Homework Problems - Math 345

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Course: Math 345 - Geometry
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Instructions: Unless stated otherwise in a particular problem, you may do any computations in these problems either by hand, by writing your own Maple program, or by using one of my Maple libraries available online or in the math lab. The choice is yours.

Logic and Proofs

Propositional Logic

1. Math Zoology 101: (1 point each) Let x, y be real numbers, f, g functions from the set of real numbers to the set of real numbers, and A, B sets. Classify each of the following expressions as either a number, statement, function, or set (assuming the expression is defined).

- **a.** $x^2 > 0$
- **b.** $x^2 + y^2$
- **c.** $A \cup B$
- **d.** $x \in A$
- **e.** $\{x\}$
- **f.** $f \circ g$
- **g.** 3
- **h.** $f(x) = 2x$
- **i.** g' (the derivative of g)
- **j.** $3 < 2$
- **k.** $A \subseteq B$
- **l.** $f(y)$
- **m.** $\sqrt{\frac{x+y}{2}}$

2. Let P, Q be statements. Use a truth table to show that each of the following is a tautology.

- **a.** $P \Rightarrow P$
- **b.** $(P \text{ and } {\sim}P) => Q$
- **c.** $P \Rightarrow (Q \Rightarrow Q)$
- **d.** $((P \text{ or } Q) \text{ and } {\sim}Q) \Rightarrow P$
- **e.** $P \text{ or } {\sim}P$
- **f.** ${\sim}(P \text{ or } Q) \Leftrightarrow {\sim}P \text{ and } {\sim}Q$

3-8. Use the rules of natural deduction to prove each of the tautologies in exercise 2.

Predicate Logic and Equality

1. Use natural deduction to prove $x = y$ and $y = z \Rightarrow x = z$

   Let $P(x)$ be a statement containing x and $Q(x,y)$ a statement containing x,y. Use the rules of natural deduction to prove the following.

2. $(\exists x, P(x)) \Rightarrow (\exists y, P(y))$
3. $(\forall x, P(x)) \Rightarrow (\exists y, P(y))$
4. $(\exists y, \forall x, Q(x,y)) \Rightarrow (\forall x, \exists y, Q(x,y))$
5. $\sim(\forall x, P(x)) \Rightarrow \exists x, \sim P(x)$
6. Give examples of particular statements $P(x)$ and $Q(x,y)$, and then translate each of the statements in exercises 2-5 into English statements using your examples.

Sets, Functions, and Sequences

Part I

In the following problems, let A, B, C, X, Y, Z be sets. Note: to *disprove* a statement is to prove its negation.

1. Prove or disprove that $A \cap B \subseteq A \cup B$.
2. Prove or disprove that if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.
3. Prove or disprove that if $A \cap B = A \cup B$ then $A = B$.
4. Let $A \xrightarrow{f} B$. Prove or disprove that $f^{-1}(B) = A$.
5. Let $A \xrightarrow{f} B$. Prove or disprove $f(A) = B$ if and only if f is surjective.
6. Prove or disprove id_A is bijective.
7. Let $f : X \to X$. Prove or disprove that $id_X \circ f = f \circ id_X = f$.

Part 2

1. Prove or disprove that $(A - B) \cup (A - C) = A - (B \cap C)$.
2. Prove or disprove that if $A - B = A$ then $B \subseteq A$.
3. Prove or disprove that composition of functions is associative, i.e. if $f : Z \to W$, $g : Y \to Z$, and $h : X \to Y$ then $f \circ (g \circ h) = (f \circ g) \circ h$.
4. Let $A \xrightarrow{f} A \times A$ by $f(x) = (x,x)$ for all $x \in A$. Prove or disprove f is injective.
5. (*left cancellation law for injective functions*) Let $Y \xrightarrow{f} Z$. Prove or disprove that f is injective if and only if for all functions $g,h : X \to Y$
   $$(f \circ g = f \circ h) \Rightarrow g = h$$
6. (*right cancellation law for surjective functions*) Let $X \xrightarrow{f} Y$. Prove or disprove that f is surjective if and only if for all functions $g,h : Y \to Z$
   $$(g \circ f = h \circ f) \Rightarrow g = h$$
7. Fun with composition! Let $A \xrightarrow{f} A$ and $A \xrightarrow{g} A$. Show that if $f \circ g \circ f = g$ and $g \circ f \circ f = f$ then $g = f$.
8. Math Zoology: Let $x \in \mathbb{R}$, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, and $A, B \subseteq \mathbb{R}$. Classify each of the following expressions as either a number, statement, function, ordered pair, or set

(assuming the expression is defined).

a. $f(A \times A)$
b. $Range(f)$
c. f^{-1}
d. $f^{-1}(x,x)$
e. $f(\{(x,x)\})$
f. $f(x,x)$
g. id_A
h. $Domain(f) = \mathbb{R} \times \mathbb{R}$
i. $A \times B$
j. $f(\mathbb{I}_4 \times \mathbb{I}_4)$

Toy Geometries

Incidence Structures

1. Let l,m be lines. Prove or disprove that if $l \parallel m$ then $m \parallel l$.
2. Let l,m,k be lines with $m \parallel l$. Prove or disprove that $l \parallel k$ or $l = k$ if and only if $m \parallel k$ or $m = k$.
3. Let $(\mathcal{P},\mathcal{L})$ be an incidence structure such that for any line t and any point A not on t there exists at most one line parallel to t containing A. Let l,m,k be lines in $\mathcal{L}$ with $m \parallel l$. Prove or disprove that $l \parallel k$ or $l = k$ if and only if $m \parallel k$ or $m = k$.
4. Let $(\mathcal{P},\mathcal{L})$ be an incidence structure such that for any line t and any point A not on t there exists at most one line parallel to t containing A. Prove or disprove that the parallel classes of lines form a partition of the set $\mathcal{L}$ by showing that the relation "*is in the parallel class of*" is an equivalence relation.

Affine Planes

Part I

1. Let $(\mathcal{P},\mathcal{L})$ be an incidence structure satisfying axiom A1 and let A,B be distinct points in $\mathcal{P}$. Prove or disprove that $\overleftrightarrow{AB} = \overleftrightarrow{BA}$.
2. Prove or disprove that every affine plane has at least four points.
3. Prove or disprove that every affine plane has at least six lines.
4. Prove or disprove that every line in an affine plane contains at least two points.
5. Prove or disprove: The pair $\left(\mathbb{Q}^2,\mathcal{L}\right)$ where

   $$\mathcal{L}=\left\{l : l = \left\{(x,y) \in \mathbb{Q}^2 : ax+by+c = 0\right\} \text{ for some } a,b,c \in \mathbb{Q} \text{ with } a \neq 0 \text{ or } b \neq 0\right\}$$

   is an affine plane.
6. Prove or disprove: The pair $(\mathbb{Z}^2,\mathcal{L})$ where

   $$\mathcal{L}=\left\{l : l = \{(x,y) \in \mathbb{Z}^2 : ax+by+c = 0\} \text{ for some } a,b,c \in \mathbb{Z} \text{ with } a \neq 0 \text{ or } b \neq 0\right\}$$

   is an affine plane.

Part II

1. Let l be a line in an affine plane. Prove or disprove that every line other than l that

intersects l also intersects every line in the parallel class of l.

2. Let l, m be distinct non-parallel lines in an affine plane. Prove or disprove that every point on m is contained in a unique line in the parallel class of l.
3. Prove or disprove that there is a bijection between any two lines in an affine plane. [Hint: Use a parallel class.] Thus prove that if any line has exactly n points, every line has n points.
4. Let l, m be distinct intersecting lines in a finite affine plane $(\mathcal{P}, \mathcal{L})$ of order n. Show that there exist a bijection from $l \times m$ to P. Conclude the the total number of points in $\mathcal{P}$ is n^2. [Hint: Map the ordered pair (A, B) in $l \times m$ to the point of intersection of the line through A parallel to m and the line through B parallel to l. Then show that this mapping is a function and is bijective.]
5. In an affine plane of order n, prove that each point is contained in exactly $n+1$ lines.
6. In an affine plane of order n, prove that there are exactly n^2+n lines.
7. In an affine plane of order n, prove that there are exactly $n+1$ parallel classes.

Projective Planes

1. Prove or disprove: Let $\mathcal{P}=\{A, B, C, D\}$ and
$$\mathcal{L} = \{\{A,B\},\{A,C\},\{A,D\},\{B,C\},\{B,D\},\{C,D\}\}$$
The incidence structure $(\mathcal{P}, \mathcal{L})$ is a projective plane.
2. Construct an incidence structure that satisfy axioms $P1$ and $P2$ but not axiom $P3$ having:
   a. two points and two lines
   b. three points and three lines
   c. four points and four lines
3. Prove or disprove that in any projective plane there are four distinct lines, no three of which are concurrent.
4. Prove or disprove that the projective completion of an affine plane is a projective plane.
5. Prove or disprove that the incidence structure formed by removing a single line and all of the points on that line from a projective plane is an affine plane.
6. In any finite projective plane, if one line consists of exactly n points, then every line consists of exactly n points. [Hint: Use the previous problem.]
7. Prove or disprove that a projective plane of order n has n^2+n+1 points, n^2+n+1 lines, and every point is contained in $n+1$ lines.
8. How many parallel classes does a projective plane or order n have? Prove your answer.
9. Let $(\mathcal{P}, \mathcal{L})$ be a projective plane. Define $\mathcal{P}' = \mathcal{L}$ and let $\mathcal{L}'$ be the set of all pencils of lines in $(\mathcal{P}, \mathcal{L})$. Prove that $(\mathcal{P}', \mathcal{L}')$ is a projective plane. This is called the *dual* projective plane to $(\mathcal{P}, \mathcal{L})$.

Elementary Euclidean Plane Geometry

Foundations

Part I

Until further notice, all points and lines are in the Euclidean plane as defined by the SMSG axioms and definitions given in the lecture notes. Prove or disprove each of the following statements. Note that you are allowed to use earlier numbered SMSG Theorems to prove later ones, but not the other way around. In every problem, be sure to include a neat and clearly labeled diagram (or more than one diagram if necessary).

1. SMSG Thm 1.
2. SMSG Thm 2.
3. SMSG Thm 3.
4. SMSG Thm 4.

Part Ia

1. SMSG Thm 7.
2. Prove or disprove: If $A.B.C$ and $C.D.E$ then $A.C.E$.
3. SMSG Thm 8.
4. SMSG Thm 9.

Part II

1. SMSG Thm 11.
2. SMSG Thm 12.
3. SMSG Thm 13.
4. SMSG Thm 14.
5. SMSG Thm 17.
6. SMSG Thm 19.
7. SMSG Thm 20.
8. SMSG Thm 21.

Part III

1. SMSG Thm 22.
2. SMSG Thm 25.
3. SMSG Thm 29.
4. SMSG Thm 30.
5. SMSG Thm 33.
6. SMSG Thm 35.
7. SMSG Thm 37.
8. SMSG Thm 38.
9. SMSG Thm 40.

Part IV

1. SMSG Thm 41.
2. SMSG Thm 42.
3. SMSG Thm 43.
4. SMSG Thm 44.
5. SMSG Thm 46.
6. SMSG Thm 47.
7. SMSG Thm 48.
8. SMSG Thm 49.

9. SMSG Thm 50.
10. SMSG Thm 51.

Part V

1. SMSG Thm 52.
2. SMSG Thm 53.
3. SMSG Thm 55.
4. SMSG Thm 56.
5. SMSG Thm 57
6. SMSG Thm 59.
7. SMSG Thm 60.
8. SMSG Thm 61.
9. SMSG Thm 62.
10. SMSG Thm 64.
11. SMSG Thm 65.
12. SMSG Thm 66.

Part VI

1. SMSG Thm 68.
2. SMSG Thm 69.
3. SMSG Thm 70.
4. SMSG Thm 72.
5. SMSG Thm 74.
6. SMSG Thm 75.
7. SMSG Thm 76.
8. SMSG Thm 77.
9. SMSG Thm 80.
10. SMSG Thm 81.

Congratulations! You are now an expert in 10th grade Euclidean geometry.

Advanced Euclidean Geometry

Chapter 1.1-1.6

Read section pages 1-11 in Baragar's text. Then do the following. You may use any of the SMSG theorems in your proofs, and additionally, any facts from algebra. Baragar assumes that the students are familiar with trigonometry and may require the use of trigonometry in some exercises, so I will allow it for exercises that cannot be done without it, but I would encourage you to use trigonometry only as a last resort for most problems. Note: In any homework problem that asks you a yes or no answer or asks you for a numerical answer you must still do a proof. In other words you must always prove your answer is correct. For example, in exercise 1.30 the answer is given in the back of the book, so I don't want you to just hand in the answer, but rather you should prove that the measure of the angle is what you claim it is. Moral: every answer to every homework problem requires a proof.

1. Exercise 1.5

2. Exercise 1.6
3. Exercise 1.7
4. Exercise 1.26
5. Exercise 1.27
6. Exercise 1.30

Chapter 1.6

1. Exercise 1.31
2. Exercise 1.32
3. Exercise 1.36
4. Exercise 1.37
5. Exercise 1.38
6. Exercise 1.39
7. Exercise 1.40

Chapter 1.7

1. Exercise 1.41
2. Exercise 1.42
3. Exercise 1.44
4. Exercise 1.45
5. Exercise 1.48
6. Exercise 1.49

Chapter 1.8

1. Exercise 1.50
2. Exercise 1.52
3. Exercise 1.55
4. Exercise 1.56
5. Exercise 1.57
6. Exercise 1.58
7. Exercise 1.59
8. Exercise 1.60
9. Exercise 1.61
10. Exercise 1.62 (Note: there is a typo in the question. It should say "Prove the P lies on the radical axis for Γ and Γ''".)

Ceva and Chapter 1.9

1. Exercise 1.67
2. Exercise 1.69
3. Exercise 1.70
4. Exercise 1.119
5. Exercise 1.123 (assume that the triangle is acute)
6. Prove that the lines containing the altitudes of an obtuse triangle are concurrent.

Chapter 1.10

1. Exercise 1.72
2. Exercise 1.73
3. Exercise 1.74
4. Exercise 1.75
5. Exercise 1.76
6. Exercise 1.77
7. Exercise 1.80
8. Exercise 1.81
9. Exercise 1.83
10. Exercise 1.84

Chapter 1.11

1. Exercise 1.88
2. Exercise 1.91
3. Exercise 1.92
4. Exercise 1.93
5. Exercise 1.94
6. Exercise 1.95

Miscellaneous Topics

1. Exercise 1.98
2. Exercise 1.99
3. Exercise 1.105
4. Exercise 1.106
5. Exercise 1.110
6. Exercise 1.111
7. Exercise 1.116
8. Exercise 1.117
9. Exercise 1.125
10. Exercise 1.126
11. Exercise 1.127

Constructions with Straightedge and Compass

Geometer's Sketchpad I

The following problems should be done in a single Geometer's Sketchpad document. The name of the document should be your last name followed by the assignment number (e.g. lastnameNN.gsp). Each exercise should be done on a separate page in your document, and the label of the page should be "Problem n" where n is the problem number below. [*To add a new page to a Sketchpad document choose File/Document Options/View Pages/Add and rename the page in the Page Name box.*]

For each question that requires you to write a script, you should (1) save a script in your document (2) on the corresponding page in the document you should have the figure that you used to define the script and (3) text boxes that document the script by giving (a) the name of the script (b) the inputs it requires (*see the script view to check this*) (c) a description of what it does and (d) any other documentation you think is helpful (i.e. related to that specific construction).

The entire file must be submitted to me as an email attachment and sent before the start of class on the due date. If you send it early and then later want to send me a revised version you may, but I will only grade the last version received. The subject line of your email message should be:

"Firstname Lastname - Assignment #NN - Attachment"

Your constructions must follow the rules for construction by compass and straightedge. **This means that you cannot use any of the Sketchpad menu items that are listed under the "Transform" or "Graph" menus**! No credit will be given for any problem that uses these menu items. You also cannot use external Sketchpad scripts in your own scripts, with the exception of those listed under Appearance Tools (which you may use to label your drawings). You may use the menu items under the "Construct" menu as any of these can be constructed with a straightedge and compass. You may also use the Arrow (pointer), Point, Circle, Segment, Line, Ray, Text, and Appearance tools buttons from the left hand side of the screen, but you cannot use the rotation or dilation pointer. You can also use any scripts that you write yourself and include in your document.

If you have any questions about what the inputs or outputs of a particular script should be, just ask.

1. Exercise 4.16.
2. Exercise 4.17.
3. Exercise 4.18.
4. Exercise 4.19.
5. Exercise 4.20.
6. Exercise 4.21.
7. Exercise 4.22.
8. Exercise 4.24.

Constructible Points, Lines, Circles, and Numbers

1. Let A,B be points with $|AB| = 1$ and consider the points, lines, and circles that are constructible from $\{A,B\}$. We can assign a nonnegative integer, called the **birthday**, to each constructible point, line, and circle as follows. On day 0 we have only two points, A and B and so we say that these are born on day 0 and have birthday zero. On day n, we construct all possible lines and circles (that are not already constructed) through already constructed points, and also construct the points of intersection of any lines and circles we construct (that are not already constructed) and we say that the birthday of all such lines, circles, and points is n. Thus, for example, there are two points with birthday zero A and B.

There is one line born on day one ($\overleftrightarrow{AB}$), two circles two circles born on day one ($\odot AB$ and $\odot BA$), and four points born on day one (the new points at the intersections of these two circles and the line). [Note: for the following questions you do not need to write a detailed formal proof, but you should explain your answer and include a clearly labeled diagram.]

- **a.** How many lines are born on the second day?
- **b.** How many circles are born on the second day?
- **c.** How many points are born on the second day?
- **d.** We say that a positive real number r is constructed by day n if it is the distance between two points that were constructed on or before day n. Thus the only distance that is constructed by day zero is 1, the distance between A and B. What real numbers are constructed by day one?
- **e.** Find at least four new real numbers are constructed by day 2 that are not constructed by day one and not a rational multiple of a number constructed by day 1?

In the following problems, use the rules of construction with straightedge and compass to construct the given figure, and explain why your construction does what it is supposed to (you do not have to write a full proof, but explain the main ideas that guarentee it works). Illustrate your construction with appropriate figures.

2. Given two circles with disjoint interiors, construct the common external tangent lines to both circles.
3. Given two circles with disjoint interiors, construct the common internal tangent lines to both circles.
4. Given two circles that are externally tangent and a segment AB, construct a third circle that is externally tangent to both circles and has radius $|AB|$.

Geometer's Sketchpad II

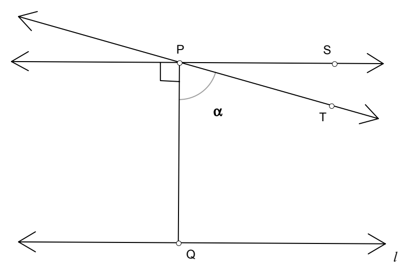
1. Exercise 4.25. (see page 33 and exercises 1.60-1.66)
2. Exercise 4.26.
3. Exercise 4.27. (just make script that shows that when they are collinear the absolute value of the product is 1)
4. Exercise 4.28. (just make script that shows that when they are concurrent the value is 1)
5. Exercise 4.31.
6. Exercise 4.32.
7. Exercise 4.33.
8. Exercise 4.35.

Introduction to Hyperbolic Geometry

1. Prove the first theorem in the Toy Euclidean Plane section of the lecture notes.
2. Prove the second theorem in the Toy Euclidean Plane section of the lecture notes.

In the following proofs, use the SMSG axioms of hyperbolic geometry (S1-S11 and ~S12) to prove the given statement. You may not use any of the theorems of hyperbolic geometry given in the lecture notes without proving them first, as we did not prove any of them.

3. Let l be a line a P a point not on l. If there are two lines through P parallel to l then there are infinitely many lines through P parallel to l.
4. Let l be a line, P a point not on l, Q the foot of the perpendicular from P to l, $\overleftrightarrow{PS}$ the perpendicular to $\overleftrightarrow{PQ}$ through P, T a point on the same side of $\overleftrightarrow{PQ}$ as S and α the measure of $\angle QPT$ (see figure below). Then there is a real number ρ such that $\overrightarrow{PT}$ intersects l if and only if $\alpha < \rho$. The number ρ is called the **angle of parallelism** for P and l.



5. It can be proven in hyperbolic geometry that the sum of the angles of any triangle is less than 180. Define the **defect** of a triangle $\triangle ABC$ to be the difference between the angle sum and 180, i.e.

$$d(\triangle ABC) = 180 - (|\angle A| + |\angle B| + |\angle C|)$$

Let AP be a cevian of $\triangle ABC$. Prove that

$$d(\triangle ABC) = d(\triangle APB) + d(\triangle APC).$$

6. Define the **area** of a triangle in hyperbolic geometry to be equal to its defect. Prove that if every triangle has positive defect then this definition of area satisfies axioms **S13-S15**.
7. Prove that in hyperbolic geometry a triangle can have arbitrarily large perimeter, but cannot have an arbitrarily large area, i.e. that for any positive real number there is a triangle having that perimeter, but that there exists a positive real number M such that the area of every triangle is less than M.

Complex Numbers

1. Let $z, w \in \mathbb{C}$ and $\theta, \gamma \in \mathbb{R}$. Prove the following.
   - **a.** $|z| \geq 0$
   - **b.** $|e^{i\theta}| = 1$
   - **c.** $e^{i\theta}e^{i\gamma} = e^{i(\theta+\gamma)}$
   - **d.** $|zw| = |z||w|$
   - **e.** The Euclidean distance from z to w is $|z - w|$
   - **f.** $\overline{zw} = \overline{z}\,\overline{w}$
   - **g.** $\overline{z+w} = \overline{z} + \overline{w}$
   - **h.** $z\overline{z} = |z|^2$
   - **i.** $|z| = |\overline{z}|$
   - **j.** $\overline{e^{i\theta}} = e^{-i\theta}$
2. Convert (with proof) the following complex numbers to polar form.
   - **a.** $1 + i$

b. $-i$
c. -2
d. 5
e. $5+6i$

3. Let $z=1+i$, $w=-2+i$, and $v=-i$. Convert (with proof) the following expressions to a complex number in standard form.
   a. zw
   b. $z+w$
   c. w^2+3w+1
   d. $\overline{w}^2+3\overline{w}+1$
   e. $w\overline{w}$
   f. $\overline{5}$
   g. $3e^i$
   h. e^z
   i. $v^{1000001}$
4. Find (with proof) a complex number α in standard form, such that multiplication of any complex number z by α has the effect of rotating z about the origin by $20°$ clockwise.

Transformations

1. Construct a transformation $T(z)$ of the complex plane (or extended complex plane) that has the following geometric effect. Express your transformations in terms of z and complex operations, not in terms of x,y where $z=x+yi$.
   a. reflection across the line through i and $1+i$.
   b. clockwise rotation about $3+4i$ by $45°$
   c. reflection across the line through $1+i$ and $2-3i$
   d. geometric inversion through the circle $\odot OA$ where $O=3+4i$ and $A=0$
2. Let T be a translation of the complex plane. Prove that T can be expressed as the composition of reflections, i.e. find reflections T_1, T_2 such that $T=T_1\circ T_2$. You can give either and analytic proof or a synthetic proof. It's up to you.
3. Let T be a rotation of the complex plane. Prove that T can be expressed as the composition of reflections, i.e. find reflections T_1, T_2 such that $T=T_1\circ T_2$. You can give either and analytic proof or a synthetic proof. It's up to you.
4. Prove that geometric inversion in a circle is not a composition of any number of reflections.

Inversion

1. Prove that if P is outside of $\odot OA$ and Q and R are the points where the tangent lines through P meet $\odot OA$, then the inverse of P is the midpoint of QR. Similarly show that if P is inside $\odot OA$ and Q and R are the points where the perpendicular to OP through P meets $\odot OA$, then the tangent lines to $\odot OA$ at Q and R intersect at the inverse P' of P with respect $\odot OA$.
2. Prove that if $\odot OA$ and point $P\neq O$ are constructible then so is the geometric inverse of P

with respect to $\odot OA$.

3. Prove that if l is a line through the center of $\odot OA$ then the image of l under inversion in the circle is the line l itself. (Do not use the theorem that inversion maps clines to clines since we didn't prove that theorem.)
4. Prove that if $\odot OA$ is constructible and points P, Q are constructible points on $\odot OA,$ then there is a unique constructible cline $\overleftrightarrow{\odot RP}$ that contains P, Q and is orthogonal to $\odot OA$ at both P and Q.

Klein's Erlanger Program I

Read chapter 4 in Henle. This is a very well written chapter and you won't want to miss it!

1. Prove that congruence is an equivalence relation on the set of all figures in a geometry.
2. Henle 4.1 (Chapter 4, Exercise number 1)
3. Henle 4.2
4. Henle 4.3
5. Henle 4.4

Klein's Erlanger Program II

1. Henle 4.5
2. Henle 4.8
3. Henle 4.11
4. Henle 4.12
5. Henle 4.16 (Do not use any of the geometries mentioned in the lecture notes or in class.)
6. Henle 4.17 (Do not use any of the geometries mentioned in the lecture notes or in class.)
7. Prove that every transformation of $\mathbb{C}$ obtained by composing one or more translations and rotations is a rigid motion and vice versa.
8. Prove that Special Euclidean geometry is a geometry. [Hint: Show that the set of all rigid motions forms a group.]
9. Prove that Affine Geometry is a geometry.

Euclidean Transformations

1. Prove Euclidean geometry is a geometry, i.e. that $\mathbf{E}^+$ is a group.
2. Prove that the functions $T(z) = e^{i\theta}z + \gamma$ and $T(z) = e^{i\theta}\overline{z} + \gamma$ with $\theta \in \mathbb{R}$ and $\gamma \in \mathbb{C}$ are isometries in $(\mathbb{C}, \mathbf{E}^+)$.
3. Show that every rotation, reflection, translation, and glide reflection of the complex plane is contained in the Euclidean transformation group. (Show this directly, not by using the Classification Theorem from the lecture notes, since we didn't prove that theorem.)
4. For which $\theta \in \mathbb{R}$ and $\gamma \in \mathbb{C}$ is $T \in \mathbf{E}^+$ a reflection? Prove your answer.
5. In Euclidean geometry, define a **line** to be any set of points fixed by a reflection, i.e. l is a line if and only if

$$l = \left\{ z : T(z) = z \text{ for some reflection } T \in \mathbf{E}^+ \right\}$$

Use the results of the previous problem to prove that every line contains more than one point.

6. Let l be a line in Euclidean geometry and let $u, v \in l$ with $u \neq v$. Show that an equation for l is

$$\frac{z-u}{u-v} = \frac{\overline{z} - \overline{u}}{\overline{u} - \overline{v}}$$

7. Find $\alpha, \beta, \gamma \in \mathbb{C}$ (in terms of u, v) so that the equation

$$\alpha z + \beta \overline{z} + \gamma = 0$$

is the equation of the line l given in the previous problem. (This shows that lines can be thought of as level curves of special kinds of affine maps.)

8. Let l be a line in Euclidean geometry. Prove that for any choice of $u, v \in l$ with $u \neq v$ the direction number $\frac{u-v}{\overline{u-v}}$ has the same value α.
9. Prove that two distinct lines in Euclidean geometry are parallel if and only if they have the same direction number.
10. Prove that two lines in Euclidean geometry are perpendicular if and only if the direction number of one is the negative of the direction number of the other.

Möbius Geometry I

Read Chapter 5 in Henle.

1. Henle 5.1
2. Henle 5.2
3. Henle 5.3
4. Henle 5.4
5. Henle 5.7
6. Henle 5.8
7. Henle 5.9
8. Henle 5.10
9. Henle 5.11
10. Henle 5.12

Möbius Geometry II

1. Henle 5.17 (explain)
2. Henle 5.19
3. Henle 5.20
4. Henle 5.22
5. Henle 5.24
6. Henle 5.25

Hyperbolic Geometry

Read chapters 6 and 7 in Henle.

1. Henle 7.1
2. Henle 7.2
3. Henle 7.3
4. Henle 7.5
5. Henle 7.7
6. Henle 7.8
7. Henle 7.9
8. Henle 7.10
9. Henle 7.11