 1 to $100.$
2. In Post's tag problem, verify that each of the following seeds have eventually cyclic orbits by listing the orbits, and determine the number of iterations required before the orbit becomes cyclic. (1 point each except for part d)
   a. aabb
   b. aaaab
   c. baabaa
   d. **The "Whose is Longest" Contest**: Find a seed whose Tag-orbit is cyclic. You must hand in the complete orbit and must indicate the number of terms in the cycle itself. The student who has the longest minimum cycle length will receive a bonus of 3 points added to their homework grade. In the event of a tie no bonus points will be awarded to any student.
3. (3 point) The Collatz function T is defined for all integers, since odd and even are defined for any integer. Compute the T-orbit if n for all integers n satisfying $-50 \le n \le 0.$ List all of the disjoint cycles you find and state their minimum period.
4. (3 points) Use the Sumerian method for computing square roots to compute the $\sqrt{7}$ accurate to five digits (counting the leading 2 as one of the five digits). Use 1 for the value of the seed. Give the approximations both as fractions and as decimals. How many iterations are required?
5. (3 points) Use the Euclidean algorithm to reduce the fraction
$$\frac{498672943}{520221547}$$
   You must do this by hand, not by Maple and show your work. You can use a calculator to do the division and remainder computations.
6. (3 points) Define a Fibonacci-like sequence
$$F(a,b) = x_0, x_1, x_2, \ldots$$
   as follows:

$$x_0 = a,$$
$$x_1 = b,$$
$$x_n = x_{n-1} + x_{n-2} \text{ for } n > 1.$$

Compute the first ten terms for the sequences, $F(1,1)$, $F(2,1)$, $F(-1,1)$, $F(-1,-3)$, and $F(\frac{1}{3}, \frac{1}{2})$. Then describe an iterative process (i.e. a discrete dynamical system) that would compute this sequence.

7. (1 point each) Below is the generator of a stick figure iterator that replaces directed line segments with the indicated collection of line segments (the seed is a single directed line segment directed to the right with the same "end points" as those shown in the figure, so the figure shown is the first iteration). Draw the next two iterations. Be accurate. You can assume all directed line segments in the figures are congruent, and angles are what they appear to be (90 degrees or 60 degrees). You must draw these by hand, no computers allowed. Tip: Draw a large copy of the image below on a piece of paper, then put a second piece on top of the first as if you are going to trace the first and us it as a guide to draw the second. When finished you can then draw/trace over the second iteration in the same manner to produce the third.



8. (2 points each) Draw the third iteration of the following grid based fractal constructions. You must draw these by hand and you must use graph paper of the correct size.

a. $GB(2;3)$

b. $GB(3;1,3,7,9)$

c. $GB(4;2,3,6,11,14,15)$

d. Make up one of your own (a new one, not one from class, the book, or listed above).

9. (3 points each) Use the coloring technique I showed you in class to draw the given iteration of the specified HeeBGB. You must use the specified graph paper (available at the web site) and use an approved highlighter marker. You must do these by hand, although you can check your answers with the `DrawDetIFS()` command in my Maple `chaos` library if you wish.

a. HeeBGB$(-Lt, Lt, Lt, none)$, 5^{th} iteration on 32x32 graph paper

b. HeeBGB$(Up, none, -Dn, -Rt, Up, none, none, Dn, -Lt)$, 3^{rd} iteration on 27x27 graph paper

c. Make up your own HeeBGB of the form HeeBGB$(a_1, a_2, a_3, none)$ and color the 5^{th} iteration on 32x32 graph paper. None of your choices for a_1, a_2, a_3 should be Up or $none$, and at least one of them must be negative.

10. Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 2 - x^2$ for all $x \in \mathbb{R}$.

a. (1 point) Find $f^2(x)$. Write your answer as an expanded polynomial.

b. (1 point) Find $f^3(x)$. Write your answer as an expanded polynomial.

c. (1 point) Use Maple to make three plots, one showing the graphs of f and $id_{\mathbb{R}}$, another showing the graphs of f^2 and $id_{\mathbb{R}}$, and third showing the graphs of f^3 and $id_{\mathbb{R}}$. Hints: The following Maple command plots the graphs of sin and cos on the same graph. Imitate this command to make your plots. (You can also plot these by hand or using some other software if you prefer.)

```
plot({sin(x),cos(x)},x=-3..3,view=[-3..3,-3..3],numpoints=1000
```

d. (1 point) Explain how you can tell from the plots in part (c) how many fixed points, 2-cycles, and 3-cycles the function f has.

e. (2 points) Use algebra to find all of the fixed points of f. Do this calculation entirely by hand and show your work. No credit will be given for estimates, decimal approximations, use of Maple, estimating from the graphs, or educated guesses. Exact answers only.

f. (4 points) Use algebra to find all of the disjoint cycles of minimum period 2 for f. Do this calculation entirely by hand and show your work. No credit will be given for estimates, decimal approximations, use of Maple, estimating from the graphs, or educated guesses. Exact answers only. Hint: If $p(x)$ is a polynomial function with real coefficients then $p(r) = 0$ if and only if $(x - r)$ is a factor of the polynomial $p(x)$.

g. (1 point) Use the `fsolve();` command in Maple (or a graphing calculator) to find decimal approximations to all disjoint cycles of minimum period three for f.

h. (4 points) Use Newton's method to find a decimal approximation to one of the points of minimum period three that you found in part (g). Use -1.2 for the seed. How many iterations does it take to find the point accurate to within 10^{-8}?

11. Define $\mathbb{Q}_{odd}$ to be the set of all reduced fractions having an odd denominator, i.e.

$$\mathbb{Q}_{odd} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}, b \text{ is odd, and } \gcd(a, b) = 1 \right\}$$

Notice that $\mathbb{Z} \subseteq \mathbb{Q}_{odd}$. We say that such a reduced fraction $\frac{a}{b}$ is even if a is even and odd if a is odd. For example, $\frac{2}{3}$ is even whereas $\frac{5}{7}$ is odd. With these definitions we can extend the Collatz function T from the integers to a function from $\mathbb{Q}_{odd}$ to itself, i.e. we can consider $T : \mathbb{Q}_{odd} \to \mathbb{Q}_{odd}$.

a. (1 point) Find all fixed points of T in $\mathbb{Q}_{odd}$.

b. (2 points) Find all disjoint cycles of minimum period 2 for T.

c. (3 points) Find all disjoint cycles of minimum period 3 for T.

d. (4 points) Find an explicit piecewise linear formula for $T^2(x)$.

12. Newton's method doesn't always work as expected. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function.

a. (2 points) Give an example where Newton's method fails to produce a root because $f'(x) = 0$ for some x in the orbit of the chosen seed. Also describe geometrically why this fails.

b. (3 points) Assume $f'(p) \neq 0$ and $f : \mathbb{R} \to \mathbb{R}$ is differentiable at p. Prove that p is a zero of f if and only if p is a fixed point of $Newt_f$.

13. (2 points each)

a. Show that the function used in the Sumerian method is just the Newton's method iterator for finding the solutions of $x^2 = a$.

b. Derive a formula for a function analogous to the Sumerian method function, whose orbits converge to $\sqrt[3]{a}$ instead of the $\sqrt{a}$.

c. Use your formula from part (b) to compute $\sqrt[3]{5}$ to six digits of accuracy. Give your approximations as both exact fractions and decimal approximations.

d. Generalize the results of part (a) to derive a formula for a function whose orbits converge to $\sqrt[n]{a}$.

Integer Base

1. (1 point each) Use the iterative method shown in class to convert 1234 to base

a. 2

b. 3

c. 4

d. 5

Fractran

1. (4 points total) Let s be the last two digits of your social security number. Use my *CollatzGame* Fractran program to compute the T-orbit of $s + 10$, where T is the $3x + 1$ function. State clearly

a. what integer you want to compute the T-orbit of,

b. what seed you are using for the $f_{CollatzGame}$-orbit in order to accomplish that,

c. how, mathematically, you are obtaining the T orbit of your number from the $f_{CollatzGame}$-orbit.

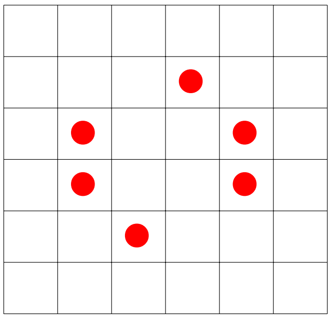
d. Do the first ten iterations (of the $f_{CollatzGame}$-orbit, not the T-orbit) by hand and show your work. [Note:When computing iterations by hand, leave your integers factored into their prime factorization rather than expanding them in the base-ten representation. Also factor the numerators and denominators of the fractions that are used to define the Fractran program function *CollatzGame* first, and then use that to do the iterations by hand. It's MUCH easier that way.]

e. The rest of the orbit you can compute by Maple, using the syntax shown in the LectureExamples worksheet. You do not have to print the whole orbit if it the orbit is very large (many are!), just have Maple count how many iterations were required before a cycle was reached.

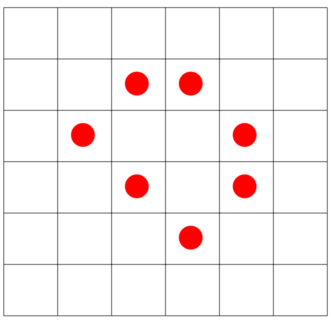
Conway’s Game of Life

1. (2 points each) Compute the entire orbit of each of the following seeds in Conway’s Game of Life. In each case compute the first iteration by hand, showing the values (neighbor counts) of each cell that you used. Also in each case classify the orbit as cyclic, eventually cyclic but not cyclic, or acyclic and for eventually cyclic orbits state the period of the cycle that is obtained and the number of iterations that were required before a cyclic point was reached. You may use Mirek’s Cellebration or Life32 to compute the orbits.

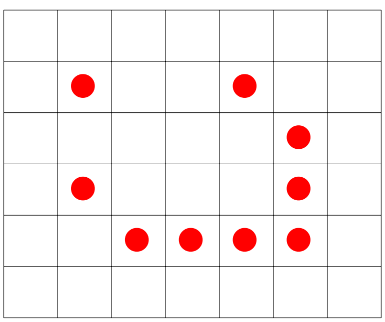
a.



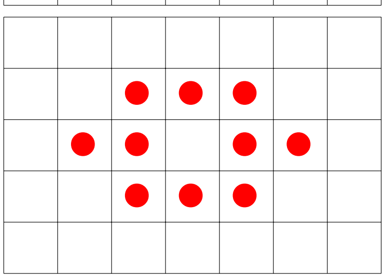
b.



c.



d.

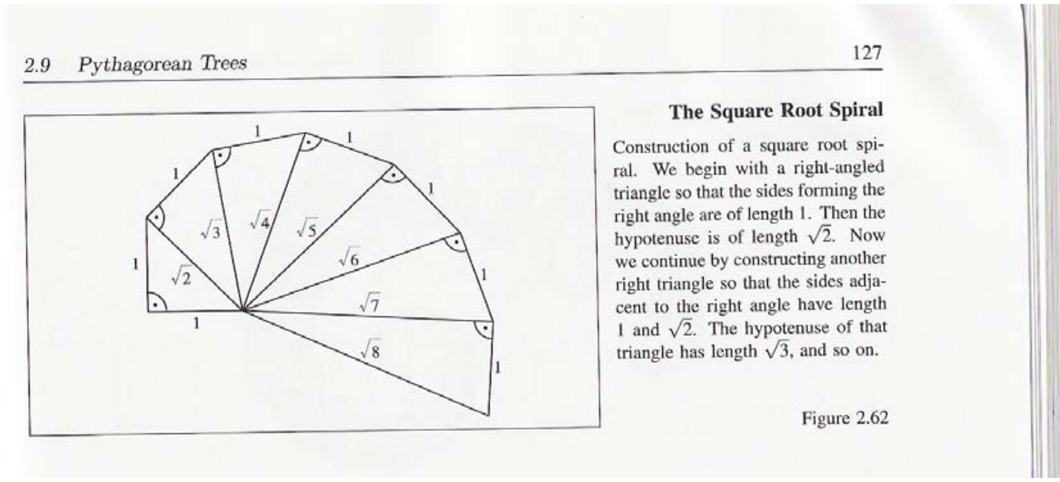


2. (3 points) Use your first name as the seed for a Game of Life, that is, in the Life32 program, draw your name in the grid of cells by hand. Hand in the seed that you used by drawing it on a grid. Then compute the Game of Life orbit of your first name. Describe what happens. What kind of orbit is it? Eventually fixed? Eventually cyclic? Acyclic? How many iterations before things stabilize, if ever? Do any cells survive or do they all eventually die? Are any gliders produced? Do the gliders live forever? Save a copy of your seed file in .lif file format and email it to me as an attachment so I can check your answer. Make sure the subject line of your email message is "<insert your name here> - Life Homework".

Maple

Note: All of the programs in this section must be handed in by putting your Maple worksheet (in `.mws` format) in the digital dropbox at our Blackboard web site and notifying me that you put it there by email (or emailing it to me as an attachment if the file is smaller than 30K). You may not hand in a printout or a floppy disk. All problems and solutions in the worksheet should be **clearly** labeled with Maple text and you should put your name, course, and Assignment number at the top of the worksheet. Before you save the worksheet to send it, choose Edit/Remove Output/from Worksheet from the menu. This will make your file much smaller. I can regenerate the output myself.

1. (0 points... just FYI) You may find it helpful to take the Maple New User's Tour. Click on the Help Menu and choose New User's Tour and then follow the instructions. It gives a nice overview of many of the features I don't have time in class to discuss.
2. (2 point) Write a Maple proc that takes a positive integer n as input and prints all positive prime numbers less than or equal to n. Your program could use either the `ithprime()` function or the `isprime()` function.
3. (2 points) Write a Maple proc L that takes as input two lists and returns as output a set containing the terms they have in common (not necessarily in the same position). For example, $L([1,3,x,5],[2,x,5,2,1,8])$ should return $\{1,x,5\}$.
4. (3 points) Write a Maple program that implements the Euclidean algorithm to compute the $\gcd(x,y)$ for any natural numbers x,y (except $x = y = 0$). Your program should not just compute the $\gcd$, but should do so by implementing the Euclidean algorithm directly (i.e. `proc(x,y) RETURN(gcd(x,y)) end;` is not what I am looking for here! :)).
5. (4 points) Write a Maple program to plot the Pythagorean spiral from Figure 2.62 from the textbook. The program should take a single positive integer n as an argument and plot the spiral until the side of length $\sqrt{n}$ is reached. A bonus of 2 points will be awarded if you use the Maple `textplot()` command to label the lengths of the sides like they do in Figure 2.62 in your textbook (you don't have to label the sides of length 1 or mark the right angles).



The Square Root Spiral

Construction of a square root spiral. We begin with a right-angled triangle so that the sides forming the right angle are of length 1. Then the hypotenuse is of length $\sqrt{2}$. Now we continue by constructing another right triangle so that the sides adjacent to the right angle have length 1 and $\sqrt{2}$. The hypotenuse of that triangle has length $\sqrt{3}$, and so on.

Figure 2.62

Metric Spaces

1. (1 point) Let $a = 1\overline{010}$ and $b = 10\overline{10}$ be 2-adic integers and d_2 the 2-adic metric. Compute $d_2(a,b)$.

2. (3 points) Let (X,d) be a metric space and $k \in \mathbb{R}^+$. Define $d_k : X \times X \longrightarrow \mathbb{R}$ by $d_k(x,y) = kd(x,y)$ for all $x,y \in X$. Prove that (X,d_k) is a metric space.

3. (2 points) Complete the proof of the Euclidean Triangle Inequality given in the lecture notes by proving the cases where either $b = 0$ or $c = 0$.

4. (3 points) Prove that $(\mathbb{R}^2, d_{taxi})$ is a metric space.

5. (3 points) Prove that $(\mathbb{R}^2, d_{max})$ is a metric space.

6. (1 point each) Let d_{taxi} be the taxicab metric on $\mathbb{R}^2$ and define the length of a line segment to be the d_{taxi} distance between it's endpoints. [Note: A subset $S \subseteq \mathbb{R}^2$ is a line segment in $(\mathbb{R}^2, d_{taxi})$ if and only if S is a line segment in $(\mathbb{R}^2, d_{Euc})$.]

- **a.** Find all equilateral triangles having the segment $\{(x,0) : x \in [0..1]\}$ for one side.
- **b.** Repeat part a, but this time use the maximum metric $d_{\max}$ instead of the taxicab metric.

7. (1 point each) Sketch the following subsets of $(\mathbb{R}^2, d_{Euc})$ and state if they are open, closed, bounded, or compact (state all the properties that apply to the given set).

- **a.** $B((1,1);2)$
- **b.** $\mathbb{R}^2 - B((0,1);1/2)$
- **c.** $\{z \in \mathbb{R}^2 : d_{Euc}(z,(2,2)) \leq 1\} \cap \{z \in \mathbb{R}^2 : d_{Euc}(z,(3,2)) \leq 1\}$
- **d.** $B((0,0);1) - \{(0,0)\}$
- **e.** $\{(x,y) : |y| > |x|\}$
- **f.** $\{(n,n) : n \in \mathbb{N}\}$
- **g.** $\{(1/n,1/n) : n \in \mathbb{Z}^+\}$
- **h.** the Middle Thirds Cantor set (i.e. the intersection of all of the iterates that produce the Cantor set).

8. (1 point each) Let (X,d) be a metric space. Prove the following.

- **a.** If $A,B \subseteq X$ are open, then $A \cup B$ is open.

b. If $A, B \subseteq X$ are open, then $A \cap B$ is open.
c. If $A, B \subseteq X$ are closed, then $A \cup B$ is closed.
d. If $A, B \subseteq X$ are closed, then $A \cap B$ is closed.

9. (1 point) Show by example that the intersection of a collection of open subsets of a metric space can be closed and not open, and that the union of a collection of closed subsets can be open and not closed.

10. (2 points each) Prove the following.
a. $(\mathbb{Z}, d_{Euc})$ is a metric space.
b. For any $x \in \mathbb{Z}$, $\{x\}$ is open in $(\mathbb{Z}, d_{Euc})$ but closed in $(\mathbb{R}, d_{Euc})$.

11. (6 points) Let (X,d) be a complete metric space, $X \xrightarrow{f} X$ a continuous map, and $x_0, x_1, x_2, \ldots$ a convergent sequence in X with $\lim_{n\to\infty} x_n = x$. Prove that $f(x_0), f(x_1), f(x_2), \ldots$ is a convergent sequence and that $\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right)$, i.e. show that limits commute with continuous maps.

Hausdorff Metric

1. (6 points) Let

$$R = \overline{B}((2,0);1)$$
$$S = \{(0,1)\}$$
$$T = \{(-x,0) : x \in [0..1]\}$$
$$U = \left\{(x,y) : y \geq 0 \text{ and } x \leq 0 \text{ and } y \leq x+2\right\}$$
$$V = \{z : d_{Euc}(z,(0,1)) = 1\}$$

be elements of $(\mathcal{K}_2, d_H)$. Find the distances between all 25 pairs of these five elements and put your answers in a table of the form:

d_H	R	S	T	U	V
R	0				
S					
T					
U					
V					

(I filled out $d_H(R,R)$ to get you started. :)) Show your work and sketch the regions.

2. (2 points) Let $z, w \in \mathbb{R}^2$. Prove that $d_H(\{z\},\{w\}) = d_{Euc}(z,w)$.

3. (3 points) Let $S = \left\{(x,y) : x \in [0..2\pi] \text{ and } y = 2\sin(x)\right\}$ and $T = \{(3,1)\}$. Compute $d_H(S,T)$. You do not have to give an exact answer but your answer must be accurate to within eight digits of accuracy. Hint: Time to break out Maple (or a graphing calculator) and your good old fashioned calculus knowledge!

Chaos

1. (3 points each) Let $(X,d), (Y,d')$ be metric spaces and $f : X \to X$, $g : Y \to Y$. Let h be a topological conjugacy between f and g.

   a. Prove that q is an attracting fixed point of f if and only if $h(q)$ is an attracting fixed point of g.

   b. Prove that q is a repelling fixed point of f if and only if $h(q)$ is a repelling fixed point of g.

   c. Prove that q is a term in an attracting (resp. repelling) n-cycle if and only if $h(q)$ is a term in an attracting (resp. repelling) n-cycle of g.

2. (1 point) Give five different examples of subsets of $\mathbb{R}^2$ which are dense in $(\mathbb{R}^2, d_{\mathrm{Euc}})$.

3. (3 points) Prove that $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x$ has sensitive dependence on initial conditions, but is not chaotic.

4. (4 points) The function $Q(x) = x^2 - 2$ is chaotic on the interval $[-2\ldots 2]$, therefore, in any open subinterval $(a\ldots b) \subseteq [-2..2]$ there must be a periodic point. Let $I_k = (0.2k\ldots 0.2(k+1))$ for $k \in \{0,1,2,\ldots 9\}$. For each such k, find a periodic point $p_k \in I_k$, and state its minimum period, n_k. List all of your points and their periods in a table and verify that they are periodic by listing their Q-orbit. You may use decimal approximations, but they should be accurate to at least 10 digits.

5. (1 point each) Draw the time series plot and the graphical analysis for the first twenty iterations starting with seed 1.5 for each of the following functions from $\mathbb{R}$ to $\mathbb{R}$.

   a. arctan

   b. f where $f(x) = 2\sin(x) + x$

   c. your own favorite nonlinear function

6. (4 points each) Give an example of differentiable functions f and g from $\mathbb{R} \to \mathbb{R}$ such that f has an attracting two cycle (that is not a fixed point) and a repelling fixed point and g has an attracting three cycle (that is not a fixed point) and a repelling fixed point.. In each case (1) explain why your function has the required properties (2) draw the time series plots for appropriate seeds to illustrate that it behaves as claimed (3) draw an animated graphical analysis in Maple for several appropriate seeds to illustrate that it behaves as claimed and (4) explain how your time series and graphical analysis plots illustrate the required properties. Do all of your work and write your explanations for this problem in Maple and hand in your solutions by emailing me a Maple worksheet in `.mws` format (NOT `.mw` format!!) with the subject line `[Maple] - Firstname Lastname`.

7. (3 points) Let $n \in \mathbb{N}^+$, $\mathbb{R} \xrightarrow{f} \mathbb{R}$ a differentiable function, and $x_0, x_1, x_2, \ldots$ the f-orbit of a cyclic point $x_0 \in \mathbb{R}$ with minimum period n. Derive a formula for $(f^n)'(x_0)$ in terms of $f'(x_0), f'(x_1), \ldots, f'(x_{n-1})$. Use this to prove that if one point in the n-cycle is attracting (resp. repelling), then they all are.

Contraction Maps

1. (1 point each) Give examples of contraction maps $\mathbb{R} \xrightarrow{f} \mathbb{R}$ which satisfy the given condition and explain why your function has the desired properties. Plot the graph of each of your functions on Maple.

   a. f is bijective and strictly decreasing

   b. f is injective but not surjective

   c. f is surjective but not injective

d. f is neither surjective nor injective

2. Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ a twice differentiable function with $f'(x) \neq 0$ for any $x \in I$. Let $N = Newt_f$ be the Newton's method iterator function for f.
   a. (1 point) Let $p \in I$ be a zero of f. Show that $N'(p) = 0$.
   b. (4 points) Let p be a zero of f in I. Prove there is an open interval $U \subseteq I$ containing p such that the N -orbit of x_0 converges to p for any $x_0 \in U$. Thus conclude that Newton's method is guaranteed to work if $f'(p) \neq 0$ and your initial guess is close enough to p.
3. (2 points each except as noted)
   a. Let $S : \mathbb{R}^+ \to \mathbb{R}^+$, be the function used in the Sumerian method for finding $\sqrt{2}$. Find the largest open interval $I \subseteq \mathbb{R}^+$ such that $|S'(x)| \leq 0.5$ for all $x \in I$.
   b. Prove that S is a contraction mapping on I. [Hint: you **must** show that $S : I \to I$ first, then use part (a).]
   c. (1 point) Let $x \in \mathbb{R}^+ - I$. Show that $S(x) \in I$.
   d. Prove that the Sumerian method for computing $\sqrt{2}$ works for **any** choice of seed in $\mathbb{R}^+$. [Hint: use parts (b) and (c).]
   e. Use the convergence estimate given by the contraction mapping theorem to compute the number of iterations required to compute $\sqrt{2}$ to six digits of accuracy starting with seed $1,000,000$. Then compute the actual number of iterations required and show that it is less than or equal to your estimate.

Complex Numbers

1. (2 points each) Let $z, w \in \mathbb{C}$ and $\theta, \gamma \in \mathbb{R}$. Prove the following.
   a. $|z| \geq 0$
   b. $|e^{i\theta}| = 1$
   c. $e^{i\theta}e^{i\gamma} = e^{i(\theta+\gamma)}$
   d. $|zw| = |z||w|$
   e. $d_{Euc}(z, w) = |z - w|$
   f. $\overline{zw} = \overline{z}\overline{w}$
   g. $\overline{z + w} = \overline{z} + \overline{w}$
   h. $z\overline{z} = |z|^2$
   i. $|z| = |\overline{z}|$
   j. $\overline{e^{i\theta}} = e^{-i\theta}$
2. (1 point each) Convert the following complex numbers to polar form.
   a. $1 + i$
   b. $-i$
   c. -2
   d. 5
   e. $5 + 6i$
3. (1 point each) Let $z = 1 + i$, $w = -2 + i$, and $v = -i$. Convert the following expressions to a complex number in standard form.

a. zw

b. $z + w$

c. $w^2 + 3w + 1$

d. $\overline{w}^2 + 3\overline{w} + 1$

e. $w\overline{w}$

f. $\overline{5}$

g. $3e^i$

h. e^z

i. $v^{1000001}$

4. (1 point) Find a complex number α in standard form, such that multiplication of any complex number z by α has the effect of rotating z about the origin by $20°$ clockwise.

Affine Maps

In what that follows when we refer to affine maps, we are referring to affine maps on $\mathbb{R}^2$ unless specifically stated otherwise.

1. (1 point each) Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 0 \end{pmatrix}, B = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}, b = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Compute each of the following.

a. $A + B$

b. AB

c. BA

d. $3Ab$

e. $Ab + 2b$

2. (2 points) Let $M \in M_{2,2}(\mathbb{R})$. Prove that for any $x, y \in \mathbb{R}^2$ and any $a, b \in \mathbb{R}$,

$$M(ax + by) = aMx + bMy.$$

3. (1 point each) For each T and p below, compute $T(p)$. Do it by hand and then check your answer in Maple using my *chaos* library.

a. $T = \mathrm{Affine}(2, 1, 30°, 45°, -1, 2),\ p = (3, 2)$

b. $T = \mathrm{affine}(2, 1, -1, 3, 0, -1),\ p = (-1, 3)$

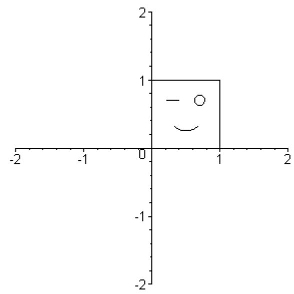
c. $T = \mathrm{affineC}(2 + i, 1 - i, i),\ p = -1 + 3i$

d. $T = \mathrm{affineM}\left(\begin{bmatrix} 2 & 0.3 \\ -\frac{1}{2} & -1 \end{bmatrix}, [2, 1]\right),\ p = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

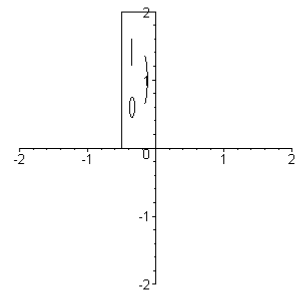
4. (4 points) Derive the conversion formulas given in lecture for converting between the four forms of affine maps: standard, matrix, geometric and complex. Show your work, don't use Maple.

5. (1 point each) For each image A below, find an affine map $T = \mathrm{Affine}(r, s, \theta, \phi, e, f)$ in geometric form so that $A = T(\mathrm{MrFace})$. Do the first six by hand and the remaining two any way you like.

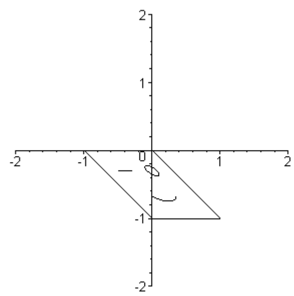
a.



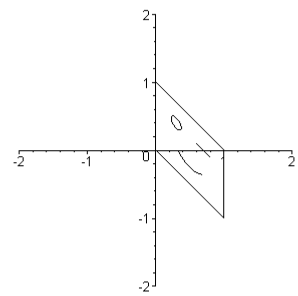
b.



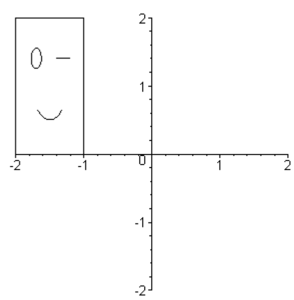
c.



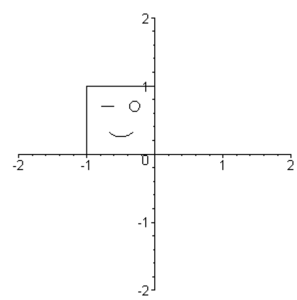
d.



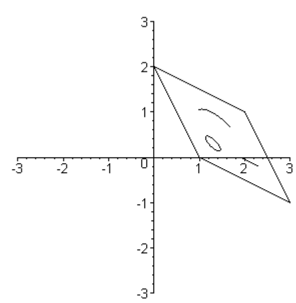
e.



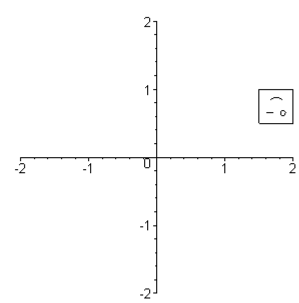
f.



g.



h.



6. Recall that there is a unique affine map that sends any three noncollinear points to any other three points.

- **a.** (2 points) Find an affine map T such that $T(0) = i$, $T(i) = 1 + i$, and $T(2+i) = -1$. Do this by hand.
- **b.** (3 points) Use Maple to convert your answer into standard, matrix, complex, and geometric form. (Note: there is a command to do this in my chaos library).
- **c.** (1 point) Use Maple to plot $T(\mathrm{MrFace})$.

7. Let $T = \mathrm{affineC}(\alpha, \beta, \gamma)$ and $C_T = |\alpha| + |\beta|$. By the theorem we proved in class, T is a contraction mapping if and only if $C_T < 1$.

a. (2 points) Suppose $T = \text{affine}(a,b,c,d,e,f)$. Find a formula for C_T in terms of a,b,c,d,e,f.

b. (2 points) Suppose $T = \text{Affine}(r,s,\theta,\phi,e,f)$. Find a formula for C_T in terms of r,s,θ,ϕ,e,f.

8. (2 points) Show that if $|r| < 1$, $|s| < 1$, and $\theta = \phi$ then $\text{Affine}(r,s,\theta,\phi,e,f)$ is a contraction mapping.

9. (4 points) Prove that affine maps send line segments to line segments (or single points, which can be thought of as very short line segments. :)), i.e. if T is an affine map and S is a line segment in the plane, show that $T(S)$ is a line segment.

Return of the Heebie GB's

1. (1 point each) Plot the fractals (i.e. the attractors of the IFS's) in problems #7,8,9 in the **Iteration** section above using the DrawIFS command in my Maple `chaos` library. Use at least 30000 points. Tips: More points will produce a better picture but will take longer and may crash Maple if you run out of memory.

   WARNING: Be sure to use the **Classic Maple 9 Worksheet**, *not the default Maple 9 worksheet to plot these! The Windows icon for the Classic Maple 9 Worksheet should be available from your Start Menu. If you use the default Maple 9 worksheet (the java based one) (a) the plots will take a very long time and your Maple worksheet might lock up and (b) the plots will look terrible.*

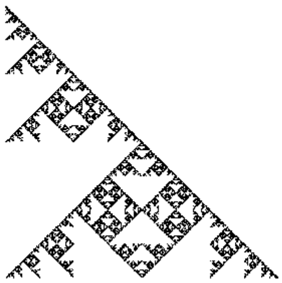
2. **Heebie Geebie Free Three for Three Contest**: Try to create the most interesting or beautiful fractal you can using the `DrawIFS` command in my Maple `chaos` library. Your fractal does not have to be the attractor of a `HeeBGB`, but can be the attractor of any IFS you like. You can also make an image that consists of a collage of more than one image using the Maple `display()` command in `plots` package. If you want to add text or a title to your Fractal Work of Art, you can do so with the Maple `textplot()` command in the Maple `plots` package. Submit your entry as a Maple worksheet via the assignment drop box at the BlackBoard page for our course (there is a link to it from our course home page). I will not accept entries via email, paper, or floppy. Files must be named `LastnameContest.mws` where `Lastname` is replaced by your actual last name in order to be considered. I will post all of the entries on a web page (anonymously) and they will be judged by a panel of "experts" of my choosing (contest participants will not be eligible for judging). The top three winners will receive a bonus of three free points added directly to their homework grade. If there are three or fewer contestants they will win by default. In the case of duplicate entries which are winners, the three point bonus will be evenly divided among those having the duplicate entries. The judges will be asked to judge the fractals using two criteria: how aesthetically pleasing the image is and how mathematically interesting it appears. You can change the color of the image if you wish or even use multiple colors (if you can figure out how to do it). Be sure you show the Maple commands you used to produce the image in your worksheet along with the image itself.

Guess My IFS

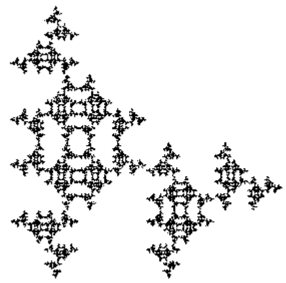
1. (2 points each) Play *Guess My IFS* with each of the following IFS attractors by drawing (or plotting) the first iteration of the IFS applied to MrFace as a seed. Then use the

DrawIFS command in Maple to confirm your guess by plotting the attractor and hand in your plots. Hint: the first five are grid based IFS's, the last three are not. You should be able to do all of these "by eye", i.e. without resorting to any calculations or measurments.

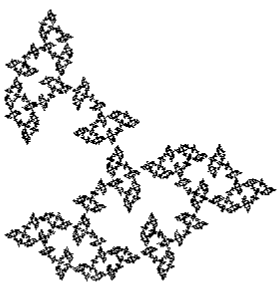
a.



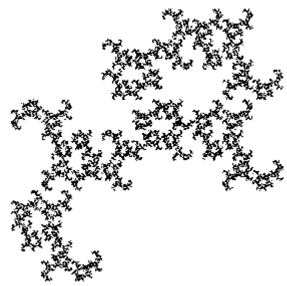
b.



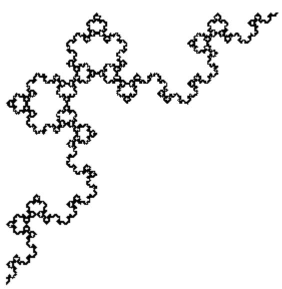
c.



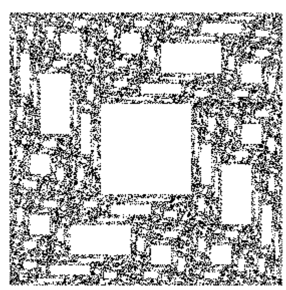
d.



e.

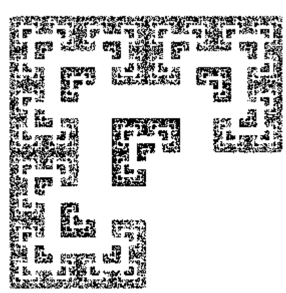


f.

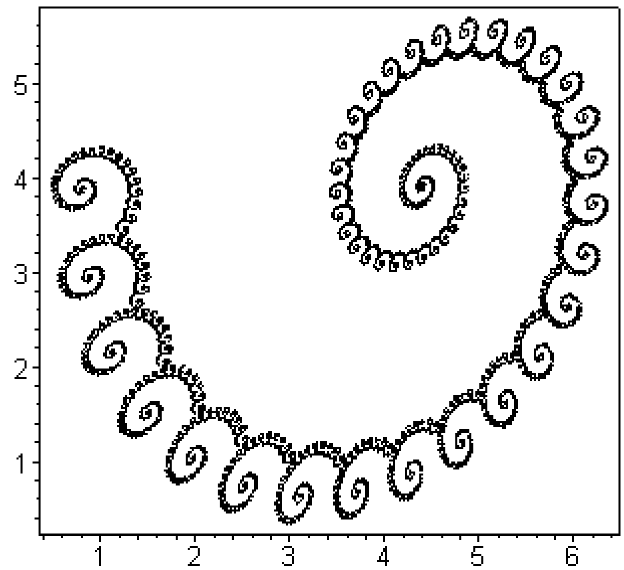


g.

h.

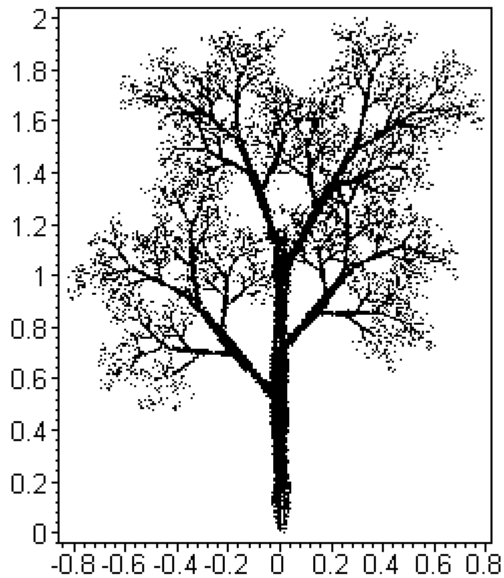


2. (5 points) Find an IFS whose attractor is the following figure. Draw the first iteration of the IFS applied to MrFace using Maple and then plot the attractor as you did in the previous problem. You may use Maple as much as you wish in this problem. Hint: You might want to print the image on graph paper to get coordinates for points and their images since guessing this IFS is not easy by eye.



5
4
3
2
1
1
2
3
4
5
6

3. (5 points) Repeat the previous problem for the fractal tree image below. Notice that the method you are using to make a fractal which looks like this particular tree can be applied to making a fractal that looks like any shape you like. Thus you have learned how to make fractals which look like a given image you are trying to model. Hints: Many trees look alike, but are all unique to some extent... be sure your fractal has all of its branches connected in the same places as the tree below!



Address My IFS

1. (2 points each) Let $W = [w_0, \ldots, w_{n-1}]$ be an IFS and $c_0, \ldots, c_{n-1}$ the contraction factors of $w_0, \ldots, w_{n-1}$ respectively. Let $a, b \in \mathbb{R}^k$ and $c = \max\{c_0, \ldots, c_{n-1}\}$. Let $t_1, \ldots, t_m \in \{0, 1, \ldots, n-1\}$ and define $J = w_{t_1} \circ w_{t_2} \circ \cdots \circ w_{t_m}$.

a. Prove that $d_{Euc}(J(a), J(b)) \leq c^m d_{Euc}(a, b)$.

b. Let $a = \Phi(t_1, \ldots, t_m, t_{m+1}, \ldots)$ and $b = \Phi(t_1, \ldots, t_m, t'_{m+1}, \ldots)$ then

$$d_{Euc}(a, b) \leq c^m \operatorname{diam}(F_W).$$

i.e. if two points have addresses that agree in the first m digits, then they will be no further than $c^m \operatorname{diam}(F_W)$ apart.

2. (1 point each) Define a **Generalized Chaos Game** as follows. Let $p_0, \ldots, p_{n-1}$ be points in $\mathbb{R}^2$ and $c_0, \ldots, c_{n-1}$ numbers in $(0\ldots 1)$. Then $ChaosGame([p_0, c_0], [p_1, c_1], \ldots, [p_{n-1}, c_{n-1}])$ represents the chaos game in which the current point is moved c_i of the distance towards a randomly selected goal point p_i. For example, in this notation the standard Chaos Game is $ChaosGame([(0,0), 0.5], [(0,1), 0.5], [(1,0), 0.5])$.

a. Find an IFS that produces the same attractor as $ChaosGame([p_0, c_0], [p_1, c_1], \ldots, [p_{n-1}, c_{n-1}])$ (in terms of the p_i and c_i).

b. Prove that p_i is the fixed point of the i^{th} affine map in the IFS you found in part a.

c. Plot the attractor of $ChaosGame([e^{2\pi i}, 0.4], [e^{2\pi i/5}, 0.5], [e^{4\pi i/5}, 0.6], [e^{6\pi i/5}, 0.7], [e^{8\pi i/5}, 0.8])$. Note: the ChaosGame command in my Maple package does not support multiple ratios c_i so it can't be used to do this.

d. Find a generalized Chaos Game that produces the Sierpinski Carpet (i.e. $HeeBGB(Up, Up, Up, Up, none, Up, Up, Up, Up)$) and express your answer in the form $ChaosGame([p_0, c_0], [p_1, c_1], \ldots, [p_{n-1}, c_{n-1}])$.

e. Use the ChaosGame in the chaos package to verify that your game works (note: the syntax is different... see ?chaos for help).

3. (1 point each) Let $W = [w_0, w_1, w_2] = HeeBGB(Rt, -Up, -Rt, none)$.

a. Plot F_w. (Note: Turn on boxed style axes before printing.)

b. Compute the fixed points of $w_0, w_1,$ and w_2 analytically.

c. Plot the fixed points found in part b on the plot of F_w.

d. Compute the exact coordinates of $\Phi(01\overline{12})$.

e. Plot $\Phi(01\overline{12})$ on the plot of F_w.

f. Indicate all of the points on the attractor which have an address starting with 201.

g. Does $f(0.t_1t_2\ldots_{(3)}) = \Phi(t_1t_2\ldots)$ define a function for this IFS? If yes, explain why and state whether or not f is continuous. If no, find a number $r \in [0\ldots1]$ such that $r = 0.r_1r_2\ldots_{(3)} = 0.s_1s_2\ldots_{(3)}$ but $\Phi(r_1r_2\ldots) \neq \Phi(s_1s_2\ldots)$.

4. Suppose your computer screen has a resolution of 800×600 pixels and you want to plot the attractor of

$$W = [Affine(0.9, 0.9, 45, 45, 0, 0), Affine(0.8, 0.8, 0, 0, 0.8, 0)].$$

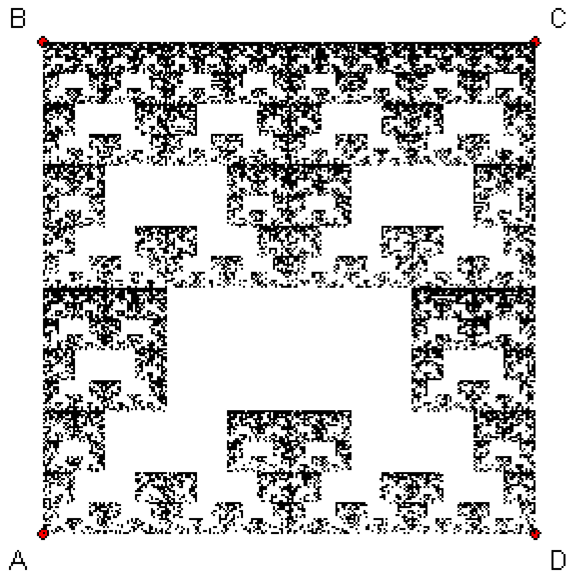
a. (3 points) What rectangular region of the plane should you display on the screen so that you are guaranteed that F_W will be entirely visible (and hopefully as large as possible)? Defend your answer.

b. (2 points) Using any point on the screen as a seed and the Address Method for producing the fractal, what is the minimum length addresses should you use in order to be guaranteed that the results will be accurate to within the size of one pixel width?

c. (1 point) Given your answer to part b, how many addresses would you need to generate and how many points would you have to plot in order to produce the attractor by the Address Method?

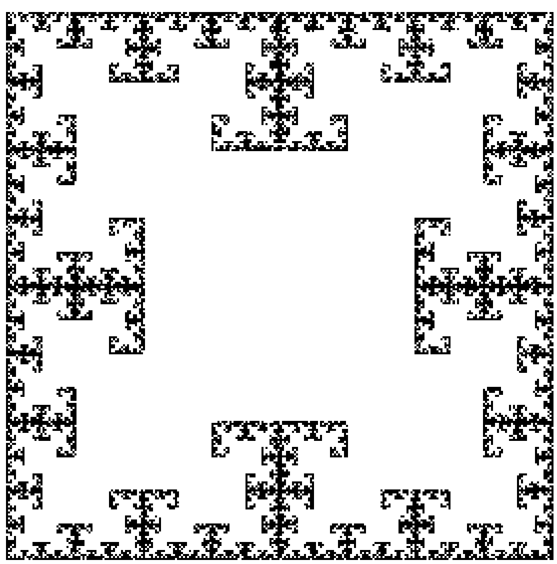
Fractal Data Analysis

1. (1 point) A radio astronomer (Jodie Foster) has a 10,000 character sequence of the letters A,B,C,D which she obtained from signals she received from outer space. The sequence appears to be random, but she decides to test it using the chaos game. She plots four points labeled A, B, C, and D at the four corners of a square and starting at the point A, she looks at the first letter in her sequence and moves half way towards the corner with that label. She then reads the next letter in the sequence and moves half way to the next corner, and so on in the usual manner for playing the chaos game. If the sequence was truly random, playing this game should fill in the square more or less uniformly. However, after plotting the 10,000 points from her sequence she obtained the following picture:



What does this picture indicate about the sequence of letters? In particular, what subsequences never appear in this sequence?

2. (1 point) Suppose you ran a fractal data analysis on your data and obtained the following image. What does that tell you about the data? In particular what finite addresses are missing from the data?



3. (up to bonus 3 points added to homework grade) Find some interesting source of data that can be grouped into four sets of data uniformly, and test it for randomness using the four corner chaos game. You should have at least 1000 data points to obtain a reasonable picture. Discuss your results intelligently.

Fractal Curves

1. (1 point each)

a. Convert $\frac{1}{5}$ to base 2.

b. Plot $\frac{1}{5}$ on a base 2 ruler.

c. How many different addresses does $\frac{1}{5}$ have with respect to the base 2 ruler IFS?

2. (1 point each) Determine if the following numbers are in the MTC by converting them to base 3. Explain.

a. $\frac{2}{9}$

b. $\frac{23}{27}$

c. $\frac{13}{81}$

d. 0.7

3. (2 points) Find an IFS that parameterizes the Sierpinski Triangle. Then use the DrawIFSCurve command in the Maple chaos package to sketch approximations to the curve using $3, 3^2, 3^3,$ and 3^4 subdivisions of the unit interval. [Hint: Map MrFace's "chin" to the left side of $w_0(F_W),$ the bottom of $w_2(F_W)$, and the "hypotenuse" of $w_1(F_W)$.]

4. (1 point) Let $P : [0\ldots 1] \to [0\ldots 1] \times [0\ldots 1]$ be the parameterization of the Peano curve. Find $P(\frac{1}{2}),\ P\left(\frac{31}{81}\right),$ and $P(0.7)$.

5. (1 point each) Determine if the following points are in the Right Sierpinski Triangle by converting them to base 2. Explain.

a. $\left(\frac{11}{32}, \frac{41}{64}\right)$

b. $\left(\frac{1}{3}, \frac{1}{6}\right)$

c. $\left(\frac{1}{3}, \frac{1}{5}\right)$

d. $(\frac{1}{2}, \frac{1}{2})$

e. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{4}\right)$

6. (1 point) Plot the points in the previous problem on the Sierpinski Triangle in Maple to verify your answers.

Fractal Interpolation

1. (3 points) Consider the data

$$\{(0,0),(2,4),(3,6),(5,3),(6,7)\}.$$

Make a Barnsley interpolation function for the given data that has Hausdorff dimension:

a. 1.0

b. 1.2

c. 1.4

d. 1.6

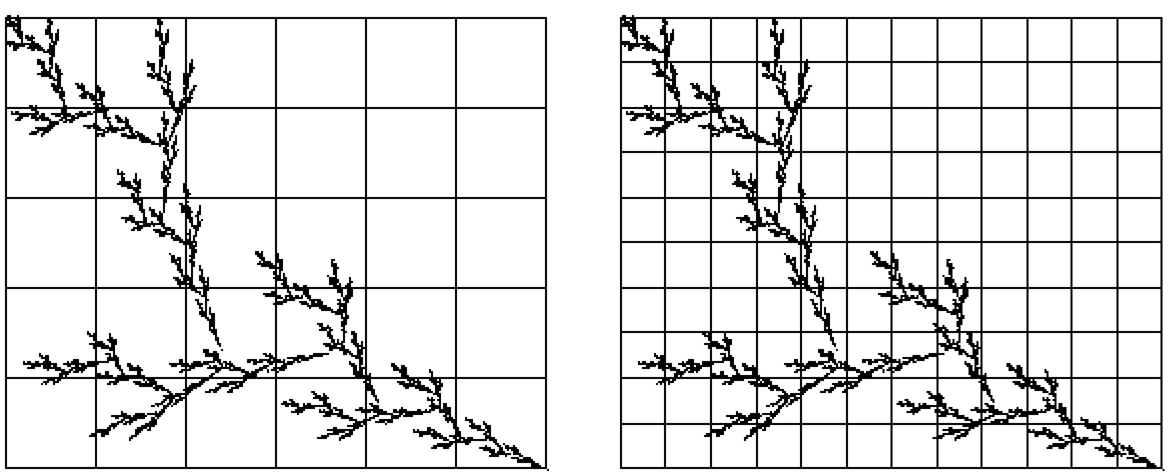
e. 1.9

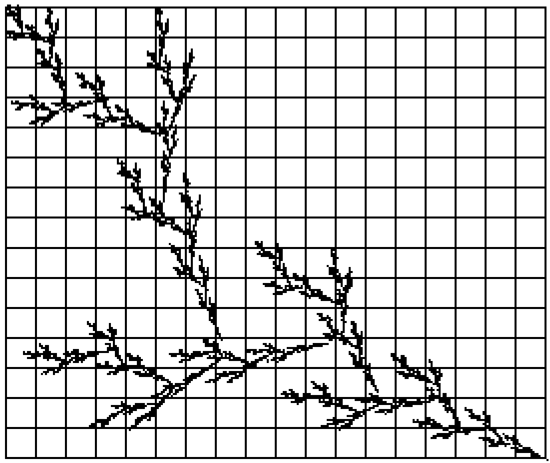
and plot the graph of each function. Show your calculations which verify that the function has the requested dimension.

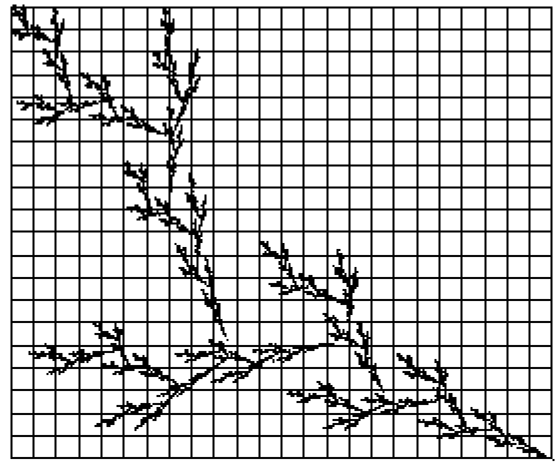
2. (8 points) Write a Maple program that will compute the value of $f(x)$ for a given $x,$ where f is a fractal interpolation function (either Mandelbrot or Barnsley type). (Note that this is not the same as computing $p(t)$ where p is a parameterization of the graph of f given by $p(0.t_1t_2\ldots_{(n)}) = \Phi(t_1t_2\ldots)$ as is done by my IFSCurve() Maple procedure.) For example, if (x_i, y_i) is one of the original data points then $f(x_i) = y_i$ and your procedure should give the value of $f(x)$ for values of x which are in between the x values of data points for the given interpolation function as well. Give examples showing that your function actually works.

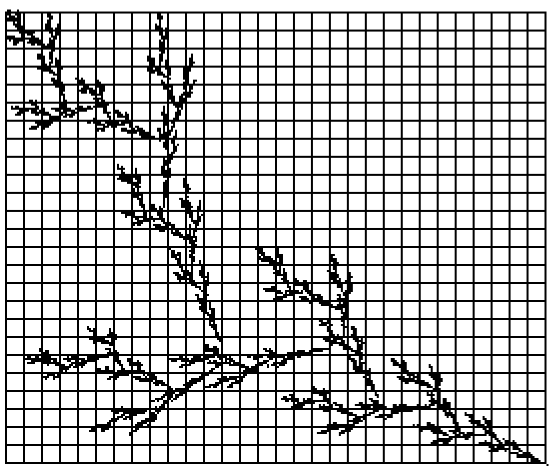
Dimension and Similarity

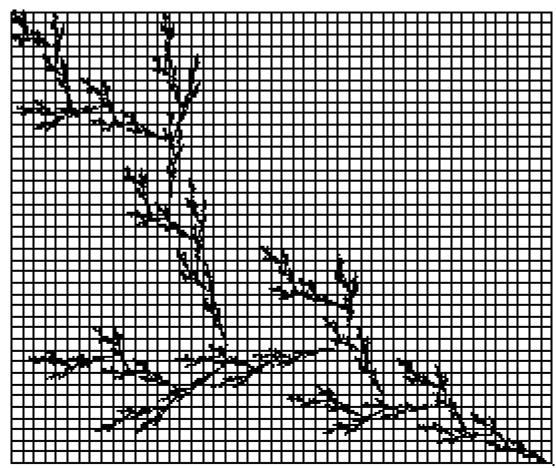
1. (3 points) Prove that the topological dimension of a metric space having two points is zero.
2. (4 points) Prove that the Hausdorff dimension of a metric space having two points is zero.
3. (1 point each) What is the topological dimension of the following shapes. You don't have to prove your answer, but give an informal explanation.
   a. $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}^+\}$.
   b. the Sierpinski Carpet (not triangle or gasket!)
   c. the attractor of GridIFS$(Up, n, Up, n, n, n, Up, n, n)$
   d. the peano curve
4. (4 points) Prove that every isometry is bijective and its inverse is also an isometry.
5. (3 points) Prove that Affine$(r, s, \theta, \phi, e, f)$ is an isometry if and only if $|r| = |s| = 1$ and $\theta = \phi + \pi k$ for some $k \in \mathbb{Z}$.
6. (1 point each) Determine the similarity dimension of the fractals in problems 1d,e,g, and h in the **Guess My IFS** section above. Give exact answers where possible, and in every case give a decimal approximation to at least 10 digits of accuracy.
7. (3 points) Compute the grid dimension of the following fractal. Use all of the data available with a least squares linear regression, don't just use two of the grids. Plot the data points and the regression line on the same set of axes using Maple. See ?stats[fit] and then click on leastsquare for examples of doing a least square regression analysis in Maple.











8. (4 points) Prove that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a similitude with scaling factor c if and only if $f = \text{Affine}(r, s, \theta, \phi, e, f)$ with $|r| = |s| = c$ and $\theta = \phi + \pi k$ for some $k \in \mathbb{Z}$.

Complex Fractals

1. (1 point) For each of the following points $c \in \mathbb{C}$, determine if z is in the Mandelbrot set.

a. 0.2

b. $-0.75 - 0.1i$

c. $-1 - i$

d. $-i$

e. $-0.919 + 0.248i$

2. (1 point each) For each of the values c in the previous problem, state whether or not the filled in Julia set K_c is connected. Plot K_c to verify your answer. Note: It takes a long time to plot these with Maple. They also appear rather small, so you might want to check the `File/Preferences/Plotting/Plot Display/Window` menu option before plotting to make them a little bigger. If you want to make really big ones you can export the image to a jpg or gif. See the examples at the bottom of the `?chaos` help screen if you want to do this. Do not send me enormous pictures via email! Print them out!

3. (4 points) Prove that the Mandelbrot set is symmetric with respect to the x-axis.
4. (8 points) **The Great Bulb Hunt**. Each of the "bulbs" (i.e. solid black regions) of the Mandelbrot set are characterized by the fact that there exists a positive integer n such that all points c in a given bulb produce a function $f(z) = z^2 + c$ having the property that the f-orbit of 0 converges to a cycle of period n. We say that this number n is the **period of the bulb**. The main cardioid is the only bulb of period 1. There is 1 bulb of period 2, 3 bulbs of period 3, 6 bulbs of period 4, 15 of period 5, and 27 bulbs of period 6. Your job is to find all the bulbs of period less than or equal to six. Find as many as possible, and your score will be determined by the number you find. Your answer should consists of
   a. A table listing a representative point from each bulb and the period of that bulb.
   b. Maple calculations showing that the given point does indeed produce a function for which the orbit of zero converges to a cycle of the stated period.
   c. A plot of the Mandelbrot set with the locations of the bulbs you found clearly marked similar to Figure 14.19 on page 809 in your book (second edition).

   Note that you will only get credit for finding bulbs which are NOT listed on the diagram on page 809, although these should also be listed in your table. Also note that just randomly hunting for such points will take you a very very very very long time, but there is a direct way to find them. It is explained in Chapter 14.1 of the book (in particular on pages 792-796). So I am finally actually testing you on whether you read the book or not!