Geometry Lecture Notes

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This is **not** a complete set of lecture notes for Math 345, Geometry. Additional material will be covered in class and discussed in the textbook.

Logic

In this section we give an informal overview of logic and proofs. For a more formal introduction see any logic textbook.

Variables, Expressions, and Statements

Definition *A **set** is a collection of items called the **members** (or **elements**) of the set.*

Remark *An element is either in a set or it is not in a set, it cannot be in a set more than once.*

Definition *An **expression** is an arrangement of symbols which represents an element of a set called the **domain** (or **type**) of the expression.*

Remark *It is not necessary that we know specifically which element of the domain an expression represents, only that it represents some unspecified element in that set.*

Definition *The element of the domain that the expression represents is called a **value** of that expression.*

Definition *A **variable** is an expression consisting of a single symbol.*

Definition *A **constant** is an expression whose domain contains a single element.*

Definition *A **statement** (or **Boolean expression**) is an expression whose domain is* $\{true, false\}$.

Remark *We do not have to know if a statement is true or false, just that it is either true or false.*

Definition *The value of a statement is called its **truth value**.*

Definition *To **solve** a statement is to determine the set of all elements for which the statement is true.*

Remark *More precisely, if a statement contains n variables, $x_1, \ldots x_n$, then to solve the statement is to find the set of all n-tuples $(a_1, \ldots, a_n)$ such that each a_i is an element of the domain of x_i and the statement becomes true when $x_1, \ldots, x_n$ are replaced by $a_1, \ldots, a_n$ respectively. Each such n-tuple is called a **solution** of the statement.*

Definition *The set of all solutions of a statement is called the* ***solution set****.*

Definition *An* ***equation*** *is a statement of the form* $A = B$ *where* A *and* B *are expressions.*

Definition *An* ***inequality*** *is a statement of the form* $A \star B$ *where* A *and* B *are expressions and* $\star$ *is one of* $\leq$, $\geq$, $>$, $<$, *or* $\neq$.

Propositional Logic

The Five Logical Operators

Definition *Let* P, Q *be statements. Then the expressions*

1. $\sim P$
2. P and Q
3. P or Q
4. $P \Rightarrow Q$
5. $P \Leftrightarrow Q$

are also statements whose truth values are completely determined by the truth values of P *and* Q *as shown in the following table*

P	Q	$\sim P$	P and Q	P or Q	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Rules of Inference and Proof

Definition *A* ***rule of inference*** *is a rule which takes zero or more statements (or other items) as input and returns one or more statements as output.*

Notation *An expression of the form*

$$\begin{array}{l} P_1 \\ \vdots \\ P_k \\ \hline Q_1 \\ \vdots \\ Q_n \end{array}$$

represents a rule of inference whose inputs are $P_1 \ldots P_k$ *and outputs are* $Q_1, \ldots, Q_n$.

Notation *The rule of inference shown above can also be expressed in* ***recipe notation*** *as*

Show P_1

$\vdots$

Show P_k

Conclude Q_1

$\vdots$

Conclude Q_n

or equivalently,

To show $Q_1, \ldots, Q_n$

Show P_1

$\vdots$

Show P_k

Definition *A **formal logic system** consists of a set of statements and a set of rules of inference.*

Definition *A **proof** in a formal logic system consists of a finite sequence of statements (and other inputs to the rules of inference) such that each statement follows from the previous statements in the sequence by one or more of the rules of inference.*

Natural Deduction

Definition *The symbol $\leftarrow$ is an abbreviation for "end assumption".*

Definition *The rules of inference for propositional logic are shown in Table 1.*

Table 1: Rules of inference for Propositional Logic

and + To show W and V 1. Show W 2. Show V	**and −** To show W 1. Show W and V **and −** To show V 1. Show W and V
$\Rightarrow$ + To show $W \Rightarrow V$ 1. Assume W 2. Show V 3. $\leftarrow$	**$\Rightarrow$ − (modus ponens)** To show V 1. Show W 2. Show $W \Rightarrow V$
$\Leftrightarrow$ + To show $W \Leftrightarrow V$ 1. Show $W \Rightarrow V$ 2. Show $V \Rightarrow W$	**$\Leftrightarrow$ −** To show $W \Rightarrow V$ 1. Show $W \Leftrightarrow V$ **$\Leftrightarrow$ −** To show $V \Rightarrow W$ 1. Show $W \Leftrightarrow V$
***or* +** To show W or V 1. Show W ***or* +** To show W or V 1. Show V	***or* − (proof by cases)** To show U 1. Show W or V 2. Show $W \Rightarrow U$ 3. Show $V \Rightarrow U$
$\sim$ + (proof by contradiction) To show $\sim W$ 1. Assume W 2. Show $\rightarrow\leftarrow$ 3. $\leftarrow$	**$\sim$ − (proof by contradiction)** To show W 1. Assume $\sim W$ 2. Show $\rightarrow\leftarrow$ 3. $\leftarrow$
$\rightarrow\leftarrow$ + To show $\rightarrow\leftarrow$ 1. Show W 2. Show $\sim W$	

Remark *Note that the inputs "Assume -" and "$\leftarrow$" are not themselves statements but rather inputs to rules of inference that may be inserted into a proof at any time. There is no reason however, to insert such statements unless you intend to use one of the rules of inference that* *requires them as inputs.*

Remark *Precedence: In order to eliminate parentheses we give the operators the following precedence (from highest to lowest):*

other math operators ($+, =, \cdot, \cup, -$, *etc*)
$\sim$
and , or
$\Rightarrow$
$\Leftrightarrow$

Example *Use Natural Deduction to prove the following tautologies.*

1. $\sim\sim P \Leftrightarrow P$
2. $\sim(P \text{ and } Q) \Leftrightarrow \sim P \text{ or } \sim Q$ *[Hint: Use* P or $\sim P$*, proven in the homework]*

Equality

Definition *The equality symbol,* $=$*, is defined by the two rules of inference given in Table 2.*

Table 2: Rules of Inference for Equality

Reflexive $=$	Substitution
To show $x = x$	To show W with the n^{th} free occurrence of x replaced by y 1. Show W 2. Show $x = y$

Remark *Note that in the Reflexive rule there are no inputs, so you can insert a statement of the form* $x = x$ *into your proof at any time. Note that there is a technical restriction on the Substitution rule that is not listed here (see the Proof Recipes sheet for details). In most situations the restriction is not a concern.*

Example *Use natural deduction to prove that* $x = y \Leftrightarrow y = x$.

Quantifiers

Definition *The symbols* $\forall$ *and* $\exists$ *are **quantifiers**. The symbol* $\forall$ *is called "for all", "for every", or "for each". The symbol* $\exists$ *is called "for some" or "there exists".*

Definition *If* W *is a statement and* x *is any variable then* $\forall x, W$ *and* $\exists x, W$ *are both statements. The rules of inference for these quantifiers are given in Table 3.*

Notation *If* x *is a variable,* t *an expression, and* $W(x)$ *a statement then* $W(t)$ *is the statement obtained by replacing every free occurance of* x *in* $W(x)$ *with* (t)*,*

Table 3: Rules of Inference for Quantifiers

$\forall +$ To show $\forall x, W(x)$ 1. Let s be arbitrary 2. Show $W(s)$	$\forall -$ To show $W(t)$ 1. Show $\forall x, W(x)$
$\exists +$ To show $\exists x, W(x)$ 1. Show $W(t)$	$\exists -$ To show $W(t)$ for some t 1. Show $\exists x, W(x)$

Remark *Note that there are restrictions on the rules of inference for quantifiers which are not listed in Table 3 (see the Proof Recipes sheet for details). In most situations they are not a concern.*

Remark *Precedence: Quantifiers have a lower precedence than $\Leftrightarrow$. Thus they quantify the largest statement to their right possible unless specifically limited by parentheses.*

Example *Prove* $(\sim\exists x, P(x)) \Rightarrow \forall x, \sim P(x)$

Example *Prove* $(\forall x, P(x) \Rightarrow Q(x))$ and $(\forall y, P(y)) \Rightarrow (\forall z, Q(z))$

Definition *Let $W(x)$ be a statement and $W(y)$ the statement obtained by replacing every free occurance of x in $W(x)$ with y. We define*

$$(\exists! x, W(x)) \Leftrightarrow \exists x, \left(W(x) \text{ and } \forall y, W(y) \Rightarrow y = x\right)$$

The statement $\exists! x, W(x)$ is read "There exists a unique x such that $W(x)$."

Table 4: Rules of Inference for $\exists!$

$\exists! +$ To show $\exists! x, W(x)$ 1. Show $W(t)$ 2. Let y be arbitrary 3. Assume $W(y)$ 4. Show $y = t$ 5. $\leftarrow$	$\exists! -$ To show $\exists x, W(x)$ and $\forall y, W(y) \Rightarrow y = x$ 1. Show $\exists! x, W(x)$

Sets, Functions, Numbers

Some Definitions from Set theory

The symbol $\in$ is formally undefined, but it means "is an element of". Many of the definitions below are informal definitions that are sufficient for our purposes.

Set notation and operations

Finite set notation:	$x \in \{x_1, \ldots, x_n\} \Leftrightarrow x = x_1$ or $\cdots$ or $x = x_n$
Set builder notation:	$x \in \{y : P(y)\} \Leftrightarrow P(x)$
Cardinality:	$\#S =$ the number of elements in a finite set S
Subset:	$A \subseteq B \Leftrightarrow \forall x, x \in A \Rightarrow x \in B$
Set equality:	$A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$
Def. of $\notin$:	$x \notin A \Leftrightarrow \sim(x \in A)$
Empty set:	$A = \emptyset \Leftrightarrow \forall x, x \notin A$
Relative Complement:	$x \in B - A \Leftrightarrow x \in B$ and $x \notin A$
Intersection:	$x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$
Union:	$x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$
Indexed Intersection:	$x \in \bigcap_{i \in I} A_i \Leftrightarrow \forall i, i \in I \Rightarrow x \in A_i$
Indexed Union:	$x \in \bigcup_{i \in I} A_i \Leftrightarrow \exists i, i \in I$ and $x \in A_i$
Two convenient abbreviations:	$(\forall x \in A, P(x)) \Leftrightarrow \forall x, x \in A \Rightarrow P(x)$ $(\exists x \in A, P(x)) \Leftrightarrow \exists x, x \in A$ and $P(x)$

Some Famous Sets

The Natural Numbers	$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$
The Integers	$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
The Rational Numbers	$\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}, b > 0, \text{ and } \gcd(a,b) = 1\right\}$
The Real Numbers	$\mathbb{R} = \{x : x$ can be expressed as a decimal number$\}$
The Complex Numbers	$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}$ where $i^2 = -1$
The positive real numbers	$\mathbb{R}^+ = \{x : x \in \mathbb{R}$ and $x > 0\}$
The negative real numbers	$\mathbb{R}^- = \{x : x \in \mathbb{R}$ and $x < 0\}$
The positive reals in a set A	$A^+ = A \cap \mathbb{R}^+$
The negative reals in a set A	$A^- = A \cap \mathbb{R}^-$
The first n positive integers	$\mathbb{I}_n = \{1, 2, \ldots, n\}$
The first $n+1$ natural numbers	$\mathbb{O}_n = \{0, 1, 2, \ldots, n\}$

Cartesian products

Ordered Pairs:	$(x,y) = (u,v) \Leftrightarrow x = u$ and $y = v$
Ordered n-tuple:	$(x_1, \ldots, x_n) = (y_1, \ldots, y_n) \Leftrightarrow x_1 = y_1$ and $\cdots$ and $x_n = y_n$
Cartesian Product:	$A \times B = \{(x,y) : x \in A$ and $y \in B\}$
Cartesian Product:	$A_1 \times \cdots \times A_n = \{(x_1, \ldots, x_n) : x_1 \in A_1$ and $\cdots$ and $x_n \in A_n\}$
Power of a Set	$A^n = A \times A \times \cdots \times A$ where there are n "A's" in the Cartesian product

Functions and Relations

Def of $\neq$	$x \neq t \Leftrightarrow \sim(x = t)$
Def of relation:	R is a relation from A to $B \Longleftrightarrow R \subseteq A \times B$
Def of function:	$f : A \to B \Leftrightarrow f \subseteq A \times B$ and $(\forall x, \exists! y, (x,y) \in f)$
Alt function notation	$X \xrightarrow{f} Y \Longleftrightarrow f : X \to Y$
Def of $f(x)$:	$f(x) = y \Leftrightarrow f : A \to B$ and $(x,y) \in f$
Domain:	$\text{Domain}(f) = A \Leftrightarrow f : A \to B$
Codomain:	$\text{Codomain}(f) = B \Leftrightarrow f : A \to B$
Image (of a set):	$f(S) = \{y : \exists x, x \in S$ and $y = f(x)\}$
Range (or Image of f):	$\text{Range}(f) = f(\text{Domain}(f))$
Identity Map:	$id_A : A \to A$ and $\forall x, id_A(x) = x$
Composition:	$f : A \to B$ and $g : B \to C \Rightarrow (g \circ f) : A \to C$ and $\forall x, (g \circ f)(x) = g(f(x))$
Injective (one-to-one):	f is injective $\Leftrightarrow \forall x, \forall y, f(x) = f(y) \Rightarrow x = y$
Surjective (onto):	f is surjective $\Leftrightarrow f : A \to B$ and $(\forall y, y \in B \Rightarrow \exists x, y = f(x))$
Bijective:	f is bijective $\Leftrightarrow f$ is injective and f is surjective
Inverse:	$f^{-1} : B \to A \Leftrightarrow f : A \to B$ and $f \circ f^{-1} = id_B$ and $f^{-1} \circ f = id_A$
Inverse Image:	$f : A \to B$ and $S \subseteq B \Rightarrow f^{-1}(S) = \{x \in A : f(x) \in S\}$

Example *Prove* $(A - B) \subseteq (A \cup B) - (A \cap B)$

Example *Prove the composition of bijective functions is bijective.*

Equivalence Relations

Definition *Let X be a set.*

$$R \text{ is a } \textbf{relation} \text{ on } X \Longleftrightarrow R \subseteq X \times X.$$

Definition *Let X be a set and $R \subseteq X \times X$. For any $x, y \in X$,*

$$xRy \Longleftrightarrow (x,y) \in R \qquad \textit{(infix notation)}$$

and

$$R(x,y) \Longleftrightarrow (x,y) \in R \qquad \textit{(prefix notation)}$$

Definition *Let X be a set and $R \subseteq X \times X$.*

R *is an* ***equivalence relation*** $\Longleftrightarrow \forall x, y, z \in X,$

(0) xRx *(reflexive)*

(1) $xRy \Rightarrow yRx$ *(symmetric)*

(2) xRy and $yRz \Rightarrow xRz$ *(transitive)*

Definition *Let $R \subseteq X \times X$ be an equivalence relation and $a \in X$.*

$$[a]_R = \{x : xRa\}$$

*This is called the **equivalence class** of a (with respect to R).*

Notation *We often abbreviate $[a]_R$ by $[a]$ when the relation R is clear from context.*

Theorem (Fundamental Theorem of Equivalence Relations) *Let $R \subseteq X \times X$ be an equivalence relation and $a, b \in X$. Then*

$$[a] = [b] \Leftrightarrow aRb.$$

Corollary (1) *Let $R \subseteq X \times X$ be an equivalence relation. Then X is a disjoint union of equivalence classes, i.e.*

$$X = \bigcup_{a \in X} [a]$$

and

$$\forall a, b \in X, [a] = [b] \text{ or } [a] \cap [b] = \emptyset.$$

Definition *If X is a set and $P = \{A_i : i \in I\}$ is a set of subsets of X such that*

$$X = \bigcup_{i \in I} A_i$$

and

$$\forall i, j \in I, i \neq j \Rightarrow A_i \cap A_j = \emptyset$$

*we say that P is a **partition** of X.*

Remark *Thus, the set of equivalence classes of an equivalence relation on X is a partition of X.*

Counting

Definition *Two sets have the same **cardinality** if and only if there is a bijection from one set to the other.*

Definition *A finite set A has n elements if and only if there is a bijection from $\{1, 2, 3, \ldots, n\}$ to A.*

Remark *If two sets have the same cardinality then they are both infinite, or both finite. If they are finite the have the same number of elements.*

Toy Geometries

Incidence Structure

Definition *An **incidence structure** is an ordered pair of sets $(\mathcal{P}, \mathcal{L})$ such that $\mathcal{L}$ is a set of subsets of $\mathcal{P}$. The elements of the set $\mathcal{P}$ are called **points** and the elements of the set $\mathcal{L}$ are called **lines**. If A is a point and l is a line then the following phrases all mean the same thing: "$A \in l$", "A is on l", "A is contained in l", "A is incident with l", "l goes through A", "l contains A".*

Example $(\{\ \}, \{\ \})$ *is a trivial incidence structure.*

Example $(\{\spadesuit,\clubsuit,\heartsuit\},\{\{\spadesuit\},\{\spadesuit,\heartsuit\},\{\clubsuit,\heartsuit\}\})$ *is an incidence structure. In this structure* $\spadesuit,\heartsuit$ *are collinear points (see below), but* $\spadesuit,\clubsuit$ *are not.*

Example $(\mathbb{R}^2,\mathcal{L})$ *where*

$$\mathcal{L} = \left\{l : l = \{(x,y) \in \mathbb{R}^2 : ax+by+c = 0\} \text{ for some } a,b,c \in \mathbb{R} \text{ with } a \neq 0 \text{ or } b \neq 0\right\}$$

is an incidence structure.

Definition *A **figure** in an incidence structure is a subset of the set of points.*

Definition *Two lines in an incidence structure **intersect** if and only if they have a point in common, i.e.* l *intersects* m *iff there exists* A *such that* A *is on* l *and* A *is on* m*. In this situation we say that the lines* l *and* m ***intersect at*** A*. A set of lines, all of which contain a point* A *are said to be **concurrent**.*

Definition *The points in a figure are **collinear** if there exists a line containing every point in the figure.*

Definition *Two lines* l,m *are **parallel** if and only if* $l \cap m = \emptyset$*. We write* $l \parallel m$ *as an abbreviation for "*l *is parallel to* m*".*

Example *Which pairs of lines are parallel in the previous examples?*

Definition *Let* l *be a line in an incidence structure. Then the **parallel class** of* l *is the set consisting of* l *and all lines parallel to* l*. We denote this set as* $\mathrm{ParallelClass}(l)$*.*

Example *What is the parallel class of* $\{\spadesuit,\heartsuit\}$ *in the second example above?*

Example *What is the parallel class of the line* $\{(x,y) : x+y+1 = 0\}$ *in the third example above?*

Definition *Let* A *be a point in an incidence structure. Then the **pencil of lines through*** A *is the set consisting of all lines containing the point* A*. We denote this set as* $\mathrm{Pencil}(A)$*.*

Example *What is the pencil of lines through* $\spadesuit$ *in the second example above?*

Example *What is the pencil of lines through the origin in the third example above?*

Notation *To simplify notation capital letters will represent points and lower case letters will represent lines unless specifically stated otherwise. This applies to variables bound by quantifiers also.*

Remark *From now on, whenever we discuss points and lines, we will be referring to elements of some incidence structure unless specifically stated otherwise.*

Example *Prove or disprove that in any incidence structure no two distinct points can have the same pencil of lines.*

Example *Let* l,m,n *be distinct lines in an incidence structure* $(\mathcal{P},\mathcal{L})$*. Prove or disprove that if* $l \parallel m$ *and* n *intersects both* l *and* m*, then there exist at least two points.*

Example *Let* $(\mathbb{R}^2,\mathcal{L})$ *be the incidence structure defined by*

$$\mathcal{L} = \left\{l : l = \{(x,y) \in \mathbb{R}^2 : y = mx + b\} \text{ for some } m,b \in \mathbb{R}\right\}$$

Prove or disprove that two nonvertical lines are parallel if and only if they have the same slope and different y-intercepts.

Affine Planes

Definition *An **affine plane** is an incidence structure satisfying the following three axioms.*

A1. There is a unique line through any two distinct points.

A2. Through any point not on a given line, there is a unique line parallel to the given line

A3. There are three points which are not collinear.

Remark *More formally an affine plane is an incidence structure $(\mathcal{P},\mathcal{L})$ satisfying the following three axioms:*

A1. $\forall A,B, A \neq B \Rightarrow \exists! l, A \in l$ and $B \in l$

A2. $\forall A \forall l,\ A \notin l \Rightarrow \exists! m, A \in m$ and $m \parallel l$

A3. $\exists A,B,C, A \neq B$ and $B \neq C$ and $A \neq C$ and $\forall l, \sim\left(A \in l \text{ and } B \in l \text{ and } C \in l\right)$

Definition *Let A,B be distinct points in any incidence structure satisfying axiom A1. Then the unique line through A and B is denoted $\overleftrightarrow{AB}$.*

Example *The Cartesian plane $(\mathbb{R}^2,\mathcal{L})$ example given above is an affine plane.*

Example *Let $\mathcal{P} = \{A,B,C,D\}$ and $\mathcal{L} = \{\{A,B\},\{A,C\},\{A,D\},\{B,C\},\{B,D\},\{C,D\}\}$. Then $(\mathcal{P},\mathcal{L})$ is an affine plane.*

Theorem *Two distinct lines in an affine plane can intersect in at most one point.*

Proof:

1.	Let l and m be lines in an affine plane with $l \neq m$	Given
2.	Assume l,m intersect at more than one point	-
3.	l,m intersect at points A,B for some A,B with $A \neq B$	def of "more than one";2
4.	A is on l and A is on m and B is on l and B is on m	def of "intersect at";3
5.	$l = \overleftrightarrow{AB}$ and $m = \overleftrightarrow{AB}$	A1 (or def of $\overleftrightarrow{AB}$);4
6.	$l = m$	subst;5,5
7.	$\rightarrow\leftarrow$	$\rightarrow\leftarrow$ +;1,6
8.	$\leftarrow$	-
9.	l,m do not intersect at more than one point	$\sim$ +;2,7,8

QED

Definition *An affine plane $(\mathcal{P},\mathcal{L})$ where $\mathcal{P}$ is a finite set of points is called a **finite** affine plane.*

Theorem *In any finite affine plane, if one line consists of exactly n points, then every line consists of exactly n points.*

Proof: Homework.

Definition *An affine plane in which every line has n points is called an **affine plane of order** n.*

Projective Planes

Definition *A **projective plane** is an incidence structure satisfying the following three axioms.*

P1. There is a unique line through any two distinct points.

P2. There is a unique point on any two distinct lines.

P3. There are four distinct points, no three of which are collinear.

Remark *More formally a projective plane is an incidence structure $(\mathcal{P},\mathcal{L})$ satisfying the following three axioms:*

P1. $\forall A,B, A \neq B \Rightarrow \exists! l, A \in l \text{ and } B \in l$

P2. $\forall l,m, l \neq m \Rightarrow \exists! A, A \in l \text{ and } A \in m$

P3. $\exists A,B,C,D, A \neq B \text{ and } A \neq C \text{ and } A \neq D \text{ and } B \neq C \text{ and } B \neq D \text{ and } C \neq D \text{ and }$
$\sim\exists l, (A \in l \text{ and } B \in l \text{ and } C \in l) \text{ or } (A \in l \text{ and } B \in l \text{ and } D \in l) \text{ or }$
$(A \in l \text{ and } C \in l \text{ and } D \in l) \text{ or } (B \in l \text{ and } C \in l \text{ and } D \in l)$

Remark *Since a projective plane satisfies P1 the unique line through A and B is still denoted $\overleftrightarrow{AB}$.*

Remark *The formal version of P3 shows why we don't use strictly formal proofs for everything! Yuk!*

Example *Let $\mathcal{P} = \{A,B,C,D,E,F,G\}$ and*

$$\mathcal{L} = \{\{A,B,C\},\{C,D,E\},\{A,E,F\},\{A,D,G\},\{B,E,G\},\{C,F,G\},\{B,D,F\}\}$$

Then $(\mathcal{P},\mathcal{L})$ is a projective plane.

Example *Let $\mathcal{P}$ be the set of (Euclidean) lines through the origin in $\mathbb{R}^3$ and $\mathcal{L}$ be the set of planes through the origin in $\mathbb{R}^3$. Then $(\mathcal{P},\mathcal{L})$ is a projective plane.*

Proof: [Note: In the following proof we assume all elementary facts about the Euclidean geometry of $\mathbb{R}^3$ are given.]

1. Let $\mathcal{P}$ be the set of (Euclidean) lines through the origin in $\mathbb{R}^3$ and $\mathcal{L}$ be the set of planes through the origin in $\mathbb{R}^3$
2. Let A,B be distinct points in $(\mathcal{P},\mathcal{L})$
3. A,B are Euclidean lines through the origin in $\mathbb{R}^3$
4. There exists a unique plane in $\mathbb{R}^3$containing A and B
5. There exists a unique line in $(\mathcal{P},\mathcal{L})$ containing A and B
6. $(\mathcal{P},\mathcal{L})$ satisfies axiom $P1$

7. Let l,m be distinct lines in $(\mathcal{P},\mathcal{L})$
8. l,m are planes in $\mathbb{R}^3$ passing through the origin
9. The intersection of l,m is a Euclidean line
10. The origin is on l and the origin is on m
11. The origin is on the intersection of l,m
12. The intersection of l,m is a unique Euclidean line through the origin
13. l,m intersect at a unique point in $(\mathcal{P},\mathcal{L})$

14. $(\mathcal{P},\mathcal{L})$ satisfies axiom $P2$

15. Let A,B,C,D be the x-axis, y-axis, z-axis, and the line through $(1,1,1)$ and the origin in $\mathbb{R}^3$ respectively
16. A,B,C,D are distinct lines through the origin in $\mathbb{R}^3$
17. A,B,C,D are distinct points in $(\mathcal{P},\mathcal{L})$
18. The planes in $\mathbb{R}^3$ determined by any two of A,B,C,D are distinct
19. The lines in $(\mathcal{P},\mathcal{L})$ determined by any two of A,B,C,D are distinct
20. No three of the points A,B,C,D are collinear in $(\mathcal{P},\mathcal{L})$
21. There exist four distinct points in $(\mathcal{P},\mathcal{L})$, no three of which are collinear
22. $(\mathcal{P},\mathcal{L})$ satisfies axiom $P3$

23. $(\mathcal{P},\mathcal{L})$ is a projective plane

QED

Example *Let $(\mathcal{P},\mathcal{L})$ be an affine plane. Construct a new incidence structure $(\mathcal{P}',\mathcal{L}')$ as follows. For each parallel class $\mathcal{C}$ in $(\mathcal{P},\mathcal{L})$ let $P_{\mathcal{C}}$ be a new point distinct from those already in $\mathcal{P}$ and each other. Define $\mathcal{P}' = \mathcal{P} \cup \{P_{\mathcal{C}} : \mathcal{C} \text{ a parallel class of } \mathcal{L}\}$ and*

$$\mathcal{L}' = \{l \cup \{P_{\mathcal{C}}\} : l \in \mathcal{C}\} \cup \{\mathcal{P}' - \mathcal{P}\}$$

*i.e. $\mathcal{P}'$ consists of all of the points in $\mathcal{P}$ plus one new point for each parallel class, and $\mathcal{L}'$ is obtained from $\mathcal{L}$ by adding to each line the new point in its parallel class and one additional line consisting of all of the new points. Then $(\mathcal{P}',\mathcal{L}')$ is a projective plane. (proof: homework). The projective plane $(\mathcal{P}',\mathcal{L}')$ is called the **projective completion** of the affine plane $(\mathcal{P},\mathcal{L})$.*

Example *Similarly if we start with a projective plane and remove a single line and all of the points on that line, while maintaining the collinearity of the points that remain, we obtain an affine plane.*

Definition *A projective plane $(\mathcal{P},\mathcal{L})$ where $\mathcal{P}$ is a finite set of points is called a **finite** projective plane.*

Theorem *In any finite projective plane, if one line consists of exactly n points, then every line consists of exactly n points.*

Proof: Homework.

Definition *A projective plane in which every line has $n+1$ elements is called a **projective plane of order** n. (It is the projective completion of an affine plane of order n.)*

Axioms for Euclidean Geometry

There are many different axiom systems for Euclidean geometry, and in particular Euclidean plane geometry. Here are a few notable ones.

- **Euclid's Axioms** - (300 BC) Euclid's *Elements* is perhaps one of the most famous works of all time. This 13 book treatise basically defined geometry for 2000 years until modern mathematicians created alternative geometries over the past two centuries. (5 axioms and

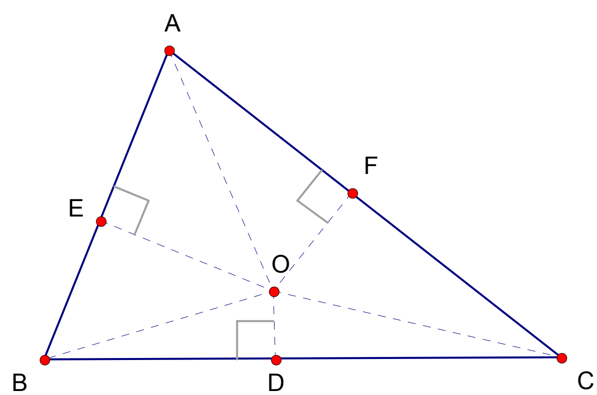
5 common notions)

- **Hilbert's Axioms** - (1899) this set of Axioms was created by David Hilbert to bring Euclid's work up to modern standards of rigor. (20 axioms and 6 undefined terms)
- **Tarski's Axioms** - (1929) developed an "minimalist" axiom system with only two undefined terms ("between" and "congruent") and eleven axioms
- **Birchoff's Axioms** - (1932) developed an axiom system using the real numbers (4 axioms and 4 undefined terms, not counting those needed to define the real numbers)
- **Bachmann's Axioms** - (1959) an axiom system for several geometries defined entirely in terms of abstract algebra, where points and lines are defined to be elements of a group. (8 axioms and two undefined terms, but more are needed to restrict to Euclidean geometry)
- **SMSG Axioms** - (~1960) the School Mathematics Study Group axioms were developed to come up with an intuitive, easy to use, but not necessarily independent set of axioms that could be used for a rigorous development of geometry that is appropriate for high school students. Like Birchoff it also uses the real numbers. (3 undefined terms, 22 axioms, not counting those needed to define the real numbers, but covers three dimensional Euclidean geometry as well as the Euclidean geometry of the plane)

The Role of Diagrams

Diagrams are very useful in Euclidean geometry to illustrate the concepts and build intuition. However much care must be taken to not rely too heavily on what seems apparent from a diagram, as it can lead to disaster as illustrated in the following "theorem".

"**Theorem**": All triangles are isosceles.



"**Proof**": Let $\triangle ABC$ be an arbitrary triangle and let O be the intersection of the angle bisector of $\angle A$ with the perpendicular bisector of BC as shown in the diagram. Constuct the feet of the perpediculars from O to the other sides and verify that $\triangle OBD \equiv \triangle OCD$, $\triangle OAE \equiv \triangle OAF$, and $\triangle OBE \equiv \triangle OCF$. Hence the triangle is isosceles.
~QED

SMSG Axioms for Euclidean Plane Geometry

Out of nothing I have created a strange new universe. -János Bolyai

The following axioms, definitions, and theorems are based on those developed by the original School Mathematics Study Group. I have modified them slightly to be consistent with other things we are doing in the course.

Definition *The **Euclidean plane**, $\mathbb{E}$, is an incidence structure $(\mathcal{P},\mathcal{L})$ satisfying the following axioms and definitions. The axioms are labeled **S1**, **S2**, **S3**, etc.*

Remark *Unless otherwise stated, upper case letters like A,B,C represent points and lower case letters such as l,m,n represent lines.*

(*two points determine a line)*
S1: For any two distinct points there is exactly one line which contains them both.

(*distance axiom)*
S2: To any two distinct points there corresponds a unique positive number.

Definition *The unique positive number corresponding to a pair of distinct points is called the **distance** between the points. The distance between a point and itself is defined to be 0. The distance between any two points A and B is denoted $d(A,B)$ or $|AB|$.*

Remark *Note that the order of the points doesn't matter so that $d(A,B) = d(B,A)$ for all points A,B.*

(*coordinate axiom*)
S3: The points on a line can be placed into a bijective correspondence with the real numbers such that the distance between two points is the absolute value of the difference between their corresponding numbers.

Definition *For each line l and correspondence with the real numbers given by **S3**, define $\Psi(A)$ to be the real number corresponding to the point A on l. By axiom S3, $\Psi : l \to \mathbb{R}$ is a bijection satisfying $d(A,B) = |\Psi(B) - \Psi(A)|$ for all points A and B on l. The function Ψ is called a **coordinate system** for the line l and $\Psi(A)$ is called the **coordinate** of point A.*

(*ruler placement axiom*)
S4: For any line l and any points A and B on l, there exists a coordinate system Ψ for l such that $\Psi(A) = 0$ and $\Psi(B) > 0$.

(*noncollinear points exist*)
S5: There exist three noncollinear points.

Definition *Point B is **between** points A and C if and only if*
(i) A,B,C are distinct collinear points and
(ii) $|AC| = |AB| + |BC|$
If B is between A and C we write $A.B.C$..

Definition *For any two distinct points A,B the **segment** AB is the set consisting of the points A,B and all points C with $A.C.B$, i.e.*

$$AB = \left\{C : A.C.B \text{ or } C = A \text{ or } C = B\right\}$$

*The points A and B are called the **endpoints** of segment AB.*

Definition *The distance $|AB|$ is called the **length** of the segment AB.*

Definition *For any two distinct points A, B then **ray** $\overrightarrow{AB}$ is the set consisting of the points A, B and all points C with $A.C.B$ or $A.B.C$, i.e.*

$$\overrightarrow{AB} = \left\{C : A.C.B \text{ or } A.B.C \text{ or } C = A \text{ or } C = B\right\}$$

*The point A is called the **endpoint** or **vertex** of ray $\overrightarrow{AB}$.*

Definition *If $B.A.C$ then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called **opposite** rays.*

Definition *A point M is called a **midpoint** of segment AB if and only if $A.M.B$ and $|AM| = |MB|$.*

Definition *Any figure whose intersection with a line segment is a midpoint of the segment is said to **bisect** the segment.*

Definition *A figure F is **convex** if and only if for any two distinct points $A, B \in F$, $AB \subseteq F$.*

(*separation axiom*)
S6: Every line l divides the set of points not on l into two disjoint convex sets L and R such that for any $P \in L$ and $Q \in R$ the segment PQ contains a point on l.

Definition *The sets L and R in axiom S6 are called **half-planes** and the line l is called an **edge** of each of them. We say that l **separates** the Euclidean plane into two half-planes. If two points A and B are contained in the same half-plane we say that they are on the **same side** of l. If A is in one half-plane and B in the other we say that they are on **opposite sides** of l.*

Definition *An **angle** is the union of two rays having a common end point which are not contained in a single line. The rays are called the **sides** of the angle and the common endpoint is called the **vertex** of the angle. If the two sides of an angle are $\overrightarrow{AB}$ and $\overrightarrow{AC}$ we denote this angle as $\angle BAC$.*

(*angle measure*)
S7: To each angle there corresponds a unique real number strictly between 0 and 180.

Definition *The number corresponding to a given angle described in axiom S7 is called the **measure** of the angle. The measure of $\angle BAC$ is denoted $|\angle BAC|$ or $m\angle BAC$. (It is also often denoted $\angle BAC$ when the distinction between the angle and its measure is clear from context.)*

Definition *If A, B, C are non-collinear points then the union of the segments AB, BC, CA is called a **triangle** and denoted $\triangle ABC$, i.e. $\triangle ABC = AB \cup BC \cup CA$. The segments AB, BC, CA are called the **sides** of the triangle. The **angles** of the triangle $\triangle ABC$ are $\angle BAC$, $\angle ABC$, and $\angle ACB$. When it is clear from context these angles are often abbreviated by $\angle A$, $\angle B$, and $\angle C$ respectively, and the lengths of sides AB, BC, and CA are often denoted as c, a, and b respectively. The side AB is called the side **opposite** angle $\angle C$ in the triangle and similarly* *for the other two pairs of sides and angles.*

Definition *The* ***interior*** *of* $\angle BAC$ *is the set of points* P *such that* B *and* P *are on the same side of* $\overleftrightarrow{AC}$ *and* C *and* P *are on the same side of* $\overleftrightarrow{AB}$. *The* ***exterior*** *of an angle is the set of all points not in the interior or on the angle itself.*

(*angle construction*)
S8: Let $\overrightarrow{AB}$ be a ray on the edge of the half plane H. For every real number r between 0 and 180 there exists exactly one ray $\overrightarrow{AP}$ with P in H such that $|\angle BAP| = r$.

Remark *Note that by axiom S8 every half-plane is nonempty, because it contains the point* P.

(*angle addition*)
S9: If D is a point on the interior of $\angle BAC$ then $|\angle BAC| = |\angle BAD| + |\angle CAD|$.

Definition *If* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ *are opposite rays and* $\overrightarrow{AD}$ *is another ray then* $\angle BAD$ *and* $\angle CAD$ *are called a* ***linear pair*** *of angles. If* $\overrightarrow{AC}$ *and* $\overrightarrow{AB}$ *are opposite rays, we will sometimes find it convenient to say* $\angle CAB$ *is a* ***straight angle*** *even though it technically is not an angle.*

Definition *Two angles are* ***supplementary*** *if the sum of their measures is* 180. *If two angles are supplementary each angle is called a* ***supplement*** *of the other.*

Definition *Two angles are* ***complementary*** *if the sum of their measures is* 90. *If two angles are supplementary each angle is called a* ***complement*** *of the other.*

(*supplement axiom*)
S10: If two angles form a linear pair, then they are supplementary.

Definition *If two angles in a linear pair have the same measure then each is called a* ***right angle***.

Definition *Two intersecting figures, each of which is either a line, ray, or segment, are* ***perpendicular*** *if the lines containing them determine a right angle.*

Definition *The* ***perpendicular bisector*** *of a segment is the line containing the midpoint that is perpendicular to the segment.*

Definition *Angles are* ***congruent angles*** *if and only if they have the same measure. Segments are* ***congruent segments*** *if and only if they have the same length. If* AB *is congruent to* CD *we write* $AB \equiv CD$ *and similarly if* $\angle ABC$ *is congruent to* $\angle DEF$ *we write* $\angle ABC \equiv \angle DEF$.

Definition *In triangles* $\triangle ABC$ *and* $\triangle A'B'C'$ *if* $AB \equiv A'B'$, $BC \equiv B'C'$, $CA \equiv C'A'$, $\angle A \equiv \angle A'$, $\angle B \equiv \angle B'$, *and* $\angle C \equiv \angle C'$ *then we say that triangle* $\triangle ABC$ *and* $\triangle A'B'C'$ *are* ***congruent triangles*** *and write* $\triangle ABC \equiv \triangle A'B'C'$. *The three pairs of segments and three pairs of angles are called* ***corresponding parts*** *of the two (not necessarily distinct) triangles. Conversely, if* $\triangle ABC \equiv \triangle A'B'C'$ *then there exists a bijective correspondence*

$\xi : \{A,B,C\} \to \{A',B',C'\}$ such that if $D = \xi(A)$, $E = \xi(B)$, and $F = \xi(C)$ and $AB \equiv DE$, $BC \equiv EF$, $CA \equiv FD$, $\angle A \equiv \angle D$, $\angle B \equiv \angle E$, and $\angle C \equiv \angle F$. In this situation we say AB and DE, BC and EF, AC and DF, $\angle A$ and $\angle D$, $\angle B$ and $\angle E$, and $\angle C$ and $\angle F$, are corresponding parts, and each pair of corresponding parts are congruent.

Remark *Note: that for convenience, whenever possible, we indicate the function ξ by listing the points in the name of the triangles involved in a congruence relation in the order that they correspond to each other, e.g. if $\triangle ABC \equiv \triangle A'B'C'$ we can and should assume that $\xi(A) = A'$, $\xi(B) = B'$, and $\xi(C) = C'$ whenever it is convenient and causes no logical problems.*

(*SAS*)
S11: If $AB \equiv A'B', BC \equiv B'C'$, and $\angle B \equiv \angle B'$ then $\triangle ABC \equiv \triangle A'B'C'$.

(*parallel axiom*)
S12: Through any point not on a given line there is exactly one line that is parallel to the given line.

Definition *A **polygon** is any figure consisting entirely of segments A_1A_2, $A_2A_3, \ldots A_{n-1}A_n, A_nA_1$ having the property that no two of these segments intersect except at their endpoints as specified, and no two segments that intersect are collinear. The segments A_iA_j are called the **sides** of the polygon and the points $A_1, A_2, \ldots A_n$ are called its **vertices**.*

Definition *A **triangular region** is the union of a triangle and its interior. A **polygonal region** is a union of finitely many triangular regions such that if any two of them intersect, they intersect only in a segment or a point.*

(*area axiom*)
S13: To every polygonal region corresponds a unique positive real number.

Definition *The positive real number associated with a polygonal region given by axiom S13 is called the **area** of the polygonal region (or simply the area of the polygon).*

(*congruence preserves area*)
S14: If two triangles are congruent then they have the same area.

(*area addition*)
S15: If two regions intersect in at most a finite number of segments and points, then the area of their union is the sum of their areas.

(*area of a rectangle axiom*)
S16: The area of a rectangle is the product of the length of two of its sides that share a vertex.

Euclidean Plane Geometry Axioms - Quick Reference

S1: (*two points determine a line)* For any two distinct points there is exactly one line which contains them both.

S2: (*distance axiom)* For each pair of distinct points there corresponds a unique positive number.

S3: (*coordinate axiom*) The points on a line can be placed into a bijective correspondence with the real numbers such that the distance between two points is the absolute value of the difference between their corresponding numbers

S4: (*ruler placement axiom*) For any line l and any points A and B on l, there exists a coordinate system Ψ_l such that $\Psi_l(A) = 0$ and $\Psi_l(B) > 0$.

S5: (*noncollinear points exist*) There exist three noncollinear points.

S6: (*separation axiom*) Every line l divides the set of points not on l into two disjoint convex sets L and R such that for any $P \in L$ and $Q \in R$ the segment PQ contains a point on l.

S7: (*angle measure*) To each angle there corresponds a unique real number between 0 and 180 inclusive.

S8: (*angle construction*) Let $\overrightarrow{AB}$ be a ray on the edge of the half plane H. For every real number r between 0 and 180 there exists exactly one ray $\overrightarrow{AP}$ with P in H such that $|\angle BAP| = r$.

S9: (*angle addition*) If D is a point on the interior of $\angle BAC$ then $|\angle BAC| = |\angle BAD| + |\angle CAD|$.

S10: (*supplement axiom*) If two angles form a linear pair, then they are supplementary.

S11: (*SAS*) If $AB \equiv A'B', BC \equiv B'C'$, and $\angle B \equiv \angle B'$ then $\triangle ABC \equiv \triangle A'B'C'$.

S12: (*parallel axiom*) Through any point not on a given line there is exactly one line that is parallel to the given line.

S13: (*area axiom*) To every polygon corresponds a unique positive real number.

S14: (*congruence preserves area*) If two triangles are congruent they have the same area.

S15: (*areas addition*) If two regions intersect in at most a finite number of segments and points, then the area of their union is the sum of their areas.

S16: (*area of a rectangle axiom*) The area of a rectangle is the product of the length of two of its sides that share a vertex.

Euclidean Plane Geometry Definitions - Quick Reference

Definition	is equivalent to
$(\mathcal{P},\mathcal{L})$ is an **incidence structure**	$\mathcal{L}$ is a set of subsets of $\mathcal{P}$
A is a **point**	$A \in \mathcal{P}$ where $(\mathcal{P},\mathcal{L})$ is an incidence structure
l is a **line**	$l \in \mathcal{L}$ where $(\mathcal{P},\mathcal{L})$ is an incidence structure
F is a **figure**	$F \subseteq \mathcal{P}$ where $(\mathcal{P},\mathcal{L})$ is an incidence structure
figure F is **collinear**	$F \subseteq l$ for some line l
lines l,m are **parallel**	$l \cap m = \emptyset$
$l \parallel m$	l is parallel to m
C is the **parallel class** of l	$C = \left\{m : m \parallel l \text{ or } m = l\right\}$
P is the **pencil of lines** through point A	$P = \{m : A \in m\}$
$\mathbb{E}$ is a **Euclidean Plane**	$\mathbb{E}$ is an incidence structure satisfying axioms S1-S16
$\lvert AB\rvert$ is the **distance** from A to B	$\lvert AB\rvert$ is the unique positive number assigned to the distinct pair of points A,B in axiom S2
$\lvert AA\rvert$ is the **distance** from A to A	$\lvert AA\rvert = 0$
Ψ_l is a **coordinate system**	$\Psi_l : l \to \mathbb{R}$ is a bijection and l is a line
x is the **coordinate** of point A	$x = \Psi_l(A)$
B is **between** A and C	points A,B,C are collinear and $\lvert AB\rvert + \lvert BC\rvert = \lvert AC\rvert$
$A.B.C$	B is between A and C
AB is a **segment**	$AB = \{P : A.P.B\} \cup \{A,B\}$
$\overrightarrow{AB}$ is a **ray**	$\overrightarrow{AB} = \{P : A.B.P\} \cup AB$
x is an **endpoint** or **vertex** of AB	$x = A$ or $x = B$
x is an **endpoint** or **vertex** of $\overrightarrow{AB}$	$x = A$
$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are **opposite rays**	$C.A.B$
M is the **midpoint** of AB	$A.M.B$ and $\lvert AM\rvert = \lvert MB\rvert$
F is a **convex** figure	for all A,B in F, $AB \subseteq F$
L,R are **half planes** with **edge** l	L,R are the disjoint sets of points not the line l given by axiom S6
A,B are on the **same side** of l	A,B are in the same half plane formed by l
A,B are on the **opposite side** of l	A,B are different half planes formed by l
$\angle ABC$ is an **angle**	$\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$
x is the **vertex** of $\angle ABC$	$x = B$

Euclidean Plane Geometry Definition - Quick Reference (cont)

Definition	is equivalent to
$\triangle ABC$ is a **triangle**	A, B, C are distinct, not collinear, and $\triangle ABC = AB \cup BC \cup CA$
$\triangle ABC$ is a **degenerate triangle**	A, B, C are collinear
P is in the **interior** of $\angle BAC$	P is on the same side of $\overrightarrow{AB}$ as C and P is on the same side of $\overrightarrow{AC}$ as B
P is in the **exterior** of $\angle BAC$	P is not on the interior of or on $\angle BAC$
$\angle BAD$ and $\angle CAD$ are a **linear pair**	$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays and D is not on $\overleftrightarrow{BC}$
$\angle A, \angle B$ are **supplementary**	$\lvert\angle A\rvert + \lvert\angle B\rvert = 180$
$\angle A, \angle B$ are **complementary**	$\lvert\angle A\rvert + \lvert\angle B\rvert = 90$
$\angle A$ and $\angle B$ are **right angles**	$\angle A, \angle B$ are a linear pair and $\lvert\angle A\rvert = \lvert\angle B\rvert$
2 collinear figures are **perpendicular**	the lines containing the figures determine a right angle
l is a **perpendicular bisector** of AB	l contains the midpoint of AB and is perpendicular to AB
$\angle A$ is **congruent** to $\angle B$	$\lvert\angle A\rvert = \lvert\angle B\rvert$
AB is **congruent** to CD	$\lvert AB\rvert = \lvert CD\rvert$
$\triangle ABC$ is **congruent** to $\triangle A'B'C'$	$\exists \xi : \{A, B, C\} \to \{A', B', C'\}$ a bijection such that $AB \equiv \xi(A)\xi(B), BC \equiv \xi(B)\xi(C), AC \equiv \xi(A)\xi(C),$ $\angle A \equiv \angle\xi(A), \angle B \equiv \angle\xi(B),$ and $\angle C \equiv \angle\xi(C)$
$A_1A_2A_3\cdots A_n$ is a **polygon**	$A_1A_2A_3\cdots A_n = (\bigcup_{i=1}^{n-1} A_iA_{i+1}) \cup A_1A_n$ and these segments intersect only at their endpoints and no intersecting segments are collinear
$[A_1A_2A_3\cdots A_n]$ is the **area**	$[A_1A_2A_3\cdots A_n]$ is the real number given by axiom S12

Review of Elementary Plane Euclidean Geometry

Some common definitions

Angles

- **obtuse angle**: an angle whose measure is greater than 90
- **acute angle**: an angle whose measure is less than 90
- **adjacent angles**: two angles that share a common ray
- **transversal**: a line k that intersects two other lines l, m at one point. The various pairs of the eight angles formed by k with l and m are:

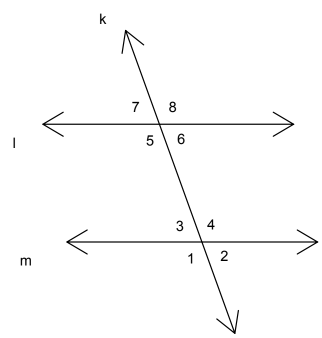


Figure 1

- **alternate interior angles**: $\{4,5\},\{3,6\}$
- **alternate exterior angles**: $\{2,7\},\{1,8\}$
- **same side interior angles**: $\{3,5\},\{4,6\}$
- **same side exterior angles**: $\{1,7\},\{2,8\}$
- **corresponding angles**: $\{1,5\},\{3,7\},\{2,6\},\{4,8\}$
- **vertical angles**: $\{1,4\},\{2,3\},\{5,8\},\{7,6\}$
- **adjacent angles**: $\{1,3\},\{3,4\},\{4,2\},\{1,2\},\{5,7\},\{7,8\},\{8,6\},\{5,6\}$

Triangles

- **isosceles triangle**: a triangle with two congruent angles
- **equilateral triangle**: a triangle with three congruent sides
- **scalene triangle**: a triangle in which no two sides are congruent
- **obtuse triangle**: a triangle containing an obtuse angle
- **acute triangle**: a triangle containing an acute angle
- **right triangle**: a triangle containing a right angle
- **legs**: the sides forming the right angle in a right triangle
- **hypotenuse**: side opposite the right angle in a right triangle
- **exterior angle**: the angle formed by the opposite ray of the ray containing one side of a triangle (or polygon) and the side of the triangle (or polygon) that shares the same vertex
- **degenerate triangle**: $AB \cup BC \cup AC$ when A, B, C are collinear (note that a degenerate triangle is not actually a triangle, but rather is thought of as what you would get if the three vertices of a triangle were moved continuously to become collinear).

Cevians and Related Segments

- **cevian**: any line segment joining a vertex of a triangle to a point on the opposite side other than the vertices
- **altitude**: the perpendicular segment from a vertex of a triangle to the line containing the opposite side. Also any segment connecting the line containing one side of a parallelogram or trapezoid to the line containing the opposite side that is perpendicular to both lines
- **base**: given an altitude or cevian of a triangle, the base of the triangle is the side opposite to the vertex containing the altitude or cevian. We say that the base and altitude/cevian correspond to each other. Similarly the vertex contained by the cevian or altitude is called its vertex. Also the parallel sides in a trapezoid are called its bases.
- **distance between parallel lines**: is the length of any segment that is connects a point on one line to a point on a line parallel to it and is perpendicular to both sides (see SMSG Thm 53 below).
- **median**: a cevian connecting a vertex to the midpoint of the opposite side
- **angle bisector**: a ray containing a point in the interior of an angle that bisects the angle (i.e. an angle bisector of $\angle ABC$ is a ray $\overrightarrow{AD}$ such that D is in the interior of $\angle ABC$ and $\angle ABD \equiv \angle DBC$). Also the cevian contained in the angle bisector of the angle of a triangle, or the line containing the angle bisector of an angle.
- **foot**: the foot of a cevian or altitude is the point where it meets the line containing the side opposite its vertex. Given a point P not on a line l the foot of the perpendicular line from P to l is the point where that line intersects l. If P is on l, the foot of the perpendicular through P to l is P itself.
- **parallel collinear figures**: two figures, each of which is a subset of a line, are parallel if the two lines containing the figures are parallel

Polygons

- **quadrilateral**: a polygon with exactly four sides
- **trapezoid**: a quadrilateral with exactly one pair of parallel sides
- **isosceles trapezoid**: a trapezoid having two angles that share one of the sides in the parallel pair
- **parallelogram**: a quadrilateral with two pairs of parallel sides
- **rectangle**: a quadrilateral with four congruent angles
- **rhombus**: a quadrilateral with four congruent sides
- **square**: a quadrilateral that is both a rectangle and a rhombus
- **pentagon**: a polygon with five sides
- **hexagon**: a polygon with six sides
- **heptagon**: a polygon with seven sides
- **octagon**: a polygon with eight sides
- **nonagon**: a polygon with nine sides
- **decagon**: a polygon with ten sides
- **regular polygon**: a polygon with n sides having all sides and all angles congruent

Circles

- **circle**: a figure consisting of all points that are a fixed positive distance r from a given point O. We if A is a point on the circle with center O and radius r we designate this circle as $\odot OA$ and write $|\odot OA| = r$.

- **center**: the point O in the definition of a circle
- **radius**: any segment having the center and a point on the circle as endpoints. The length of any radius is also referred to as the radius of the circle
- **circle congruence**: two circles are congruent if and only if their radii are the same length. Every circle is congruent to itself.
- **chord**: a segment connecting two distinct points on a circle
- **diameter**: a chord that contains the center. The length of any diameter is also called the diameter of the circle.
- **tangent**: a line that intersects a circle at exactly one point
- **tangent circles**: two circles that intersect at exactly one point
- **central angle**: an angle whose vertex is the center of a given circle
- **arc**: the set of points on a circle that are in the interior of or on a central angle or the set of points on a circle that are in the exterior of or on a central angle
- **measure of an arc**: the measure of the corresponding central angle if the arc is in the interior of the central angle and 360 minus the measure of the central angle otherwise

Similarity

- **similar triangles**: Two triangles are similar if and only if there is a correspondence between them such that the corresponding angles are congruent and lengths of the corresponding sides are proportional. If $\triangle ABC$ is similar to $\triangle DEF$ we write $\triangle ABC \sim \triangle DEF$.

Remark *Sometimes we will refer to a segment and its length interchangeably when there can be no possiblility of confusion from context. For example we might say that the area of a triangle is half the product of an altitude and the corresponding base instead of saying that it is half the product of the length of an altitude and the length of the corresponding base. However we must take care when using this in situations where it might be ambiguous, for example, saying that two triangles have altitudes of equal length is not the same as saying that two triangles have equal altitudes (i.e. equal as sets of points).*

Review of Some Elementary Theorems

The following list of theorems are numbered so we can reference them by number as we need them in the course. If the number is followed by a phrase in parentheses, that denotes the name of the theorem. We should refer to a theorem by name whenever a name is available and use the number when no name is available. When referencing these theorems in your proofs refer to them as, e.g. SMSG Thm 4 to distinguish them from other proofs we prove in this course.

* proven for homework

Thm

1. * Every line contains infinitely many points.
2. * The Euclidean plane is an affine plane.
3. * ($\equiv$ *is an equivalence relation*) Congruence of segments (respectively angles) is an equivalence relation on the set of all segments (respectively angles).
4. * If l, m are distinct lines and $F \subseteq l$ and $G \subseteq m$ then F and G intersect in at most one

point. (so for example, if F and G are segments or rays they can intersect in at most one point).

5. If $A.B.C$ then $C.B.A$.
6. (*order thm*) Let A, B, C be three distinct collinear points and Ψ a coordinate system for the line containing them. Then
$$\Psi(A) < \Psi(B) < \Psi(C) \text{ or } \Psi(C) < \Psi(B) < \Psi(A) \Leftrightarrow A.B.C$$
7. * If $A.B.C$ and $B.C.D$ then $A.B.D$ and $A.C.D$.
8. * If A, B, C are distinct collinear points then exactly one of the three statements $A.B.C$, $B.A.C$, $A.C.B$ is true.
9. * (*point plotting thm*) For any ray $\overrightarrow{AB}$ and positive real number r, there exists exactly one point P on $\overrightarrow{AB}$ such that $|AP| = r$. In particular, we can extend any segment to any length in either direction.

   (*corollary*) If CD is a segment and $\overrightarrow{AB}$ a ray there is a unique point P on $\overrightarrow{AB}$ such that $AP \equiv CD$.
10. Midpoints exist and are unique.
11. * If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays then $\overrightarrow{AB} \cup \overrightarrow{AC} = \overleftrightarrow{BC}$ and $\overrightarrow{AB} \cap \overrightarrow{AC} = \{A\}$.
12. * Every point A on l divides the points other than A on l into two disjoint convex sets L and R such that for any $P \in L$ and $Q \in R$ the segment QP contains A.

    (*point plotting thm*) If l, m are distinct lines that meet at A and r is a positive real number then m has points on both sides of l that are distance r from A (and similarly if CD is a given segment there are points P, P' on m on opposite sides of l such that $AP \equiv AP' \equiv CD$).
13. * If a line l intersects two sides of a triangle at points between the vertices, then l does not intersect the third side of the triangle.
14. * (*ray-half plane*) If l is any line containing A, and B is not on l, then all points on the ray $\overrightarrow{AB}$ except A are on the same side of l as B.
15. (*Pasch*) If l meets side AC in $\triangle ABC$ at exactly one point between A and C then l intersects AB or BC.
16. (*crossbar*) If D is in the interior of $\angle A$ in $\triangle ABC$ then $\overrightarrow{AD}$ intersects BC.
17. * In any triangle $\triangle ABC$ every point between B and C is in the interior of $\angle A$.
18. (*SOCAC*) Supplements of the same or congruent angles are congruent.
19. * (*COCAC*) Complements of the same or congruent angles are congruent.
20. * Vertical angles are congruent.
21. * (*right angles are 90*) All right angles are congruent. An angle is a right angle if and only if it has measure 90.
22. * If l, m intersect at one point and one of the four angles formed is a right angle then all four angles are right angles.
23. (*ASA*) If $\angle A \equiv \angle D$, $AB \equiv DE$, $\angle B \equiv \angle E$ then $\triangle ABC \equiv \triangle DEF$.
24. (*isosceles* $\triangle$) Two sides in a triangle are congruent if and only if the angles opposite those sides are congruent.
25. * A triangle is equilateral if and only if it is equilangular.

26. Angle bisectors exist and are unique.

27. (*SSS*) If $AB \equiv DE,\ BC \equiv EF, AC \equiv DF$ then $\triangle ABC \equiv \triangle DEF$.

28. (*uniqueness of perpendiculars*) Through a given point there exists exactly one line perpendicular to a given line.

29. * (*right triangle*) At most one angle of a triangle can be a right angle.

30. * (*perpendicular bisector*) A point P is equidistant from two distinct points A, B if and only if P is on the perpedicular bisector of AB.

31. (*exterior angle*) The measure of an exterior angle of a triangle is greater than the measure of either of the two opposite angles.

32. (*AAS*) If $\angle A \equiv \angle D,\ \angle B \equiv \angle E,$ and $BC \equiv EF$ then $\triangle ABC \equiv \triangle DEF$.

33. * (*HL*) If the hypotenuse and leg of one right triangle are congruent, respectively, to the hypotenuse and leg of another, then the right triangles are congruent.

34. (*big angle, big side*) In $\triangle ABC$, $|\angle A| > |\angle B|$ if and only if $a > b$ (i.e. $|BC| > |AC|$).

35. * (*point to line distance*) The shortest segment joining a point to a line is the perpendicular segment.

36. (*triangle inequality*) The sum of the lengths of two sides of a triangle is greater than the length of the third side.

37. * (*bigger angle, bigger side*) If two sides of one triangle are congruent respectively to two sides of a second triangle, then the included angle of the first is larger than the included angle of the second if and only if the remaining side of the first is longer than the remaining side of the second.

38. * (*common perpendicular*) Two distinct lines that are perpendicular to the same line are parallel.

39. (*alternate interior angle*) If two lines are cut by a transversal then the two lines are parallel if and only if the alternate interior angles formed are congruent.

40. * (*corresponding angles*) If two lines are cut by a transversal then the two lines are parallel if and only if any pair of corresponding angles formed are congruent. In this situation all pairs of corresponding angles are congruent.

41. * (*same side interior angles*) If two lines are cut by a transversal then the two lines are parallel if and only if any pair of same side interior angles formed are supplementary. In this situation both pairs of same side interior angles formed are supplementary.

42. * Two distinct lines parallel to the same line are parallel to each other.

43. * (*parallel class*) The set of all parallel classes is a partition of the set of all lines.

44. * (*common perpendicular*) If line l is perpendicular to line m then l is perpendicular to all lines in the parallel class of m.

45. ($\triangle$ *sum*) The sum of the measures of the angles in a triangle is 180.

46. * (*exterior angle*) The measure of an exterior angle of a triangle is equal to the sum to the measures of the angles of the triangle that are not adjacent to it.

47. * (*parallelogram*) Either diagonal of a parallelogram divides it into two congruent triangles. Any pair of parallel sides in a parallelogram are congruent. The opposite angles are congruent and consecutive angles are supplementary. The diagonals of a parallelogram bisect each other.

48. * (*distance between parallels*) If $l \parallel m$ and P, Q are on l then the distance from P to m

and

the distance from Q to m are equal.

49. * (*parallelogram*) If both pairs of opposite sides in a quadrilateral are congruent, then the quadrilateral is a parallelogram.

50. * (*parallelogram*) If two sides of a quadrilateral are parallel and congruent then the quadrilateral is a parallelogram.

51. * (*rectangle*) If a parallelogram has a right angle then it is a rectangle.

52. * (*rhombus*) The diagonals of a parallelogram are perpendicular if and only if the parallelogram is a rhombus.

53. * (*parallel projection*) Let A, B, C be distinct points on line l with $A.B.C$ and A', B', C' points on m such that $AA' \parallel BB' \parallel CC'$ then $A'.B'.C'$.

54. (*parallel projection*) Let A, B, C, D be distinct points on line l with $AB \equiv CD$ and A', B', C', D' points on m such that $AA' \parallel BB' \parallel CC' \parallel DD'$ then $A'B' \equiv C'D'$.

55. * (*equidistant parallels*) If three parallel lines intercept congruent segments on one transversal then they intercept congruent segments on every transversal.

56. * (*midpoint connector*) Let $\triangle ABC$ be a triangle and M, N the midpoints of AB and AC respectively. Then $MN \parallel BC$ and $|MN| = \frac{1}{2}|BC|$.

57. * (*area of a right triangle*) The area of a right triangle is half the product of the lengths of its legs.

58. (*area of a triangle*) The area of a triangle is half of the product of an altitude and its corresponding base.

59. * (*area of a parallelogram*) The area of a parallelogram is the product of one side and the distance from that side to the side parallel to it.

60. * (*area of a trapezoid*) The area of a trapezoid is the product of the average of the lengths of its bases and the distance between the bases.

61. * If two triangles have altitudes of the same length, the ratio of their areas is the same as the ratio of the lengths of the sides opposite the vertex containing their altitudes.

62. * Two triangles having equal length altitudes and corresponding bases have equal area.

63. (*Pythagorean Theorem*) In any right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.

64. * (*converse of the Pythagorean Theorem*) In any triangle if the square of the length of the one side is equal to the sum of the squares of the other two sides then the triangle is a right triangle with right angle opposite the longest side.

65. * (*30-60-90 triangle*) A $\triangle ABC$ has $|\angle A| = 90, |\angle B| = 60$ (and thus $|\angle C| = 30$) if and only if $|BC| = 2|AB|$ and $|AC| = \sqrt{3}\,|AB|$.

66. * (*isosceles right triangle*) A right triangle is isosceles if and only if the ratio of the length of the hypotenuse to the length of a leg is $\sqrt{2}$.

67. (*basic proportionality*) A segment connecting points on two sides of a triangle is parallel to the third side if and only if the segments it cuts off are proportional to the sides.

68. * (*AA*) If two triangles have two congruent corresponding angles then the triangles are similar.

69. * (*basic proportionality*) A line parallel to a side of a triangle that intersects the two other sides at distinct points cuts off a triangle that is similar to the original triangle.

70. * (*similarity SAS*) If $\angle A \equiv \angle D$ and $\frac{|AB|}{|DE|} = \frac{|AC|}{|DF|}$ then $\triangle ABC \sim \triangle DEF$.

71. (*similarity SSS*) If $\frac{|AB|}{|DE|} = \frac{|AC|}{|DF|} = \frac{|BC|}{|EF|}$ then $\triangle ABC \sim \triangle DEF$.

72. * (*altitude to the hypotenuse*) In any right triangle the altitude to the hypotenuse separates the triangle into two smaller triangles which are similar to each other and the original triangle.

73. (*fundamental theorem for circles*) Let l be a line, $\odot OA$ a circle, and F the foot of the perpendicular to l through O. Then either

(a) Every point of l is outside the circle, or

(b) F is on the circle and every other point of l is outside the circle (and thus l is a tangent line), or

(c) F is inside the circle and l intersects the circle in two points which are equidistant from F.

74. * (*tangent line*) A line l through point A is tangent to $\odot OA$ at A if and only if $OA \perp l$.

75. * (*chord*) In any circle, a radius bisects a chord if and only if the radius is perpendicular to the chord, and the perpendicular bisector of any chord contains the center.

76. * (*circle cutting*) Any line that contains a point in the interior of a circle intersects the circle in two points.

77. * (*chord congruence*) Two chords in the same or congruent circles are congruent if and only if they are the same distance from the center of the circle containing them.

78. (*Two Circle Theorem*) If two circles having radii a and b have centers that are a distance c apart, and if each of a, b, c is less than the sum of the other two, then the two circles intersect at exactly two points, one on each side of the line through their centers.

79. (*triangle existance*) For any positive real numbers a, b, c such that the sum of any two is greater than the third there is a triangle $\triangle ABC$ having side lengths a, b, c.

80. * (*tangent circles*) If two distinct circles having radii a and b have centers that are a distance c apart, then the two circles intersect at exactly one point if and only if the largest of a, b, c is equal to the sum of the other two. In this situation the point of intersection lies on the line through the centers.

81. * (*disjoint circles*) If two circles having radii a and b have centers that are a distance c apart, then the two circles do not intersect if and only if the largest of a, b, c is greater than the sum of the other two.

Further Study of Euclidean Plane Geometry

Angles Intercepting Circles

We now begin our study of some of the more advanced topics of Euclidean geometry given in the textbook. Note that the author assumes that we have an understanding of trigonometry and also allows us to determine betweenness relationships from diagrams rather than proving them from the separation axiom. However care must be taken to avoid errors in proofs obtained by taking this shortcut.

Theorem *(Star Trek Lemma) The measure of an angle inscribed in a circle is half of the measure of the arc it subtends.*

Corollary *If triangle $\triangle ABC$ is inscribed in a circle then $\angle A$ is a right angle if and only if*

BC *is a diameter.*

Corollary *(Bow Tie Lemma) Two inscribed angles that subtend the same arc are congruent.*

Theorem *(Tangential Case of Star Trek Lemma) If* AT *is tangent to circle* $\odot OT$ *at* T *and* B *is another point on* $\odot OT$ *then the measure of* $\angle ATB$ *is half of the measure of the arc it subtends.*

Theorem *(interior Star Trek Lemma) If chords* AA' *and* BB' *meet at a point* P *in the interior of* $\odot OA$ *and the measures of the minor arcs* AB *and* $A'B'$ *are* α *and* β *respectively, then*

$$|\angle APB| = \frac{\alpha + \beta}{2}$$

Theorem *(exterior Star Trek Lemma) If chords* AA' *and* BB' *are extended to meet at a point* P *in the exterior of* $\odot OA$ *so that* $P.A'.A$ *and* $P.B'.B$ *and the measures of the minor arcs* AB *and* $A'B'$ *are* α *and* β *respectively, then*

$$|\angle APB| = \frac{\alpha - \beta}{2}$$

Similarity

Remark *Baragar defines two triangles to be similar if they have corresponding angles that are congruent in pairs, so his definition is slightly different than ours. However, in Euclidean geometry the two definitions are equivalent by the AA theorem.*

Recall SMSG theorem 67:

Theorem *(basic proportionality) A segment connecting points on two sides of a triangle is parallel to the third side if and only if the segments it cuts off are proportional to the sides.*

Lemma *(fun with fractions) Let* a, b, x, y *be real numbers with* $y, b, y-b,$ *and* $y+b$ *nonzero. Then*

$$\frac{x}{y} = \frac{a}{b} \Rightarrow \frac{x}{y} = \frac{a}{b} = \frac{x-a}{y-b} = \frac{x+a}{y+b}$$

Theorem *(Angle Bisector Theorem) If* D *is the point where the angle bisector of* $\angle A$ *in* $\triangle ABC$ *meets* BC *then*

$$\frac{|BD|}{|BA|} = \frac{|CD|}{|CA|}$$

Also recall SMSG Thm 68,70,71

Theorem *(AA) If two triangles have two congruent corresponding angles then the triangles are similar.*

Theorem *(similarity SAS) If* $\angle A \equiv \angle D$ *and* $\frac{|AB|}{|DE|} = \frac{|AC|}{|DF|}$ *then* $\triangle ABC \sim \triangle DEF$.

Theorem *(similarity SSS) If* $\frac{|AB|}{|DE|} = \frac{|AC|}{|DF|} = \frac{|BC|}{|EF|}$ *then* $\triangle ABC \sim \triangle DEF$.

Power of a Point

Theorem *(power of a point) Let* $\odot OA$ *be a circle and* P *a point not on the circle. Then for every line* l *through* P *that meets the circle in two points* Q, R *the product*

$$|PQ| \cdot |PR|$$

has the same value.

Definition *Let* $\odot OA$ *be a circle and* P *a point. Define* $\Pi(P) = |OP|^2 - r^2$ *where* $r = |OA|$.

Corollary *Let* $\odot OA$ *be a circle and* P *a point. Then the value of the product given in the previous theorem is* $\Pi(P)$ *if* P *is outside or on the circle and* $-\Pi(P)$ *if* P *is inside the circle.*

Remark *Note that* $\Pi(P)$ *is zero if* P *is on the circle, which is the degenerate case of the power of a point theorem in which all of the products are zero.*

Theorem *(tangential version of power of a point) Let* $\odot OA$ *be a circle,* P *a point in the exterior of the circle, and* $\overleftrightarrow{PQ}$ *a tangent line to the circle meeting the circle at* Q. *Then*

$$|PQ|^2 = \Pi(P).$$

Remark *Thus the tangential version can be thought of as a special case of the power of a point theorem where we have* $Q = R$.

Ceva' Theorem

Theorem *Let* D, E, F *be three points, respectively, on sides* BC, AC, *and* AB *of* $\triangle ABC$. *Then* AD, BE, *and* CF *are concurrent at a point* P *if and only if*

$$\frac{|BD|}{|CD|} \cdot \frac{|CE|}{|AE|} \cdot \frac{|AF|}{|BF|} = 1$$

Medians and Centroid

Definition *Let* $\triangle ABC$ *be a triangle and* A', B', C' *the midpoints of sides* $BC, AC,$ *and* AB *respectively. The cevians* AA', BB' *and* CC' *are called the* ***medians*** *of the triangle.*

Theorem *The medians of a triangle are concurrent.*

Definition *The intersection of the medians of a triangle is called the* ***centroid*** *of the triangle and is usually denoted by the letter* G.

Recall SMSG Thm 56,

Theorem *(midpoint connector) Let $\triangle ABC$ be a triangle and M,N the midpoints of AB and AC respectively. Then $MN \parallel BC$ and $|MN| = \frac{1}{2}|BC|$.*

Theorem *The medians of a triangle intersect at a point $2/3$ of the way from the vertex to the midpoint of the opposite side. In other words, if G is the centroid of $\triangle ABC$ and A', B', and C' are the midpoints of the sides BC, AC, and AB respectively, then $|AG| = 2|GA'|$, $|BG| = 2|GB'|$, and $|CG| = 2|GC'|$.*

Incircle and Excircles

Lemma *A point P is equidistant from two distinct lines l,m that intersect at A if and only if is on an angle bisector of one of the angles formed by the lines at point A.*

Theorem *The angle bisectors of the angles of a triangle are concurrent.*

Definition *The point of intersection of the angle bisectors of the angles of a triangle is called the **incenter** and is usually denoted by the letter I.*

Corollary *The incenter is the unique point that is equidistant from the three sides of a triangle.*

Definition *Let r be the distance from the incenter I to the sides of $\triangle ABC$. The circle centered at I with radius r is called the **incircle** of the triangle.*

Remark *By the Fundamental Theorem of Circles the incircle is tangent to all three sides of the circle.*

Theorem *The external angle bisectors of the angles of a triangle $\triangle ABC$ intersect at points I_A, I_B, and I_C which are respectively on the angle bisectors of angles A, B and C. These three points are also equidistant from the lines containing the three sides of the triangle.*

Definition *The points I_A, I_B, I_C are called excenters of $\triangle ABC$. They are the centers of three circles that are tangent to all three lines containing the sides of $\triangle ABC$. These circles are called the **excircles**, and their radii are called the **exradii** and denoted r_A, r_B, r_C respectively.*

Area, Inradius, Exradii

Definition *If $\triangle ABC$ has sides of length a,b,c then the **semiperimeter** of the triangle is*

$$s = \frac{a+b+c}{2}$$

Theorem * *(Vertex to incircle) If* $\triangle ABC$ *has sides of length* a, b, c, *semiperimeter* s, *and* P *is the point where the incircle meets* AB *then* $|AP| = s - a$.

Theorem * *(Vertex to excircle) If* $\triangle ABC$ *has sides of length* a, b, c, *semiperimeter* s *and exradius* r_A *and* Q *is the point where the excircle corresponding to* A *meets* AB *then* $|AQ| = s$ *and* $|BQ| = s - c$.

Theorem *If* $\triangle ABC$ *has semiperimeter* s *and inradius* r *then*

$$|\triangle ABC| = rs$$

Theorem * *If* $\triangle ABC$ *has sides of length* a, b, c, *semiperimeter* s *and exradii* r_A, r_B, r_C *then*

$$|\triangle ABC| = r_A(s-a) = r_B(s-b) = r_C(s-c)$$

Heron's Formula

Theorem *(Heron's Formula) In any triangle* $\triangle ABC$,

$$|\triangle ABC| = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = (a+b+c)/2$.

Orthocenter

Theorem * *The lines containing the three altitudes of a triangle are concurrent.*

Definition *The point of intersection of the lines containing the altitudes of a triangle is called the* ***orthocenter*** *and is usually denoted by the letter* H.

Circumcircle, Circumcenter, Circumradius

Recall SMSG Thm 30:

Theorem * *(perpendicular bisector) A point* P *is equidistant from two distinct points* A, B *if and only if* P *is on the perpedicular bisector of* AB.

Theorem *(circumcenter exists) The perpendicular bisectors of the three sides a triangle are concurrent.*

Definition *The point of intersection of the perpendicular bisectors of the sides of a triangle is called the* ***circumcenter*** *and is usually denoted by the letter* O.

Corollary *The circumcenter is the unique point that is equidistant from the three vertices of* *a triangle.*

Definition *Let O be the circumcenter of $\triangle ABC$. Then $\odot OA$ is called the* ***circumcircle*** *of $\triangle ABC$ and the length of its radius is called the* ***circumradius*** *and denoted R.*

Extended Law of Sines

Theorem *(Extended Law of Sines) In any triangle $\triangle ABC$*

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} = 2R$$

where R is the circumradius.

Law of Cosines

Theorem *(Law of Cosines) In any triangle $\triangle ABC$,*

$$c^2 = a^2 + b^2 - 2ab\cos(|\angle C|)$$

Stewart's Theorem

Theorem *(Stewart) If AP is a cevian of $\triangle ABC$ and $|AP| = l$, $|BP| = m$, $|CP| = n$ then*

$$a(l^2 + mn) = b^2m + c^2n$$

Cyclic Quadrilaterals and Ptolemy's Theorem

Definition *A quadrilateral is* ***cyclic*** *if it can be inscribed in a circle.*

Remark *Since any three vertices of a quadrilateral can be circumscribed by a circle, the quadrilateral is cyclic if and only if the fourth vertex is on the circumcircle of the triangle formed by the other three vertices. Also, since the circumcircle of a triangle is unique, so is the circumcircle of a cyclic quadrilateral (because if there were two there would be two circumcircles for the triangle formed by three of the vertices).*

Theorem * *A quadrilateral is cyclic if and only if its opposite angles are supplementary.*

Theorem *(Ptolemy's Theorem) $\square ABCD$ is a cyclic quadrilateral if and only if*

$$|AC||BD| = |AB||CD| + |AD||BC|$$

More Fun with Triangles

The Euler Line

Theorem *The circumcenter, orthocenter, and centroid of a triangle are collinear.*

Definition *The line containg the circumcenter, orthocenter and centroid is called the Euler Line.*

Theorem *In $\triangle ABC$ we have $H.G.O$ and $|HG| = 2|GO|$ (where H is the orthocenter, G is the centroid, and O is the circumcenter as usual).*

Remark *We often refer to the line segment HO as the Euler line even though it is technically a segment, not a line.*

The Nine Point Circle

Theorem *In any triangle, the midpoints of the sides, the feet of the altitudes, and the midpoints of the segments connecting the vertices to the orthocenter are all contained in a circle whose center is the midpoint of the Euler line.*

Pedal Triangles

Definition *Let P be a point and $\triangle ABC$ a triangle. Let X, Y, Z be the feet of the perpendiculars through P to $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{AC}$ respectively. Then $\triangle XYZ$ is called the **pedal triangle** with repect to P and $\triangle ABC$.*

Theorem *(Simson line) The pedal triangle with respect to P and $\triangle ABC$ is degenerate if and only if P is on the circumcircle of $\triangle ABC$.*

Definition *When the pedal triangle is degenerate, the line through $X, Y,$ and Z is called the **Simson line**.*

Theorem *The pedal triangle of the pedal triangle of the pedal triangle with respect to a point P is similar to the original triangle.*

Menelaus Theorem

Definition *Let A and B be distinct points and Ψ a coordinate system on $\overleftrightarrow{AB}$ with $\Psi(A) = 0$ and $\Psi(B) > 0$. Then for any points C, D we define the **signed length** of the segment CD to be $|CD|$ if $\Psi(C) < \Psi(D)$ and $-|CD|$ if $\Psi(D) < \Psi(C)$. We denote the signed length of CD (with respect to this coordinate system) as $\|CD\|$.*

Remark *Notice that* $\|CD\| = -\|DC\|$ *for any choice of coordinate system.*

Theorem *(Menelaus) Let* D,E,F *be three points on, respectively, the lines* $\overleftrightarrow{BC}$, $\overleftrightarrow{AC}$, *and* $\overleftrightarrow{AB}$ *containing the sides of* $\triangle ABC$. *Then* D,E,F *are collinear if and only if*

$$\frac{\|AF\|}{\|FB\|}\frac{\|BD\|}{\|DC\|}\frac{\|CE\|}{\|EA\|} = -1$$

Remark *The signed lengths in the previous theorem are independent of the coordinate systems chosen, but all segments on the same line must use the same choice of coordinate system.*

The Gergonne Point

Theorem *The cevians from the vertices to the points of intersection of the incircle with the sides of a triangle are concurrent.*

Definition *The point of concurrency in the previous theorem is called the* ***Gergonne*** *point.*

The Nagel Point

Theorem *The cevians from the vertices to the points of intersection of the three excircles with the sides of a triangle are concurrent.*

Definition *The point of concurrency in the previous theorem is called the* ***Nagel*** *point.*

Morley's Theorem

Theorem *(Morley) The points of intersection of consecutive angle trisectors of the angles of a triangle are the vertices of an equilateral triangle.*

Contructions with Straightedge and Compass

Definition *Let* O *and* A *be distinct points. The following figures and points are* ***constructible****.*

1. O *and* A *are constructible.*

2. If X,Y *are distinct constructible points then the line* $\overleftrightarrow{XY}$ *is constructible.*

3. If X,Y *are distinct constructible points then the circle* $\odot XY$ *is constructible.*

4. If f,g *are distinct constructible figures (i.e. circles or lines), then their points of intersection are constructible.*

5. All constructible figures are obtained by a finite number of applications of rules #1-4.

Remark *A more general definition of constructible can be obtained by replacing* $\{O, A\}$ *in the previous definition with another set of given constructible points* S. *In this case we can say that the points, lines, and circles are* ***constructible from*** S. *However, unless otherwise stated we will always use the term 'constructible' to mean 'constructible from* $\{O, A\}$ *with* $|OA| = 1$.

Remark *We also say that a segment, ray, triangle, etc is constructible if the points that define it are constructible. For example,* $\overrightarrow{AB}$, AB, *and* $\triangle ABC$ *are constructible figures if* A, B, C *are constructible points. Also the union of constructible figures can be said to be constructible as well.*

Definition *A positive real number* r *is* ***constructible*** *if there exist constructible points* P, Q *with* $|PQ| = r$.

Constructions with Geometer's Sketchpad

Here are a few useful and important tips to keep in mind when using Geometer's Sketchpad.

- The four rules of construction in the definition of contructible can be accomplished in Sketchpad in the following way:
  1. To construct arbitrary given points such as O, A use the Point tool (the button with the dot on it on the left hand side of the screen) and just click anywhere to make the point.
  2. To construct a line (or a segment or ray) through two points X, Y select the Line Tool (the button with the segment on it on the left hand side of the screen). If you click on that button and hold down the mouse button another row of buttons pops out and you can select between segment, ray, or line. Then move the cursor over point X until it turns blue and click the left mouse button, then move the cursor over point Y until it turns blue and then click the left button again.
  3. To construct a circle with a given center X and containing point Y select the Circle Tool from the left hand side of the screen. Then click first on X and then on Y.
  4. To construct the point of intersection of two figures (lines or circles), select the Point tool, and move the cursor over the point of intersection until both circles/lines turn blue. Then click the left mouse button.
- The Text tool (the button with the A on it) can be used to type text and mathematics into your document. Double click on the document to open up a text box. You can resize the text boxes with the Selection tool by dragging the lower right hand corner of the box. You can select the entire box and reposition it on the screen with the Selection tool also (just click on the box and then drag it whereever you like). You can insert math symbols into your text using the toolbar that appears at the bottom of the screen.
- To select lines, segments, rays, circles, arcs, or points, use the Selection tool (the button with the arrow on it) on the left hand side of the screen. You can select items by clicking on them or by dragging a box around them with the mouse to select everything inside the box (press and hold the left mouse button while dragging).
- Selected objects can be hidden by pressing CTRL-H. This is very useful for un-cluttering

your drawing and making scripts. To unhide objects choose Display/Show All Hidden from the menu.

- If you select two or more points (and NOTHING else!!) and press CTRL-L, it will construct line segments between all of the points.
- If you select a segment and press CTRL-M it will construct the midpoint of the segment.
- If your mouse has a middle wheel button, you can use it to change between the various buttons on the left hand side of the screen. This is very efficient compared with clicking on the buttons themselves. Clicking the middle mouse wheel changes the selection for the line button between segment, ray, and line.
- If you right click on an object you can change it's color, make lines or circles dashed, solid, or thick, and label the object.
- If you label a point, the label remains attached to the point, but you can drag it to a more convenient position with the selection tool by moving it over the label until it changes from an arrow to a hand and then clicking and dragging the label.
- Most menu items, such as those on the Construct menu, require that very specific inputs be selected in your diagram before the menu item can be used. So if you have a menu item and it is greyed-out (not available) it is because you have not selected the correct inputs for it to be available. For example, if you want to construct a Circle-by-Center-and-Radius you must select one point and one line segment. A common mistake I make over and over again is to have something ELSE selected and not notice it, for example, if I have selected a point, a segment, and a text box. So if your menu item is not available the reason is simple: *you have to select the correct inputs for that menu item, nothing more, and nothing less.*
- To create a script construct a figure by hand. Then select (a) the points you want as inputs to the script and (b) the objects you want the script to construct from your given points. Then click and hold on the Tool button on the left hand side of the screen (the one with the double black triangles) and choose Create New Tool... . You do not need to highlight everything you have used to do the construction, only your starting points and your ending points. Note that the order that you click on the input points while highlighting will match the order that those points will be matched when you execute the script.
- To show the script itself (i.e. to read the description of your construction), you can turn on the Script View from the Tool button.

Basic Constructions

Remark *Every constructible line contains at least two constrictible points. The center of every constructible circle is constructible and every constructible circle contains at least one constructible point. This follows immediately from the definition of constructible, since no line or circle can be constructed without such points.*

Theorem *(perpendiculars are constructible) If l is a constructible line and P a constructible point then the line perpendicular to l through P is constructible.*

Theorem *(parallels are constructible) If l is a constructible line and P a constructible point* *then the line parallel to l through P is constructible.*

Theorem *(noncollapsible compass) If A, B, P are distinct constructible points and $A \neq B$ then the circle centered at P with radius $|AB|$ is constructible.*

Corollary *(copy segments) Given a contructible segment and constructible ray $\overrightarrow{AB}$, we can construct a point C on $\overrightarrow{AB}$ so that AC is congruent to the given segment.*

Remark *Thus we can copy any constructible segment onto any constructible ray.*

Theorem *(copy triangles) If $\triangle ABC$ is constructible and $\overrightarrow{DE}$ are constructible then we can construct P on $\overrightarrow{DE}$ and Q such that $\triangle ABC \equiv \triangle DPQ$.*

Corollary *(copy angles) Given a contructible angle and constructible ray $\overrightarrow{AB}$, we can construct a ray $\overrightarrow{AC}$ so that $\angle A$ is congruent to the given angle.*

Theorem *(midpoints) If AB is constructible then the midpoint of AB is constructible.*

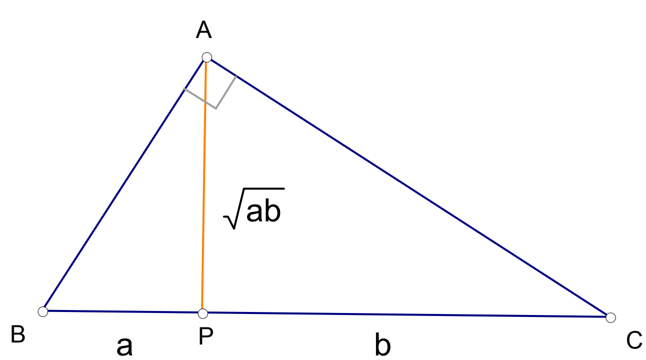
Theorem *(angle bisectors) If $\angle A$ is constructible then so is its angle bisector.*

Geometric Mean

Definition *The geometric mean of two positive real numbers a, b is $\sqrt{ab}$.*

Theorem *In any right triangle $\triangle ABC$ with right angle at A, the length of the altitude AP is the geometric mean of the lengths of the segments BP and CP, i.e.*

$$|AP|^2 = |BP||CP|$$



Doing Arithmetic with Geometry

Theorem *If x,y are constructible real numbers then*
(1) $x+y$ is constructible.
(2) $|x-y|$ is constructible.
(3) x/y is constructible.
(4) xy is constructible.
(5) $\sqrt{x}$ is constructible.

Corollary *The positive rational numbers are constructible.*

Non-Euclidean Geometry

A First look at Hyperbolic Geometry

Definition ***Neutral Geometry*** *is the collection of theorems that can be proven using all of the axioms of Euclidean geometry except the parallel axiom, i.e. any theorem that can be proven using only axioms S1-S11 is a theorem of neutral geometry.*

Definition ***Hyperbolic Geometry*** *consists of all theorems that can be proven from the axiom system for Euclidean geometry with the parallel axiom replaced by its negation, i.e. any theorem that can be proven from axioms S1-S11 and ~S12.*

Remark *Every theorem of Neutral Geometry is thus a theorem of both Euclidean Geometry and Hyperbolic Geometry.*

Definition *A* ***Hyperbolic plane***, $\mathbf{H}$, *is an incidence structure $(\mathcal{P},\mathcal{L})$ satisfying* ***S1-S11*** *and the negation of* ***S12***.

Remark *Technically speaking, when we refer to the Euclidean plane and the Hyperbolic plane as incidence structures satisfying certain axioms, we are actually talking about* ***models*** *for these axiom systems.*

A Toy Euclidean Plane

There are many bijections from the Euclidean plane to the open unit disk. Here are two.

Theorem ** Label the points in the plane uniquely in polar coordinates (r,θ) by requiring that $r>0$ and $0\le\theta<360$ for points not on the origin and label the origin $(0,0)$. Then the function*

$$f(r,\theta)=\left(1-\frac{1}{2^r},\theta\right)$$

is a bijective function from the entire plane to the interior of the unit circle.

Theorem * *Label the points in the plane as in the previous theorem. Then the function*

$$f(r,\theta) = \left(\frac{r}{\sqrt{1+r^2}}, \theta\right)$$

is a bijective function from the entire plane to the interior of the unit circle.

Statements Equivalent to the Euclidean Parallel Axiom

Theorem *In neutral geometry, the following statements are equivalent.*

1. Through any point not on a line there is exact one line parallel to the given line.
2. If two lines and a transveral form same side interior angles whose sum is less than 180 then the lines intersect on the same side of the transversal as those two angles.
3. If l, m, n are three distinct lines with $l \parallel m$ and $m \parallel n$ then $l \parallel n$.
4. A line that intersects one of two parallel lines must also intersect the other.
5. A line perpendicular to one of two parallel lines is perpendicular to the other.
6. The perpendicular bisectors of the sides of a triangle are concurrent.
7. Any triangle can be circumscribed by a circle.
8. A line perpendicular to one ray of an acute angle intersects the other ray.
9. Through any point in the interior of an angle there exists a line intersecting both rays of the angle at points other than the vertex.
10. There exists an acute angle such that every point in the interior is on a line that intersects both rays.
11. The sum of the measures of the angles of any triangle is 180.
12. There exists a triangle whose angle sum is 180.
13. The perpendicular bisectors of the legs of a right triangle intersect.
14. There exists a pair of similar, noncongruent triangles.
15. Rectangles exist.

Some theorems of Hyperbolic Geometry

Theorem *In hyperbolic geometry:*

1. Rectangles do not exist.
2. AAA
3. The angle sum of every triangle is less than 180.
4. Through any point not on a line there are infinitely many lines which are parallel to the given line.
5. There exist four lines l, m, n, r such that $l \perp m$, $m \perp n$, $n \perp r$, and $l \parallel r$.
6. The interior of any angle contains two perpendicular lines.

A Toy Hyperbolic Plane

Definition *The **Poincare Disk** is an incidence structure $(\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$ is the set of points in the interior of the unit circle in the complex plane, and $\mathcal{L}$ consists of the intersections of $\mathcal{P}$ with lines or circles in the complex plane that are perpendicular to the unit circle at both points of intersection. The hyperbolic distance between points z, w is*

$$d(z,w) = \operatorname{arctanh}\left(\frac{z-w}{1-\overline{z}w}\right).$$

The hyperbolic measure of hyperbolic angles is the same as the Euclidean angle measure of the hyperbolic rays (where we use the Euclidean angle between the Euclidean tangents when the hyperbolic rays are arcs of Euclidean circles).

Theorem *The Poincare disk is a model of hyperbolic geometry.*

Analytic Geometry: the Complex Plane

Definition *Let* $\mathbb{C} = \mathbb{R}^2$. *For each* $(x,y) \in \mathbb{C}$ *we formally write* $(x,y) = x + yi$. *This form,* $x + yi$, *is called the* ***standard form*** *of the complex number* (x,y).

Definition *Let* $x + yi, a + bi \in \mathbb{C}$, *then:*

1. $\overline{x+yi} = x - yi$. *(This is called the* ***complex conjugate****.)*
2. $|x+yi| = \sqrt{x^2+y^2}$. *(This is called the* ***complex norm****.)*
3. $\operatorname{Arg}(x+yi) =$ *the angle in* $[0\ldots 2\pi)$ *of* (x,y) *in polar form (not defined for* $x = y = 0$*). (This is called the* ***Argument*** *of* $x + yi$.*)*
4. $\operatorname{Re}(x+yi) = x$. *(This is called the* ***real part*** *of* $x + yi$.*)*
5. $\operatorname{Im}(x+yi) = y$. *(This is called the* ***imaginary part*** *of* $x + yi$.*)*
6. $(x+yi) + (a+bi) = (x+a) + (y+b)i$. *(This is the definition of* ***addition*** *in* $\mathbb{C}$.*)*
7. $(x+yi)(a+bi) = (xa-yb) + (ya+xb)i$. *(This is the definition of* ***multiplication*** *in* $\mathbb{C}$.*)*

Notation *We can abbreviate* $0 + yi$ *as* yi, $x + 0i$ *as* x, $x + 1i$ *as* $x + i$, *and* $x - 1i$ *as* $x - i$ *with no ambiguity in the above definitions. With this notation* $i = (0,1)$ *and* $i^2 = -1$. *It is easy to verify that the usual laws of addition and multiplication (associative, commutative, distributive, identity, etc.) hold for the complex numbers as well.*

Definition *Let* $\theta \in \mathbb{R}$. *Then* $e^{i\theta} = \cos\theta + i\sin\theta$

Definition *Let* $x + yi \in \mathbb{C} - \{0\}$. *The* ***standard polar form*** *of* $x + yi$ *is* $re^{i\theta}$ *where* $r = |x+yi|$ *and* $\theta = \operatorname{Arg}(x+yi)$.

Definition *The* ***distance*** *between two complex numbers* z, w *is denoted* $d(z,w)$ *and is defined to be* $d(z,w) = |z-w|$.

Theorem $e^{i\pi} + 1 = 0$ *(The most beautiful theorem in mathematics?)*

Theorem *Let* $\theta, \gamma \in \mathbb{R}$

1. $e^{i\theta}e^{i\gamma} = e^{i(\theta+\gamma)}$.
2. $|e^{i\theta}| = 1$.
3. $\overline{e^{i\theta}} = e^{i(-\theta)}$.

Theorem *Let* $z, z_1, z_2 \in \mathbb{C}$. *Then:*

1. $|z_1z_2| = |z_1||z_2|$
2. $\overline{z_1z_2} = \overline{z_1}\ \overline{z_2}$ *i.e. the conjugate of a product is the product of conjugates.*
3. $\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$ *i.e. the conjugate of a sum is the sum of the conjugates.*
4. $z\,\overline{z} = |z|^2$

5. $|z| = |\overline{z}|$
6. If $z = re^{i\theta}$ in polar form, then $\overline{z} = re^{i(-\theta)}$

Transformations

Definition *A **transformation** of a set S is a bijection from S to S.*

Remark *In other branches of mathematics a transformation of S is often called a **permutation** of S.*

Some Useful Geometric Transformations Let $w \in \mathbb{C}$ and $\theta, k \in \mathbb{R}$.

Translation by w: $T(z) = z + w$

Rotation by θ radians counterclockwise about the origin: $T(z) = e^{i\theta}z$

Reflection across the x-axis: $T(z) = \overline{z}$

Homothety by positive factor k with respect to the origin: $T(z) = kz$

Inversion* **with respect to the unit circle**: $T(z) = \frac{1}{\overline{z}}$

*Inversion is a transformation of the extended complex plane $\mathbb{C}^+ = \mathbb{C} \cup \{\infty\}$ with $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

Remark *You can compose these functions to obtain many useful transformations!*

Inversion

Remark *Henle defines inversion in the unit circle by $T(z) = \frac{1}{z}$. We will call this **algebraic inversion** as opposed to the inversion $T(z) = \frac{1}{\overline{z}}$ which we will call **geometric inversion**. The term inversion will refer to geometric inversion in these lecture notes.*

Theorem *Geometric inversion in the unit circle sends $re^{i\theta}$ to $\frac{1}{r}e^{i\theta}$ for $r \neq 0$ and it sends 0 and ∞ to each other.*

Definition *Given a circle $\odot OA$ with radius R, if P is a point other than O we define the **geometric inverse** of P with respect to $\odot OA$ to be the point P' on $\overrightarrow{OP}$ such that $|OP||OP'| = R^2$. The inverse of O is ∞ and vice versa.*

Theorem * *If P is outside of $\odot OA$ and Q and R are the points where the tangent lines through P meet $\odot OA$, then the inverse of P is the midpoint of QR. Similarly if P is inside*

$\odot OA$ and Q and R are the points where the perpendicular to OP through P meets $\odot OA$, then the tangent lines to $\odot OA$ at Q and R intersect at the inverse P' of P with respect $\odot OA$.

Definition *A **cline** is either a circle or a line.*

Theorem *Geometric inversion with respect to a circle maps clines to clines.*
In particular,
1. Points on the circle of inversion map to themselves
2. The center of the circle of inversion maps to infinity and vice versa.
3. Points inside the circle of inversion map to points outside the circle of inversion and vice versa.
4. The only clines which map to themselves are those that are orthogonal to the circle of inversion at both points of intersection, and the circle of inversion itself.

Geometry

Groups

Definition *A **group** is a pair $(G,*)$ where G is a set and $* : G \times G \to G$ is a binary operation such that:*
1. $*$ is associative (i.e. $\forall a,b,c \in G, (a*b)*c = a*(b*c)$)
2. There is an identity element for $*$ in G (i.e. $\exists e \in G, \forall a \in G,\ a*e = a \text{ and } e*a = a$)
3. Every element has an inverse (i.e. $\forall a \in G, \exists a^{-1} \in G, a*a^{-1} = e \text{ and } a^{-1}*a = e$)

Theorem *For any nonempty set G of transformations of a set S, if*
1. G is closed under composition (i.e. $\forall g,h \in G, g \circ h \in G$)
2. G is closed under taking inverses (i.e. $\forall g \in G, g^{-1} \in G$)
then $(G,\circ)$ is a group.

Corollary *The set of all transformations of a set S forms a group with composition as the operator.*

Definition *The group in the previous corollary is called the **symmetric group** on S and is denoted $\mathrm{Sym}(S)$.*

Remark *It is commonplace to refer to the group $(G,*)$ by the set G and vice versa, when the operation $*$ is understood. For geometry we will only be concerned with the case where G is a set of transformations and $*$ is composition.*

Definition *If G,H are groups of transformations and $G \subseteq H$ we say G is a **subgroup** of H.*

Remark *This does not imply that every subset of a group is a subgroup. The subset must be* *closed under composition and inverses for it to be a subgroup.*

Corollary *Every group of transformations is a subgroup of a symmetric group.*

From now on we will only talk about transformation groups, i.e. subgroups of a symmetric group, i.e. groups of transformations of a set S with composition as the operation.

Klein's Erlanger Program

Felix Klein - your greatgreatgreatgreatgreatgreatgrandpop!
(see: `http://math.scranton.edu/monks/misc/Lineage.html`)

Definition *A **geometry** is pair (S,G) where S is a set and G is a group of transformations of S. The set S is called the **underlying space** of the geometry and G is called the **group of transformation** or the **transformation group** of the geometry.*

Definition *The elements of the underlying set of a geometry are called **points**. A **figure** is a set of points in a geometry.*

Definition *Two figures are **congruent** in a geometry if and only if there is a transformation in that geometry that maps one to the other, i.e. if (S,G) is a geometry and $U,V \subseteq S$, then*

$$U \equiv V \Leftrightarrow \exists T \in G,\ T(U) = V$$

Theorem * *Congruence is an equivalence relation on the set of all figures in a geometry.*

Invariants

Definition *A set of figures $\mathcal{F}$ in a geometry (S,G) is said to be **invariant** if and only if the image of any figure in $\mathcal{F}$ under any transformation in G is also an element of $\mathcal{F}$, i.e. $\mathcal{F}$ is invariant if and only if*

$$\forall U \in \mathcal{F}, \forall T \in G, T(U) \in \mathcal{F}$$

Remark *Thus a set of figures is invariant if whenever a figure is in the set, so is every figure that is congruent to it.*

Definition *More generally, if $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots, \mathcal{F}_n$ are sets of figures in a geometry (S,G) and $\mathcal{F}$ is a subset of $\mathcal{F}_1 \times \mathcal{F}_2 \times \cdots \times \mathcal{F}_n$ then $\mathcal{F}$ is said to be an **invariant set of n-tuples** if and only for any transformation in G and any tuple $(A_1, A_2, \ldots, A_n)$ in $\mathcal{F}$, the tuple $(T(A_1), T(A_2), \ldots, T(A_n))$ is also an element of $\mathcal{F}$, i.e. $\mathcal{F}$ is invariant if and only if*

$$\forall (A_1, A_2, \ldots, A_n) \in \mathcal{F}, \forall T \in G, (T(A_1), T(A_2), \ldots, T(A_n)) \in \mathcal{F}$$

Definition *A function f whose domain is an invariant set of figures $\mathcal{F}$ in a geometry is* ***invariant*** *if and only if the value of f on a figure is the same as the value of f on the image of a figure under any transformation, i.e. f is invariant in geometry (S,G) if and only if*

$$\forall U \in \mathcal{F}, \forall T \in G,\ f(U) = f(T(U))$$

Remark *Thus a function on a set of figures is invariant if whenever figure U is congruent to figure V, $f(U) = f(V)$.*

Definition *More generally for multivariable functions, let (S,G) be a geometry and $\mathcal{F}$ be an invariant set of n-tuples figures in (S,G). A function $f : \mathcal{F} \to C$ where C is any set is* ***invariant*** *if and only if for every T in G, and every $(A_1, A_2, \ldots, A_n) \in \mathcal{F}$*

$$f(A_1, A_2, \ldots, A_n) = f(T(A_1), T(A_2), \ldots, T(A_n))$$

The Study of a Particular Geometry: *is the study of its invariant sets and functions*!

Examples of Geometries

Euclidean geometry: $(\mathbb{C}, \mathbf{E}^+)$ where $\mathbf{E}^+$ is the set of all transformations of $\mathbb{C}$ of the form

$$T(z) = e^{i\theta}z + \gamma \text{ or } T(z) = e^{i\theta}\overline{z} + \gamma$$

where $\theta \in \mathbb{R}$ and $\gamma \in \mathbb{C}$.

Definition *An* ***isometry*** *is a distance preserving transformation, i.e. T is an isometry if and only if for all z, w, $d(z,w) = d(T(z), T(w))$.*

Theorem (Classification) *The transformation group for Euclidean geometry consists of the set of all reflections, rotations, translations, and glide reflections (a translation followed by reflection in a line parallel to the direction of translation).*

Theorem * *A transformation T of $\mathbb{C}$ is an isometry if and only if there exists $\theta \in \mathbb{R}$ and $\gamma \in \mathbb{C}$ such that*

$$T(z) = e^{i\theta}z + \gamma \text{ or } T(z) = e^{i\theta}\overline{z} + \gamma$$

Special Euclidean Geometry: (Henle's Euclidean geometry) $(\mathbb{C}, \mathbf{E})$ where $\mathbf{E}$ is the set of all **rigid motions**, i.e. all transformations of $\mathbb{C}$ of the form

$$T(z) = e^{i\theta}z + \gamma$$

where $\theta \in \mathbb{R}$ and $\gamma \in \mathbb{C}$.

Theorem * *Every transformation of $\mathbb{C}$ obtained by composing one or more translations and* *rotations is a rigid motion and vice versa.*

Translational Euclidean Geometry: (Henle calls this translational geometry) $(\mathbb{C}, \mathcal{T})$ where $\mathcal{T}$ is the set of all translations, i.e. all transformations of $\mathbb{C}$ of the form

$$T(z) = z + \gamma$$

where $\gamma \in \mathbb{C}$.

Rotational Euclidean Geometry: (Henle calls this rotational geometry) $(\mathbb{C}, \mathcal{R})$ where $\mathcal{R}$ is the set of all rotations about the origin, i.e. all transformations of $\mathbb{C}$ of the form

$$T(z) = e^{i\theta} z$$

where $\theta \in \mathbb{R}$.

Trivial Geometry: $(\mathbb{C}, \{id_{\mathbb{C}}\})$

Note: A group with one element is called a **trivial group**.

Extreme Geometry: $(S, \mathrm{Sym}(S))$ where S is any set.

Hyperbolic Geometry: $(\mathbf{D}, \mathbf{H})$ where $\mathbf{D}$ is the open unit disk

$$\mathbf{D} = \{z \in \mathbb{C} : |z| < 1\}$$

and $\mathbf{H}$ is the set of transformations of $\mathbf{D}$ of the form

$$T(z) = e^{i\theta} \frac{z - \gamma}{1 - \overline{\gamma} z}$$

where $\gamma \in \mathbb{C}$ with $|\gamma| < 1$ and $\theta \in \mathbb{R}$.

[Note: this is actually **Special Hyperbolic Geometry**, i.e. hyperbolic geometry without reflections.]

Elliptic Geometry: $(\mathbb{C}^+, \mathbf{S})$ where $\mathbf{S}$ is the set of transformations of $\mathbb{C}^+$ of the form

$$T(z) = e^{i\theta} \frac{z - \gamma}{1 + \overline{\gamma} z}$$

where $\gamma \in \mathbb{C}$ with $|\gamma| < 1$ and $\theta \in \mathbb{R}$.

Möbius Geometry: $(\mathbb{C}^+, \mathbf{M})$ where $\mathbf{M}$ is the set of transformations of $\mathbb{C}^+$ of the form

$$T(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Affine Geometry: $(\mathbb{C}, \mathbf{L})$ where $\mathbf{L}$ is the set of transformations of $\mathbb{C}$ of the form

$$T(z) = \alpha z + \beta \overline{z} + \gamma$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ and $|\alpha| \neq |\beta|$.

Projective Geometry: $(\mathbb{P}^2, \mathbf{P})$ where $\mathbf{P}$ is the set of projective transformations (defined later) of the real projective plane $\mathbb{P}^2$.

Complex Analytic Geometry

Direction Number

Definition *Let z be a nonzero complex number. The **direction number** of z is the complex number*

$$\operatorname{dir}(z) = \frac{z}{\overline{z}}$$

Theorem *Let $z \in \mathbb{C} - \{0\}$ and $\theta = \operatorname{Arg}(z)$. Then*

$$\operatorname{dir}(z) = e^{2\theta i}$$

Complex Equation of a Euclidean Line

Definition *A T transformation in $\mathbf{E}^+$ is a **reflection** if $T(z) = e^{i\theta}\overline{z} + \gamma$ and T has fixed points.*

Remark *The transformations $T(z) = e^{i\theta}\overline{z} + \gamma$ that do not have fixed points are the glide reflections.*

Definition *A **line** in Euclidean geometry $(\mathbb{C}, \mathbf{E}^+)$ is the set of fixed points of a reflection.*

Remark *Note that while we can develop Euclidean geometry this way, Henle assumes we already know what a Euclidean line is in $\mathbb{C}$ without having to prove everything from this definition.*

Theorem *For any two distinct points there is exactly one line which contains them both.*

Remark *Thus axiom S1 in the SMSG axiomatic development of Euclidean Geometry is a theorem in the Erlanger Program development of Euclidean Geometry. Similiarly, by making appropriate definitions, we can prove the rest of the SMSG axioms as theorems in the Erlanger Program view..*

Theorem *Let v, w be distinct complex numbers. The solution set of*

$$\operatorname{dir}(z - w) = \operatorname{dir}(v - w)$$

is the Euclidean line through the points v, w.

Definition *Let l be a Euclidean line in $\mathbb{C}$. Then the **direction number of the line** l is defined* to be

$$\operatorname{dir}(l) = \operatorname{dir}(v - w)$$

where v, w are any two distinct points on l.

Theorem * *The direction number of a line does not depend on which points v, w on the line are used.*

Direction Number of a Angle

Definition *Let P, Q, R be distinct complex numbers and $\theta = |\angle PQR|$. The the **direction number of the angle** $\angle PQR$ is*

$$\operatorname{dir}(\angle PQR) = e^{2\theta i}$$

Theorem *Let P, Q, R be distinct points. Then*

$$\operatorname{dir}(\angle PQR) = \frac{\operatorname{dir}\left(\overleftrightarrow{QP}\right)}{\operatorname{dir}\left(\overleftrightarrow{QR}\right)} \text{ or } \operatorname{dir}(\angle PQR) = \frac{\operatorname{dir}\left(\overleftrightarrow{QR}\right)}{\operatorname{dir}\left(\overleftrightarrow{QP}\right)}$$

Remark *$\angle PQR$ is acute if and only if $\operatorname{dir}(\angle PQR)$ is equal to whichever of $\frac{\operatorname{dir}\left(\overleftrightarrow{QP}\right)}{\operatorname{dir}\left(\overleftrightarrow{QR}\right)}$ and $\frac{\operatorname{dir}\left(\overleftrightarrow{QR}\right)}{\operatorname{dir}\left(\overleftrightarrow{QP}\right)}$ has positive imaginary part.*

Theorem *Two Euclidean angles are congruent if and only their direction numbers are the same.*

The Universal Proving Machine!

To prove a theorem in a geometry which only deals with functions and figures which are invariant for that geometry, it suffices to prove it for one well-chosen example in each congruence class, since by definition of invariance, if the theorem is verified for one example it must be true for all situations that are congruent to that example.

Example *Prove that the medians of a triangle are concurrent at a point 2/3 of the way from each vertex to the midpoint of the opposite side by the Erlanger Program method.*

Möbius Geometry

Conformal Maps

Definition *If c, d are smooth curves in the Euclidean plane that intersect at a point P, then* *the angle between the curves at point P is defined to be the angle between their respective tangent lines at P.*

Definition *A continuous transformation of the Euclidean plane is **conformal at a point** P if it preserves angles at P, i.e. if T is a conformal map and smooth curves c, d meet at a point P at an angle θ then the angle between $T(c)$ and $T(d)$ at $T(P)$ is also θ. A transformation is **conformal** if it is conformal at every point in its domain.*

Definition *In Möbius geometry and its subgeometries, the **angle** between two curves is defined to be the same as it is in Euclidean geometry.*

Remark *Clearly translation, rotation, reflection, and glide reflection are all conformal.*

Theorem *Algebraic inversion is conformal at every point except the origin.*

Corollary *Geometric inversion is conformal at every point except the center of inversion.*

Möbius Transformations

Definition *A **Möbius Transformation** is a transformation of the extended complex plane $\mathbb{C}^+$ of the form*

$$T(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Remark *Möbius transformations include all translations, rotations, homotheties, and algebraic inversions in circles.*

Remark *If $c \neq 0$ then*

$$\frac{az+b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c^2}\left(\frac{1}{z+\frac{d}{c}}\right)$$

and so these Möbius transformations are a composition of a translation, inversion, homothety and rotation, and another translation.

If $c = 0$ then

$$\frac{az+b}{cz+d} = \frac{a}{d}z + \frac{b}{d}$$

and so these Möbius transformations are a composition of a translation followed by a homothety and rotation.

Thus every Möbius transformation is a composition of rotations, translations, homotheties, and algebraic inversions.

Matrix of a Möbius Transformation

Lemma *If* $T(z) = \frac{az+b}{cz+d}$ *and* $S(z) = \frac{ez+f}{gz+h}$ *are Möbius transformations then*

$$T \circ S(z) = \frac{(ae+bg)z+(af+bh)}{(ce+dg)z+(cf+dh)}$$

Definition *If* $T(z) = \frac{az+b}{cz+d}$ *is a Möbius transformation then the matrix*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is called the ***matrix associated with the Möbius transformation*** T.

Corollary *The matrix associated with* $T \circ S$ *is the matrix product of the matrix associated with* T *and the matrix associated with* S.

Lemma *If* $T(z) = \frac{az+b}{cz+d}$ *is a Möbius transformation then* $T^{-1}(z) = \frac{dz-b}{-cz+a}$ *and* T^{-1} *is also a Möbius transformation.*

Corollary *Möbius geometry is a geometry.*

Fixed Points

Theorem *A Möbius transformation other than the identity has either one or two fixed points.*

Corollary *The only Möbius transformation that has three or more fixed points is the identity map.*

Fundamental Theorem

Theorem *(Fundamental Theorem of Möbius Geometry) Given six points* $z_1, z_2, z_3, w_1, w_2, w_3 \in \mathbb{C}^+$ *with* z_1, z_2, z_3 *distinct and* w_1, w_2, w_3 *distinct there is a unique Möbius transformation* T *mapping* z_1 *to* w_1, z_2 *to* w_2, *and* z_3 *to* w_3.

Cross Ratio

Definition *The cross ratio of* $z_1, z_2, z_3, z_4 \in \mathbb{C}^+$ *is*

$$\langle z_1, z_2, z_3, z_4 \rangle = \frac{z_1 - z_3}{z_1 - z_4} \frac{z_2 - z_4}{z_2 - z_3}$$

Remark *If z_2, z_3, z_4 are distinct then the function*

$$T(z) = \langle z, z_2, z_3, z_4 \rangle$$

is the unique Möbius transformation sending z_2 to 1, z_3 to 0, and z_4 to ∞.

Theorem *The cross ratio of four distinct points is an invariant of Möbius geometry.*

Corollary *The cross ratio of four distinct points is an invariant of special Euclidean, Hyperbolic, and Elliptic geometry.*

Theorem *Let a, b, c be distinct points. The cross ratio $\langle z, a, b, c \rangle$ is real if and only if z is on the Euclidean circle or line containing a, b, c.*

Corollary *The set of clines is an invariant of Möbius geometry.*

Corollary *In Möbius geometry all clines are congruent.*

Symmetry

Definition *Two points z^* and z in the extended complex plane are symmetric with respect to the cline containing the distinct points z_2, z_3, z_4 if and only if*

$$\langle z^*, z_2, z_3, z_4 \rangle = \overline{\langle z, z_2, z_3, z_4 \rangle}$$

Theorem *If z_2, z_3, z_4 are distinct and collinear then the map that sends z to z^* is reflection in the line containing the three points. If z_2, z_3, z_4 are distinct and not collinear then the map that sends z to z^* is geometric inversion in the circle containing the three points.*

Theorem *Symmetry is an invariant of Möbius geometry.*

Hyperbolic Geometry

Definition *Hyperbolic geometry is $(\mathbf{D}, \mathbf{H})$ where $\mathbf{D}$ is the open unit disk*

$$\mathbf{D} = \{z \in \mathbb{C} : |z| < 1\}$$

and $\mathbf{H}$ is the set of transformations of $\mathbf{D}$ of the form

$$T(z) = e^{i\theta} \frac{z - \gamma}{1 - \overline{\gamma} z}$$

where $\gamma \in \mathbb{C}$ with $|\gamma| < 1$ and $\theta \in \mathbb{R}$.

Theorem $\mathbf{H}$ *is the group of all Möbius transformations that map the open unit disk to itself.*

Definition *A **hyperbolic straight line** is the intersection with the open unit disk of a Euclidean circle or line in the complex plane that meets the unit circle at two right angles.*

Definition *The points on the Euclidean unit circle (which are not points in hyperbolic geometry) are called **ideal points**.*

Remark *Henle says that two hyperbolic lines are **parallel** if the Euclidean clines that contain them intersect at an ideal point. He calls two hyperbolic lines that do not intersect either at a hyperbolic or ideal point **hyperparallel**.*

Definition *Two distinct clines are **orthogonal** if they intersect at right angles.*

Theorem *For any cline Γ and distinct points $A, B \in \mathbb{C}^+$ there exists a unique cline Δ through A, B which is orthogonal to Γ. Furthermore, in this situation the geometric inverse of any point on Δ is also on Δ.*

Theorem *All hyperbolic straight lines are congruent in hyperbolic geometry.*

Theorem *In hyperbolic geometry, through any two distinct points there is exactly one hyperbolic straight line.*

Theorem *The sum of the measures of the angles in any triangle in hyperbolic geometry is less than 180.*

Example Proofs

Euclidean Geometry

In the following theorems all points, lines, etc are assumed to be in $\mathbb{E}$. Proofs use the SMSG axioms and definitions given in the Euclidean geometry sections of the lecture notes above. Proofs of theorems from the *Review of Some Elementary Theorems* section above are labled *(Thm n.)* where n is the theorem number.

Lemma *(relabeling) If A, B, C, D are collinear points and $A \neq B$ and $C \neq D$ then $\overleftrightarrow{AB} = \overleftrightarrow{BA}$ and $\overleftrightarrow{AB} = \overleftrightarrow{CD}$.*

Proof:

1. A, B, C, D are collinear — given

2. $A \neq B$ and $C \neq D$ given
3. $A \in l$ and $B \in l$ and $C \in l$ and $D \in l$ for some line l def collinear;1
4. $\overleftrightarrow{AB}$ is the unique line containing A,B 2 pts det a line (Axiom S1);2
5. $l = \overleftrightarrow{AB}$ def unique;4;3
6. $\overleftrightarrow{BA}$ is the unique line containing B,A 2 pts det a line (Axiom S1);2
7. $l = \overleftrightarrow{BA}$ def unique;6;3
8. $\overleftrightarrow{CD}$ is the unique line containing C,D 2 pts det a line (Axiom S1);2
9. $l = \overleftrightarrow{CD}$ def unique;8;3
10. $\overleftrightarrow{AB} = \overleftrightarrow{BA}$ and $\overleftrightarrow{AB} = \overleftrightarrow{CD}$ substitution;5,7,9

QED

Remark *Because of this Lemma, we won't fuss over relabeling lines in our proofs, and will treat all such relabelings as equivalent names for the same line. A similar comment will apply when we prove other relabeling lemmas in the future.*

Theorem *(Thm 5.)If $A.B.C$ then $C.B.A$.*

Proof:
1. $A.B.C$ Given
2. A,B,C are distinct collinear point def between;1
3. $|AC| = |AB| + |BC|$ def between;1
4. $|AC| = |CA|$ and $|AB| = |BA|$ and $|BC| = |CB|$ distance axiom
5. $|CA| = |BA| + |CB|$ substitution;4,3
6. $\quad = |CB| + |BA|$ commutativity of $+$
7. $C.B.A$ def between;2,5,6

QED

Lemma *(relabeling) If A,B are distinct points then $AB = BA$.*

Proof:
1. A,B are distinct points Given
2. $AB = \{C : C = A \text{ or } C = B \text{ or } A.C.B\}$ def segment
3. $\quad = \{C : C = B \text{ or } C = A \text{ or } B.C.A\}$ SMSG Thm 5 (twice)
4. $\quad = BA$ def segment

QED

Lemma *(distinct pts, distinct coords) If A,B are distinct points and Ψ is a coordinate system for the line containing them, then $\Psi(A) \neq \Psi(B)$.*

Proof:

1. A, B are distinct points | Given

2. $\Psi = \Psi_{\overleftrightarrow{AB}}$ | Given

3. $A \neq B$ | def distinct;1

4. Ψ is bijective | coordinate axiom (S3)

5. Ψ is injective | def bijective

6. Assume $\Psi(A) = \Psi(B)$

7. $A = B$ | def injective;5;6

8. $\rightarrow\leftarrow$ | $\rightarrow\leftarrow$ +;7,3

9. $\leftarrow$

10. $\Psi(A) \neq \Psi(B)$ | $\sim$ +;6,8

QED

Theorem *(order thm) Let A, B, C be three distinct collinear points and Ψ a coordinate system for the line containing them. Then*

$$\Psi(A) < \Psi(B) < \Psi(C) \text{ or } \Psi(C) < \Psi(B) < \Psi(A) \Leftrightarrow A.B.C$$

Proof:

1. A, B, C are distinct collinear points | Given

2. Ψ a coordinate system for $\overleftrightarrow{AB}$ | Given

(⇒)

3. Assume $\Psi(A) < \Psi(B) < \Psi(C)$ or $\Psi(C) < \Psi(B) < \Psi(A)$

(case 1)

4. Assume $\Psi(A) < \Psi(B) < \Psi(C)$

5. $0 < \Psi(B) - \Psi(A)$ and $0 < \Psi(C) - \Psi(A)$ and $0 < \Psi(C) - \Psi(B)$ | algebra

6. $|AC| = |\Psi(C) - \Psi(A)|$ | coordinate axiom (S3)

7. $= \Psi(C) - \Psi(A)$ | def absolute value;5

8. $= \Psi(C) - \Psi(B) + \Psi(B) - \Psi(A)$ | algebra

9. $= |\Psi(C) - \Psi(B)| + |\Psi(B) - \Psi(A)|$ | def absolute value;5

10. $= |\Psi(B) - \Psi(A)| + |\Psi(C) - \Psi(B)|$ | algebra

11. $= |AB| + |BC|$ | coordinate axiom (S3)

12. $A.B.C$ | def between;1,6,11

13. $\leftarrow$

(case 2)

14. Assume $\Psi(C) < \Psi(B) < \Psi(A)$

15. $0 < \Psi(B) - \Psi(C)$ and $0 < \Psi(A) - \Psi(C)$ and $0 < \Psi(A) - \Psi(B)$ | algebra

16. $|CA| = |\Psi(A) - \Psi(C)|$ | coordinate axiom (S3)

17. $= \Psi(A) - \Psi(C)$ | def absolute value;15

18. $= \Psi(A) - \Psi(B) + \Psi(B) - \Psi(C)$ | algebra

19. $= |\Psi(A) - \Psi(B)| + |\Psi(B) - \Psi(C)|$ | def absolute value;15

20. $= |\Psi(B) - \Psi(C)| + |\Psi(A) - \Psi(B)|$ algebra
21. $= |BC| + |CA|$ coordinate axiom (S3)
22. $C.B.A$ def between;1,16,21
23. $A.B.C$ SMSG Thm 5
24. $\leftarrow$
25. $A.B.C$ or-;3,4,12,14,23
26. $\leftarrow$
27. $\Psi(A) < \Psi(B) < \Psi(C)$ or $\Psi(C) < \Psi(B) < \Psi(A) \Rightarrow A.B.C$ $\Rightarrow +$;3,25

($\Leftarrow$)

28. Assume $A.B.C$
29. $\Psi(A), \Psi(B), \Psi(C)$ are distinct dist pts,dist coords lemma;1
30. $\Psi(A) < \Psi(B) < \Psi(C)$ or
$\Psi(C) < \Psi(B) < \Psi(A)$ or
$\Psi(B) < \Psi(A) < \Psi(C)$ or
$\Psi(C) < \Psi(A) < \Psi(B)$ or
$\Psi(A) < \Psi(C) < \Psi(B)$ or
$\Psi(B) < \Psi(C) < \Psi(A)$ trichotomy;29

(case 1 or 2)

31. Assume $\Psi(A) < \Psi(B) < \Psi(C)$ or $\Psi(C) < \Psi(B) < \Psi(A)$
32. $\leftarrow$

(case 3 or 4)

33. Assume $\Psi(B) < \Psi(A) < \Psi(C)$ or $\Psi(C) < \Psi(A) < \Psi(B)$
34. $B.A.C$ ($\Rightarrow$) proof above
35. $|BC| = |BA| + |AC|$ and $|AC| = |AB| + |BC|$ def between;28,34
36. $|BC| = |BA| + |AB| + |BC|$ substitution;35,35
37. $0 = 2|BA|$ algebra
38. $|BA| = 0$ algebra
39. $|BA| > 0$ coordinate axiom (S3);1
40. $|BA| \neq 0$ trichotomy
41. $\rightarrow\leftarrow$ $\rightarrow\leftarrow +$;40,38
42. $\Psi(A) < \Psi(B) < \Psi(C)$ or $\Psi(C) < \Psi(B) < \Psi(A)$ $\rightarrow\leftarrow$ implies anything
43. $\leftarrow$

(case 5 or 6)

44. (the proof is similar to case 3 or 4 and is omitted here)
45. $\Psi(A) < \Psi(B) < \Psi(C)$ or $\Psi(C) < \Psi(B) < \Psi(A)$ *or* −;30,31,31,33,42,44
46. $\leftarrow$
47. $A.B.C \Rightarrow \Psi(A) < \Psi(B) < \Psi(C)$ or $\Psi(C) < \Psi(B) < \Psi(A)$ $\Rightarrow +$;28,45

QED

Theorem *(Thm 10) Midpoints exist and are unique.*

Proof:

1. Let S be a segment
2. $S = AB$ for some distinct points A, B — def segment
3. There exists a coordinate system Ψ on $\overleftrightarrow{AB}$ with $\Psi(A) = 0$ and $0 < \Psi(B)$ — ruler placement axiom (S4)
4. $\overleftrightarrow{AB} \xrightarrow{\Psi} \mathbb{R}$ and Ψ is bijective — coordinate axiom (S3)
5. Ψ is surjective — def of bijective
6. $\Psi(B)/2 \in \mathbb{R}$ — def of $f(x)$ and closure of real numbers;4
7. There exists $M \in \overleftrightarrow{AB}$ such that $\Psi(M) = \Psi(B)/2$ — def surjective;5,4
8. $0 < \Psi(B)/2 < \Psi(B)$ — arithmetic
9. $\Psi(A) < \Psi(M) < \Psi(B)$ — substitution;3,7,8
10. $A.M.B$ — order theorem (Thm 6);9
11. $|AM| = |\Psi(M) - \Psi(A)|$ — coordinate axiom (S3)
12. $\quad = |\Psi(B)/2 - 0|$ — substitution;3,7,11
13. $\quad = |\Psi(B)/2|$ — arithmetic;2
14. $\quad = |\Psi(B) - \Psi(B)/2| = |\Psi(B) - \Psi(M)|$ — arithmetic
15. $\quad = |MB|$ — coordinate axiom (S3)
16. M is a midpoint of S — def midpoint;10,11,15
17. Every segment has a midpoint — $\forall +$;1,16

    ($\star$Now we show that it's unique$\star$)
18. Let N be a midpoint of S
19. $A.N.B$ and $|AN| = |NB|$ — def of midpoint
20. $\Psi(A) < \Psi(N) < \Psi(B)$ — order theorem (Thm 6);3,19
21. $0 < \Psi(N)$ and $0 < \Psi(B) - \Psi(N)$ — arithmetic and substitution;3
22. $\Psi(N) = |\Psi(N) - 0|$ — arithmetic
23. $\quad = |\Psi(N) - \Psi(A)|$ — substitution;3
24. $\quad = |AN|$ — coordinate axiom (S3)
25. $\quad = |NB|$ — substitution;19
26. $\quad = |\Psi(B) - \Psi(N)|$ — coordinate axiom (S3)
27. $\quad = \Psi(B) - \Psi(N)$ — arithmetic;21
28. $2\Psi(N) = \Psi(B)$ — algebra;22,27
29. $\Psi(N) = \Psi(B)/2$ — algebra
30. $\quad = \Psi(M)$ — substitution;7
31. Ψ is injective — def of bijective;4
32. $N = M$ — def injective;29-31
33. Midpoints are unique — def of unique;18,32

QED

Lemma *(alternate def of ray) Let A, B be distinct points. Then $X \in \overrightarrow{AB}$ if and only if $X \in \overleftrightarrow{AB}$*

and $\sim X.A.B$.

Proof:

1. (do in class)

Lemma *(sides) If* l *is a line and points* A,B *are on the opposite side of* l *from point* C *then* A,B *are on the same side of* l*. Similarly, if* A,B *are on the same side of* l *and* C *is on the opposite side of* l *as* A *then* C *is on the opposite side of* l *as* B*.*

Proof:

1. l is a line, A,B,C points — given
2. A is on the opposite side of l as C — given
3. B is on the opposite side of l as C — given
4. l separates the points of the plane not on l into a disjoint union half-planes L,R — separation axiom (S6)
5. $C \in R$ or $C \in L$ — def union, opposite;2,3,4

   (case 1:)
6. Assume $C \in R$
7. $A \in L$ and $B \in L$ — def opposite side;2,3
8. A,B are on the same side of l — def same side
9. $\leftarrow$

   (case 2:)
10. Assume $C \in L$
11. $A \in R$ and $B \in R$ — def opposite side;2,3
12. A,B are on the same side of l — def same side
13. $\leftarrow$
14. A,B are on the same side of l — or-;5,6,8,10,12

QED

Theorem *(Pasch) If* l *meets side* AC *in* $\triangle ABC$ *at exactly one point between* A *and* C *then* l *intersects* AB *or* BC*.*

Proof:

1. l meets side AC in $\triangle ABC$ at exactly one point D between A and C — given
2. $B \in l$ or $B \notin l$ — P or not P thm

   (case 1:)
3. Assume $B \in l$
4. $B \in AB$ — def segment
5. l intersects AB — def intersect;3,4
6. l intersects AB or l intersects BC — or+
7. $\leftarrow$

(case 2:)

8. Assume $B \notin l$

9. $A \notin l$ — def exactly one;1

10. (B is on the same side of l as A) or (B is on the opposite side of l as A) — separation axiom (S6);8,9

(case a:)

11. Assume B is on the opposite side of l as A

12. l intersects AB — separation axiom (S6);11

13. l intersects AB or l intersects BC — or+

14. $\leftarrow$

15. (case b:)

16. Assume B is on the same side of l as A

17. AC intersects l at D — copy;1

18. A is on the opposite side of l as C — separation axiom (S6)

19. B is on the opposite side of l as C — sides lemma;16,18

20. l intersects BC — separation axiom (S6)

21. l intersects AB or l intersects BC — or+

22. $\leftarrow$

23. l intersects AB or l intersects BC — or-;10,11,13,16,21

24. $\leftarrow$

25. l intersects AB or l intersects BC — or-;2,3,6,8,23

QED

Lemma *(segment subset ray subset line) Let A,B be distinct points. Then $AB \subseteq \overrightarrow{AB} \subseteq \overleftrightarrow{AB}$.*

Proof:

1. Let A,B be distinct points — given

2. Let $X \in AB$

3. $X = A$ or $X = B$ or $A.X.B$ — def segment

4. $X = A$ or $X = B$ or $A.X.B$ or $A.B.X$ — or+

5. $X \in \overrightarrow{AB}$ — def ray

6. $AB \subseteq \overrightarrow{AB}$ — def subset;2,5

7. Let $Y \in \overrightarrow{AB}$

8. $Y = A$ or $Y = B$ or $A.Y.B$ or $A.B.Y$ — def ray

(case 1)

9. Assume $Y = A$ or $Y = B$

10. $Y \in \overleftrightarrow{AB}$ — def $\overleftrightarrow{AB}$

11. $\leftarrow$

(case 2)

12. Assume $A.Y.B$ or $A.B.Y$

13. $Y \in \overleftrightarrow{AB}$ — def between

14. $\leftarrow$

15. $Y \in \overleftrightarrow{AB}$ — or-;8,9,10,12,13

16. $\overrightarrow{AB} \subseteq \overleftrightarrow{AB}$ — def subset;7,15

QED

Lemma *(intersect all supersets) If F, G, H are figures with $G \subseteq H$. If F intersects G then F intersects H.*

Proof:

1. F, G, H are figures — given
2. $G \subseteq H$ — given
3. F intersects G — given
4. There exists X with $X \in F$ and $X \in G$ — def intersects
5. $X \in H$ — def subset;2,4
6. There exists X with $X \in F$ and $X \in H$ — $\exists$ +;4,5
7. F intersects H — def intersects

QED

Lemma *(opposites don't attract) Let A, B be distinct points and C, D on opposite sides of $\overleftrightarrow{AB}$. Then $\overrightarrow{AC}$ does not intersect $\overrightarrow{BD}$.*

Proof:

1. Let A, B be distinct points and C, D on opposite sides of $\overleftrightarrow{AB}$ — given
2. Let X be an arbitrary point on $\overrightarrow{AC}$
3. $X = A$ or X is on the same side of $\overleftrightarrow{AB}$ as C — ray-half plane thm
4. Assume X is on $\overrightarrow{BD}$
5. $X = B$ or X is on the same side of $\overleftrightarrow{AB}$ as D — ray-half plane thm

   (case 1)

6. Assume $X = B$
7. $X \neq A$ — def distinct;1
8. X is not on the same side of $\overleftrightarrow{AB}$ as C — separation axiom (S6);6
9. $\sim\left(X = A \text{ or } X \text{ is on the same side of } \overleftrightarrow{AB} \text{ as } C\right)$ — DeMorgan's Law;7,8
10. $\rightarrow\leftarrow$ — $\rightarrow\leftarrow$ +;3,9
11. $\leftarrow$

    (case 2)

12. Assume X is on the same side of $\overleftrightarrow{AB}$ as D
13. $X \neq A$ — separation axiom (S6)

14. X is on the oppositite side of $\overleftrightarrow{AB}$ as C — sides lemma;1,12

15. X is not on the same side of $\overleftrightarrow{AB}$ as C — separation axiom (S6)

16. $\sim\left(X = A \text{ or } X \text{ is on the same side of } \overleftrightarrow{AB} \text{ as } C\right)$ — DeMorgan's Law;7,8

17. $\rightarrow\leftarrow$ — $\rightarrow\leftarrow$ +;3,9

18. $\leftarrow$

19. $\rightarrow\leftarrow$ — or-;5,6,10,12,17

20. $\leftarrow$

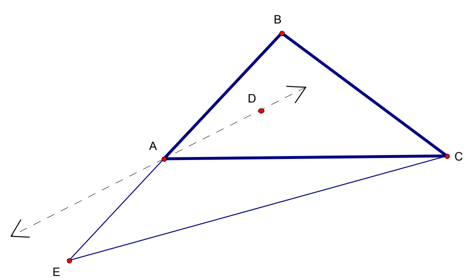
21. X is not on $\overrightarrow{BD}$ — $\sim$ +;4,19

22. No point X on $\overrightarrow{AC}$ is on $\overrightarrow{BD}$ — $\forall$ +;2,21

23. $\overrightarrow{AC}$ does not intersect $\overrightarrow{BD}$ — def of intersect

QED

Theorem *(crossbar) If D is in the interior of $\angle A$ in $\triangle ABC$ then $\overrightarrow{AD}$ intersects BC.*



Proof:

1. D is in the interior of $\angle A$ in $\triangle ABC$ — given

2. A, B, C are not collinear and distinct — def $\triangle$

3. There exists E on $\overleftrightarrow{AB}$ such that $E.A.B$ and $|AB| = |AE|$ — point plotting thm

4. $A \in EB$ — def segment

5. $A \in \overleftrightarrow{EB}$ — segment subset line lemma

6. $A \in \overleftrightarrow{AD}$ — def of $\overleftrightarrow{AD}$

($\star$show BEC forms a triangle$\star$)

7. Assume $E \in \overleftrightarrow{BC}$

8. $\overleftrightarrow{BC} = \overleftrightarrow{EB}$ — 2pts det a line (S1)

9. $A \in \overleftrightarrow{BC}$ — substitution;8,5

10. A, B, C are collinear — def collinear

11. $\rightarrow\leftarrow$ — $\rightarrow\leftarrow$ +;10,2

12. $\leftarrow$

13. $E \notin \overleftrightarrow{BC}$ — $\sim$ +;7,11

14.	E, B, C are not collinear	def collinear
	(⋆ok, now we know it's a triangle⋆)	
15.	$\overleftrightarrow{AD}$ intersects side BE of $\triangle BEC$ at A	def intersects;4,5
	(⋆Here's a key idea⋆)	
16.	$\overleftrightarrow{AD}$ intersects BC or $\overleftrightarrow{AD}$ intersects CE	Pasch's Theorem
	(⋆So all we have to show is that it doesn't intersect CE ⋆)	
17.	D is on the same side of $\overleftrightarrow{AC}$ as B	def interior;1
18.	$A \in \overleftrightarrow{AC}$	def $\overleftrightarrow{AC}$
19.	EB intersects $\overleftrightarrow{AC}$	def intersects;4,18
20.	E is on the opposite side of $\overleftrightarrow{AC}$ as B	separation axiom (S6)
21.	D is on the opposite side of $\overleftrightarrow{AC}$ as E	sides lemma;17,20
22.	$\overrightarrow{AD}$ does not intersect $\overrightarrow{CE}$	opposites don't attract lemma;2;20;21
23.	$CE \subseteq \overrightarrow{CE}$	segment subset ray lemma
24.	$\overrightarrow{AD}$ does not intersect CE	contrapositive of intersect all supersets lemma
25.	D is on the same side of $\overleftrightarrow{AB}$ as C	def interior;1
26.	There exists a point P such that $P.A.D$	point plotting thm
27.	$A \in PD$	def segment
28.	P is on the opposite side of $\overleftrightarrow{AB}$ as D	separation axiom (S6)
29.	P is on the opposite side of $\overleftrightarrow{AB}$ as C	sides lemma25,28
30.	$\overrightarrow{AP}$ does not intersect $\overrightarrow{EC}$	opposites don't attract lemma;2;20;21
31.	$EC \subseteq \overrightarrow{EC}$	segment subset ray lemma
32.	$\overrightarrow{AP}$ does not intersect CE	contrapositive of intersect all supersets lemma;30,31
33.	$\overleftrightarrow{AD} = \overrightarrow{AD} \cup \overrightarrow{AP}$	SMSG Thm 11
34.	$\overleftrightarrow{AD} \cap CE = \left(\overrightarrow{AD} \cup \overrightarrow{AP}\right) \cap CE$	substitution
35.	$= \left(\overrightarrow{AD} \cap CE\right) \cup \left(\overrightarrow{AP} \cap CE\right)$	distributivity of $\cap$ over $\cup$
36.	$= \varnothing \cup \varnothing$	def intersect;32;24
37.	$= \varnothing$	def union
38.	$\overleftrightarrow{AD}$ does not intersect CE	def intersects;34,37
39.	$\overleftrightarrow{AD}$ intersects BC	alt form of or-;16,38
40.	There exists a point Q on $\overleftrightarrow{AD}$ and BC	def of intersects
41.	$Q \in \overrightarrow{BC}$	segment subset ray thm
42.	Q is on the same side of $\overleftrightarrow{AB}$ as C	ray half-plane thm
43.	$Q.A.D$ or $A.Q.D$ or $A.D.Q$ or $Q = D$ or $Q = A$	SMSG Thm 8
44.	$Q.A.D$ or $Q \in \overrightarrow{AD}$	def ray
	(case 1)	

45. Assume $Q \in \overrightarrow{AD}$

46. $\leftarrow$

(case 2)

47. Assume $Q.A.D$

48. $A \in QD$ — def segment

49. $A \in \overleftrightarrow{AB}$ — def $\overleftrightarrow{AB}$

50. QD intersects $\overleftrightarrow{AB}$ — def intersects;48,49

51. Q is on the opposite side of $\overleftrightarrow{AB}$ as D — separation axiom (S6)

52. Q is on the opposite side of $\overleftrightarrow{AB}$ as C — sides lemma

53. Q is not on the same side of $\overleftrightarrow{AB}$ as C — definition of opposite

54. $\rightarrow\leftarrow$ — $\rightarrow\leftarrow$ +;53,42

55. $Q \in \overrightarrow{AD}$ — $\rightarrow\leftarrow$ implies anything

56. $\leftarrow$

57. $Q \in \overrightarrow{AD}$ — or-;44,45,45,47,55

58. $\overrightarrow{AD}$ intersects BC — def intersects;57,40

QED!

Remark *At this point we are going to make another of our transitions from formal to informal proofs. From now on we will only number lines when referring to them in a reason is a key idea in the proof and they are not referred to immediately as illustrated in the following example proofs.*

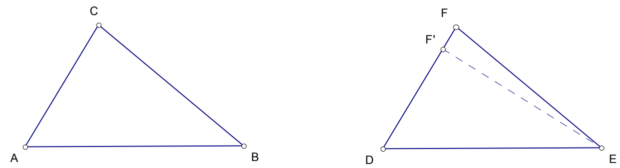
Theorem *(SOCAC) Supplements of the same or congruent angles are congruent.*

Proof:

Let $\angle A$ and $\angle B$ be congruent angles	given
Let $\angle C$ be a supplement of $\angle A$	
Let $\angle D$ be a supplement of $\angle B$	
$\lvert\angle C\rvert + \lvert\angle A\rvert = 180$	def supplement
$= \lvert\angle D\rvert + \lvert\angle B\rvert$	def supplement
$\lvert\angle A\rvert = \lvert\angle B\rvert$	def $\equiv$ angles
$\lvert\angle C\rvert + \lvert\angle A\rvert = \lvert\angle D\rvert + \lvert\angle A\rvert$	substitution
$\lvert\angle C\rvert = \lvert\angle D\rvert$	algebra
$\angle C \equiv \angle D$	def $\equiv$ angles

QED

Theorem *(ASA) If $\angle A \equiv \angle D$, $AB \equiv DE$, $\angle B \equiv \angle E$ then $\triangle ABC \equiv \triangle DEF$.*



Proof:

1. A, B, C are noncollinear and DEF are noncollinear — given
2. $\angle A \equiv \angle D$ — given
3. $AB \equiv DE$ — given
4. $B \equiv \angle DEF$ — given
5. There exists F' on $\overrightarrow{DF}$ such that $DF' \equiv AC$ — point plotting SMSG Thm 9

   $\triangle ABC \equiv \triangle DEF'$ — SAS axiom (S11);4,1,2
6. $\angle B \equiv \angle DEF'$ — CPOCTAC

   F' is on the same side of $\overleftrightarrow{DE}$ as F — ray half-plane thm

   Define $r = |\angle DEF'|$
7. $\overrightarrow{EF'}$ is the unique ray in same half-plane of $\overleftrightarrow{DE}$ as F such that $r = |\angle DEF'|$ — angle construction axiom (S8)

   $|\angle DEF'| = |\angle B|$ — def $\equiv$ angles;5

   $= |\angle DEF|$ — def $\equiv$ angles;3

   $= r$ — copy

   $\overrightarrow{EF} = \overrightarrow{EF'}$ — def unique;6

   $F' \in \overrightarrow{EF}$ — def ray

   $F = F'$ — SMSG Thm 4

   $\triangle ABC \equiv \triangle DEF$ — substitution

QED

Theorem *(isosceles $\triangle$) Two sides in a triangle are congruent if and only if the angles opposite those sides are congruent.*

Proof:

Let A, B, C be noncollinear points forming $\triangle ABC$

Define the correspondence $\xi(A) = A,\ \xi(B) = C,\ \xi(C) = B$

($\Rightarrow$)

Assume $AB \equiv BC$

$\angle A \equiv \angle A$ — $\equiv$ is an equiv reln (SMSG Thm 3)

$\triangle ABC \equiv \triangle ACB$ — SAS axiom (S11) using ξ

$\angle B \equiv \angle C$ — CPOCTAC via ξ

$\leftarrow$

($\Leftarrow$)

Assume $\angle B \equiv \angle C$

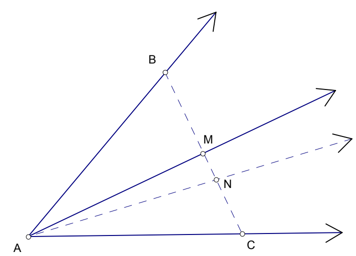
$BC \equiv BC$ — $\equiv$ is an equiv reln (SMSG Thm 3)

$\triangle ABC \equiv \triangle ACB$ ASA (SMSG Thm 23) using ξ
$AB \equiv AC$ CPOCTAC via ξ
$\leftarrow$
QED

Remark *In the following proofs we will start to add "connector" words from English to connect the reasons to the statements and to make grammatical transitions from one statement connect to the next. We also add punctutation like periods, etc.*

Theorem *Angle bisectors exist and are unique.*

Comment: You might think that the easy way to prove this is to imitate what we did for the proof that midpoints exist, by using the angle construction axiom to construct a ray of measure half that of the given angle on the appropriate side of one of the rays. But you would be wrong. Try it and see!



Proof:
Let $\angle BAC'$ be an angle.
There exists a point C on $\overrightarrow{AC'}$ with $AB \equiv AC$ by the point plotting theorem.
$\triangle ABC$ is isosceles by the definition of isosceles.
So $\angle ABC \equiv \angle ACB$ by the isosceles triangle theorem.
There exists a midpoint M of segment BC by SMSG Thm 10.
Hence $BM \equiv MC$ by the definition of midpoint.
Thus $\triangle AMB \equiv \triangle AMC$ by the SAS axiom (S11).
So $\angle BAM \equiv \angle CAM$ since CPOCTAC.
Now $B.M.C$ by the definition of midpoint.
So M is on $\overrightarrow{BC}$ and M is on $\overrightarrow{CB}$ by the definition of ray.
But M is on the same side of $\overleftrightarrow{AC}$ as B and M is on the same side of $\overleftrightarrow{AB}$ as C by the ray half-plane theorem.
So M is in the interior of $\angle BAC$ by the definition of angle interior.
So $\overrightarrow{AM}$ is the angle bisector of $\angle BAC$ by the definition of angle bisector.

Now suppose $\overrightarrow{AN}$ is also an angle bisector of $\angle BAC$
Then N is in the interior of $\angle BAC$ and $\angle BAN \equiv \angle CAN$ by the definition of angle bisector.
So $\overrightarrow{AN}$ intersects BC at some point P by the crossbar theorem.
But $AP \equiv AP$ since $\equiv$ is an equivalence relation.
So $\triangle PAB \equiv \triangle PAC$ by the SAS axiom (S11).
Therefore $PB \equiv PC$ since CPOCTAC,
and thus P is a midpoint of BC by the definition of midpoint.

But midpoints are unique	by SMSG Thm 10.
So $P = M$	by the definition of unique.
Thus, $\overrightarrow{AP} = \overrightarrow{AM}$	by substitution
and $\overrightarrow{AP} = \overrightarrow{AN}$	by the relabeling lemma.
So $\overrightarrow{AN} = \overrightarrow{AM}$	by substitution.
So angle bisectors are unique	by the definition of unique.

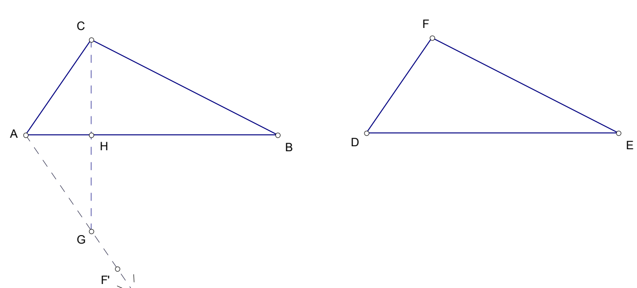
QED

Remark *Notice that if we eliminate the space between the statements and reasons, and word wrap this we get a typical informal math proof such as those found in most textbooks:*

Proof: *Let $\angle BAC'$ be an angle. There exists a point C on $\overrightarrow{AC'}$ with $AB \equiv AC$ by the point plotting theorem. $\triangle ABC$ is isosceles by the definition of isosceles. So $\angle ABC \equiv \angle ACB$ by the isosceles triangle theorem. There exists a midpoint M of segment BC by SMSG Thm 10. Hence $BM \equiv MC$ by the definition of midpoint. Thus $\triangle AMB \equiv \triangle AMC$ by the SAS axiom (S11). So $\angle BAM \equiv \angle CAM$ since CPOCTAC. Now $B.M.C$ by the definition of midpoint. So M is on $\overrightarrow{BC}$ and M is on $\overrightarrow{CB}$ by the definition of ray. But M is on the same side of $\overleftrightarrow{AC}$ as B and M is on the same side of $\overleftrightarrow{AB}$ as C by the ray half-plane theorem. So M is in the interior of $\angle BAC$ by the definition of angle interior. So $\overrightarrow{AM}$ is the angle bisector of $\angle BAC$ by the definition of angle bisector.*

Now suppose $\overrightarrow{AN}$ is also an angle bisector of $\angle BAC$. Then N is in the interior of $\angle BAC$ and $\angle BAN \equiv \angle CAN$ by the definition of angle bisector. So $\overrightarrow{AN}$ intersects BC at some point P by the crossbar theorem. But $AP \equiv AP$ since $\equiv$ is an equivalence relation. So $\triangle PAB \equiv \triangle PAC$ by the SAS axiom (S11). Therefore $PB \equiv PC$ since CPOCTAC and thus P is a midpoint of BC by the definition of midpoint. But midpoints are unique by SMSG Thm 10. So $P = M$ by the definition of unique. Thus, $\overrightarrow{AP} = \overrightarrow{AM}$ by substitution and $\overrightarrow{AP} = \overrightarrow{AN}$ by the relabeling lemma. So $\overrightarrow{AN} = \overrightarrow{AM}$ by substitution. So angle bisectors are unique by the definition of unique.
QED

Theorem *(SSS) If $AB \equiv DE$, $BC \equiv EF$, $AC \equiv DF$ then $\triangle ABC \equiv \triangle DEF$.*



Proof:
Let $\triangle ABC$ and $\triangle DEF$ be triangles with $AC \equiv DF, AB \equiv DE,$ and $BC \equiv EF$.
There exists a point F' on the opposite side of $\overleftrightarrow{AB}$ as C such that $\angle BAF' \equiv \angle D$ by the angle

congstruction axiom (S8).

There exists a point G on $\overrightarrow{AF'}$ such that $AG \equiv DF$ by the point plotting theorem.

Every point of $\overrightarrow{AF'}$ other than A is on the same side of $\overleftrightarrow{AB}$ by the ray half-plane thm.

G is on the same side of $\overleftrightarrow{AB}$ as F' by $\forall -$.

G is on the opposite side of $\overleftrightarrow{AB}$ as C by the sides lemma.

CG intersects $\overleftrightarrow{AB}$ at some point H by the separation axiom (S6).

$H = A$ or $H = B$ or $H.A.B$ or $A.H.B$ or $A.B.H$ by SMSG Thm 8.

(case 1)

Assume $A.H.B$

$AG \equiv DF$ from above (aka copy).

$\angle BAF' \equiv \angle D$ also from above.

$AB \equiv AB$ by reflexivity of $\equiv$.

$\triangle ABG \equiv \triangle DEF$ by SAS (S11).

H is in the interior of $\angle ACB$ and $\angle AGB$ by SMSG Thm 17.

$BG \equiv EF$ since CPOCTAC.

$BG \equiv BC$ by transitivity of $\equiv$.

$\angle ACG \equiv \angle AGC$ by the isosceles $\triangle$ thm applied to $\triangle ACG$.

$\angle BCG \equiv \angle BGC$ by the isosceles $\triangle$ thm applied to $\triangle BCG$.

$|\angle ACG| = |\angle AGC|$ and $|\angle BCG| = |\angle BGC|$ by the definition of congruent segments.

$|\angle ACB| = |\angle ACG| + |\angle BCG|$ by the angle addition axiom (S7).

$= |\angle AGC| + |\angle BGC|$ by substitution.

$= |\angle AGB|$ by the angle addition axiom (S7).

$\angle ACB \equiv \angle AGB$ by the definition of congruent segments.

$AG \equiv AC$ by transitivity of $\equiv$.

$BC \equiv EF$ from above.

So $\triangle ABC \equiv \triangle ABG$ by SAS (S11).

Thus $\triangle ABC \equiv \triangle DEF$ by transitivity of $\equiv$.

$\leftarrow$

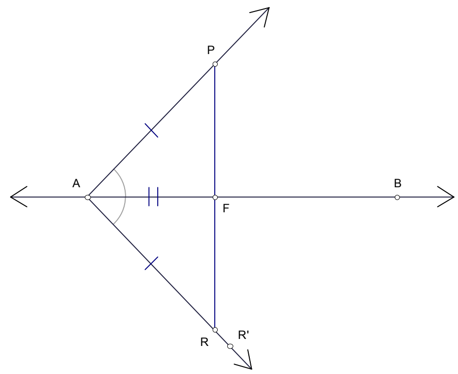
($\star$the other cases are similar and are omitted$\star$)

QED

Remark *In the following proof we continue to make our proofs more informal. In additions to omitting all but essential line numbers, we now eliminate all but essential reasons, and in addition skip non-essential steps. For example, if a statement only has one main logical operation, or only one thing has been modified in going from one statement to the next, then it is usually obvious what the reason is. Also the rules of logic are so well used by this point that it is usually clear when you are doing an or-, implies+, proof by contradiction, etc. at this point and so no reason need to be given.*

Another shortcut we introduce here is that instead of referring to a previous line or lines by giving their line numbers in a reason, we might simply restate the needed information as part of the reason for a statement. For example, "so $\angle BAR \equiv \angle BAP$ by the definition of congruent angles since they have the same measure." in the following proof.

Theorem *(existance of perpendiculars) Through a given point there exists a line* *perpendicular to a given line.*



Proof:

Let $\overleftrightarrow{AB}$ be a line and P a point.

Either $P \in \overleftrightarrow{AB}$ or $P \notin \overleftrightarrow{AB}$.

(case 1)

Assume $P \in \overleftrightarrow{AB}$.

There exists a unique ray $\overrightarrow{CP}$ such that $|\angle BPC| = 90$ (or $|\angle APC| = 90$ if $P = B$) by the angle construction axiom (S8).

$\overleftrightarrow{CP} \perp \overleftrightarrow{AB}$ by def of perpendicular.

So there is a perpendicular to $\overleftrightarrow{AB}$ through P.

$\leftarrow$

(case 2)

Assume $P \notin \overleftrightarrow{AB}$.

There is a ray $\overrightarrow{AR'}$ with R' on the opposite side of $\overleftrightarrow{AB}$ as P such that $|\angle BAR'| = |\angle BAP|$ by the angle construction axiom (S8).

There exists a point R on $\overrightarrow{AR'}$ such that $AR \equiv AP$ by the point plotting theorem.

($\star$we will show that $\overleftrightarrow{RP} \perp \overleftrightarrow{AB}$ $\star$)

R is on the same side of $\overleftrightarrow{AB}$ as R by the ray half-plane theorem.

So R is on the opposite side of $\overleftrightarrow{AB}$ as P by the sides lemma.

Thus RP intersects $\overleftrightarrow{AB}$ at some point F with $R.F.P$ by the separation axiom (S6).

$\angle BAR = \angle BAR'$ by the relabeling lemma,

so $|\angle BAR| = |\angle BAR'|$ by substitution,

$= |\angle BAP|$ from above.

so $\angle BAR \equiv \angle BAP$ by the definition of congruent angles since they have the same measure.

Either $F = A$ or $F \neq A$.

(case a)

Assume $F = A$

A is on RP and $R.F.P$ by substitution.

R, A, P are collinear by definition of between.

$\angle BAR \equiv \angle BAP$ are a linear pair by definition of linear pair.

$\angle BAR, BAP$ are right angles by definition of right angle since they have equal measure.

So $\overleftrightarrow{RP} \perp \overleftrightarrow{AB}$ by definition of perpendicular.

$\leftarrow$
(case b)
Assume $F \neq A$.
$F.A.B$ or $A.F.B$ or $A.B.F$ or $F = B$ by SMSG Thm 8.
$F.A.B$ or $F \in \overrightarrow{AB}$ by definition of ray.
(case i)
Assume $F \in \overrightarrow{AB}$
$\angle BAP = \angle FAP$ and $\angle BAR = \angle FAR$ by relabeling.
$\angle FAR \equiv \angle FAP$ by substitution.
$\leftarrow$
(case ii)
Assume $F.A.B$
$\angle FAP, \angle BAP$ are a linear pair and
$\angle FAR, \angle BAR$ are a linear pair by the definition of linear pair.
$\angle FAP, \angle BAP$ are supplementary and
$\angle FAR, \angle BAR$ are supplementary by the supplement axiom (S10).
$\angle FAR \equiv \angle FAP$ since SOCAC.
$\leftarrow$
$\angle FAR \equiv \angle FAP$ in both cases.
$AF \equiv AF$ since congruence is reflexive.
$AR \equiv AP$ by above.
So $\triangle AFP \equiv \triangle AFR$ by SAS axiom (S11).
$\angle AFP \equiv \angle AFR$ because CPOCTAC.
$\overrightarrow{FR}, \overrightarrow{FP}$ are opposite by definition of opposite.
$\angle AFP, \angle AFR$ are a linear pair by definition of linear pair.
$\overleftrightarrow{RP} \perp \overleftrightarrow{AB}$ by definition of perpendicular.
$\leftarrow$
$\overleftrightarrow{RP} \perp \overleftrightarrow{AB}$ in both cases.
So there is a perpendicular to $\overleftrightarrow{AB}$ through P.
$\leftarrow$
So there is a perpendicular to $\overleftrightarrow{AB}$ through P in every case.
QED

Remark *Note to prove that the perpendicular through a point to a line is unique we will need the following theorem.*

Theorem *(exterior angle) The measure of an exterior angle of a triangle is greater than the measure of either of the two opposite angles.*

Proof:
Let $\triangle ABC$ be a triangle and D a point on $\overrightarrow{AC}$ with $A.C.D$.
There exists a unique midpoint M of BC by SMSG Thm 10.
$BM \equiv CM$ by definition of midpoint.
There exists a point P on $\overrightarrow{AM}$ with $MP \equiv MA$ and $A.M.P$ by the point plotting theorem.

$\angle AMB \equiv \angle CMP$ since vertical angles are equal.
Thus, $\triangle AMB \equiv CMP$ by SAS.
(*) $\angle PCB \equiv \angle B$ since CPOCTAC.

   ($\star$*we now want to show that* P is in the interior of $\angle DCB$ $\star$)

M is in the interior of $\angle BAC$ by SMSG Thm 17.
M is on the same side of $\overleftrightarrow{AC}$ as B by definition of interior.
P is on the same side of $\overleftrightarrow{AC}$ as M by the ray half-plane theorem.
P is on the same side of $\overleftrightarrow{AC}$ as B by the sides lemma.
D is on the opposite side of $\overleftrightarrow{BC}$ as A by the separation axiom, since AD intersects $\overleftrightarrow{BC}$ at C.
P is on the opposite side of $\overleftrightarrow{BC}$ as A by the separation axiom, since AP intersects $\overleftrightarrow{BC}$ at M.
P is on the same side of $\overleftrightarrow{BC}$ as D by the sides lemma.
$|\angle PCD| > 0$ by the angle measure axiom.
$|\angle BCD| = |\angle PCD| + |\angle PCB|$ by the angle addition axiom
$\quad > |\angle PCB|$ by algebra.
$\quad = |\angle B|$ by definition of congruent angles.
Similarly, if E is a point on $\overrightarrow{BC}$ with $B.C.E$ then $|ECA| > |CAB|$.
But $|\angle ECA| = |\angle BCD|$ since vertical angles are congruent.
So $|\angle BCD| > |\angle CAB|$ by substitution.
Thus, $|\angle BCD|$ is greater than either of the measures of the two opposite angles in $\triangle ABC$.
QED

Theorem *(uniqueness of perpendiculars) Through a given point there exists a unique line perpendicular to a given line.*

Proof:
Let $\overleftrightarrow{AB}$ be a line and P a point.
Either $P \in \overleftrightarrow{AB}$ or $P \notin \overleftrightarrow{AB}$.

(case 1)
Assume $P \in \overleftrightarrow{AB}$
There exists a unique ray $\overrightarrow{CP}$ such that $|\angle BPC| = 90$ (or $|\angle APC| = 90$ if $P = B$) by the angle construction axiom (S8).
$\overleftrightarrow{CP} \perp \overleftrightarrow{AB}$ by def of perpendicular.
So there is a unique perpendicular to $\overleftrightarrow{AB}$ through P since $\overrightarrow{CP}$ is unique.
$\leftarrow$

(case 2)
Assume $P \notin \overleftrightarrow{AB}$.
There exists a line $\overleftrightarrow{PF} \perp \overleftrightarrow{AB}$ with $F \in \overleftrightarrow{AB}$ by the existance of perpendiculars theorem.
Let $\overleftrightarrow{AG}$ be a arbitrary line perpendicular to $\overleftrightarrow{AB}$ at G.

   Assume $G \neq F$.
   There exists $H \in \overleftrightarrow{AB}$ with $F.G.H$ by the point plotting theorem.
   $\angle HGA$ and $\angle GFA$ are right angles.
   All right angles have equal measure by SMSG Thm 21.
   So $|\angle HGA| = |\angle GFA|$ by forall minus.

$\angle GFA$ is an exterior angle of $\triangle AFG$ by definition of exterior angle.
$|\angle GFA| > |\angle HGA|$ by the exterior angle theorem.
$|\angle GFA| \neq |\angle HGA|$ by trichotomy.
$\rightarrow\leftarrow$ since we showed these measures are both equal and not equal.
$\leftarrow$
$G = F$ by proof by contradiction.
$\overleftrightarrow{AG} = \overleftrightarrow{AF}$ by substitution.
So there is a unique perpendicular to $\overleftrightarrow{AB}$ through P by definition of unique.
$\leftarrow$
So there is a unique perpendicular to $\overleftrightarrow{AB}$ through P by proof by cases.
QED

Theorem *(AAS) If $\angle A \equiv \angle D,\ \angle B \equiv \angle E,$ and $BC \equiv EF$ then $\triangle ABC \equiv \triangle DEF$.*

Proof:
Let $\triangle ABC$ and $\triangle DEF$ be triangles with $\angle A \equiv \angle D,\ \angle B \equiv \angle E,$ and $BC \equiv EF$.
There exists P on $\overrightarrow{AB}$ with $AP \equiv ED$ by the point plotting theorem.
$\triangle PAC \equiv \triangle DEF$ by SAS.
$\angle BPC \equiv \angle D$ by CPOCTAC.
$\angle BPC \equiv \angle A$ since $\equiv$ is transitive.
$|\angle BPC| = |\angle A|$ by definition of $\equiv$ angles.
$B.P.A$ or $B.A.P$ or $P = A$
(case 1)
Assume $B.P.A$
$\angle BPC$ is an exterior angle of $\triangle PAC$ by definition of exterior angle.
$|\angle BPC| > |\angle A|$ by the exterior angle theorem.
$|\angle BPC| \neq |\angle A|$ by trichotomy.
$\rightarrow\leftarrow$ since the measures are both equal and not equal.
$P = A$ since a contradiction implies anything.
$\leftarrow$
(case 2)
Assume $B.A.P$
$\angle A$ is an exterior angle of $\triangle PAC$ by definition of exterior angle.
$|\angle A| > |\angle BPC|$ by the exterior angle theorem.
$|\angle BPC| \neq |\angle A|$ by trichotomy.
$\rightarrow\leftarrow$ since the measures are both equal and not equal.
$P = A$ since a contradiction implies anything.
$\leftarrow$
(case 3)
Assume $P = A$
$\leftarrow$
So $P = A$ by proof by cases.
$\triangle ABC \equiv \triangle DEF$ by substitution.
QED

Lemma *(order lemma) If C is on $\overrightarrow{AB}$ then $|AC| > |AB|$ if and only if $A.B.C$.*

Proof: The proof is left as an easy exercise in using axioms S2, S3, and S4.

Theorem *(big angle, big side) In $\triangle ABC$, $|\angle A| > |\angle B|$ if and only if $a > b$ (i.e. $|BC| > |AC|$).*

Proof:
Let $\triangle ABC$ be a triangle.
($\Leftarrow$)
Assume $|BC| > |AC|$
There exists D on $\overrightarrow{CA}$ such that $|DC| = |BC|$ by the point plotting theorem.
$|DC| > |AC|$ by substitution.

$C.A.D$ by the order lemma.
A is in the interior of $\angle CBD$ by SMSG Thm 17.
$|\angle ABD| > 0$ by the angle measure axiom (S7).
$|\angle CBD| = |\angle CBA| + |\angle ABD|$ by the angle addition axiom (S9).
$> |\angle CBA|$ by algebra.

$\triangle BCD$ is isosceles by the definition of isoceles.
$\angle CBD \equiv \angle D$ since CPOCTAC.
$|\angle CBD| = |\angle D|$ by the definition of $\equiv$ angles.
$|\angle D| > |\angle CBA|$ by substitution.

$\angle BAC$ is an external angle of $\triangle ABD$ by definition of external angle.
$|\angle BAC| > |\angle D|$ by the external angle theorem.
$> |\angle CBA|$ from above.

So $|\angle BAC| > |\angle CBA|$ by the transitivity of $>$.
$\leftarrow$
($\Rightarrow$)
($\star$ Note: This is an interesting example of how you can sometimes use one half of an $\Leftrightarrow$ theorem to prove the other $\star$)
Assume $|\angle A| > |\angle B|$.
$|BC| > |AC|$ or $|BC| = |AC|$ or $|BC| < |AC|$ by trichotomy.
(case 1)
Assume $|BC| > |AC|$.
$\leftarrow$
(case 2)
Assume $|BC| = |AC|$
$\triangle ABC$ is isosceles by definition of isosceles.
$|\angle A| = |\angle B|$ by the isosceles triangle theorem.
$|\angle A| \neq |\angle B|$ by the trichotomy law since $|\angle A| > |\angle B|$.
$\rightarrow\leftarrow$ since these lengths are equal and not equal.
$|BC| > |AC|$ since a contradiction implies anything.

$\leftarrow$
(case 3)
Assume $|BC| < |AC|$
$|\angle A| < |\angle B|$ by the ($\Leftarrow$) direction of this theorem.
$|\angle A| \nless |\angle B|$ by the trichotomy law since $|\angle A| > |\angle B|$.
$\rightarrow\leftarrow$ since the previous statement is the negation of the one before it.
$|BC| > |AC|$ since a contradiction implies anything.
$\leftarrow$

$|BC| > |AC|$ by proof by cases.
QED

Theorem *(triangle inequality) The sum of the lengths of two sides of a triangle is greater than the length of the third side.*

Proof:
Let $\triangle ABC$ be a triangle.
($\star$WLOG it suffices to show $|AB| + |BC| > |AC|$ $\star$)
There exists D on $\overrightarrow{AB}$ with $BD \equiv BC$ and $A.B.D$ by the point plotting theorem.
So $\triangle CBD$ is isosceles.
$\angle D \equiv \angle BCD$ by the isosceles triangle theorem.
B is in the interior of $\angle ACD$ by SMSG Thm 17.
$|\angle ACB| > 0$ by the angle measure axiom (S7).
$|\angle ACD| = |\angle ACB| + |\angle BCD|$ by the angle addition axiom (S9).
$> |\angle BCD|$ by algebra.
$= |\angle D|$ by the definition of $\equiv$ angles.
So $|AC| < |AD|$ by the big side, big angle theorem applied to $\triangle DAC$.
$= |AB| + |AC|$ by the definition of between.
QED

Lemma *(converse of the vertical angles theorem) If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays and P, R are on opposite sides of $\overleftrightarrow{AB}$ such that $\angle CAR \equiv \angle BAP$ then P, A, R are collinear and hence $\angle CAR$ and $\angle BAP$ are vertical angles.*

Proof:
Let $\overrightarrow{AB}$ and $\overrightarrow{AC}$ be opposite rays and P, R on opposite sides of $\overleftrightarrow{AB}$ such that $\angle CAR \equiv \angle BAP$.
$|\angle CAR| = |\angle BAP|$ by definition of $\equiv$ angles.
$\angle CAR, \angle BAR$ form a linear pair by definition of linear pair.
$\angle CAR, \angle BAR$ are supplementary by the supplement axiom (S10).
$|\angle CAR| + |\angle BAR| = 180$ by the definition of supplement.
$|\angle BAP| + |\angle BAR| = 180$ by substitution.
$|\angle BAP| = 180 - |\angle BAR|$ by algebra.
Define P' such that $\overrightarrow{AP}$ and $\overrightarrow{AP'}$ are opposite rays.
$\angle BAP'$ and $\angle BAP$ are a linear pair.
$\angle BAP', \angle BAP$ are supplementary by the supplement axiom (S10).

$|\angle BAP| + |\angle BAP'| = 180$ by the definition of supplement.
$|\angle BAP| = 180 - |\angle BAP'|$ by algebra.
So $180 - |\angle BAR| = 180 - |\angle BAP'|$ by substitution and
$|\angle BAR| = |\angle BAP'|$ by algebra.
Define $r = |\angle BAR|$.
There is a unique ray with vertex A on the half-plane of $\overleftrightarrow{AB}$ making an angle of measure r with $\overrightarrow{AB}$ by the angle construction axiom.
$\overrightarrow{AR} = \overrightarrow{AP'}$ by definition of unique.
$\overrightarrow{AR}$ is the opposite ray to $\overrightarrow{AP}$ by substitution.
P, A, R are collinear by the definition of opposite.
$\angle PAB$ and $\angle CAR$ are vertical angles by the definition of vertical angle.
QED

Theorem *(alternate interior angle) If two lines are cut by a transversal then the two lines are parallel if and only if the alternate interior angles formed are congruent.*

Proof:
Let l, m, n be lines with $m \parallel n$ and l a transversal meeting m, n at A and B respectively.
($\Leftarrow$)
Assume $C \in m$ and $D \in n$ are points on opposite sides of l with $\angle CAB \equiv \angle ABD$.
$|\angle ABD| = |\angle CAB|$ by definition of $\equiv$.
There exists a midpoint M of AB by the midpoint theorem.
$AM \equiv BM$ by definition of midpoint.
$|\angle CAB| = 90$ or $|\angle CAB| \neq 90$.
(case 1)
Assume $|\angle CAB| = 90$.
$|\angle ABD| = 90$ by substitution.
$l \perp m$ and $l \perp n$ by definition of perpendicular.
$m \parallel n$ by the common perpendicular theorem.
$\leftarrow$
(case 2)
Assume $|\angle CAB| \neq 90$.
$|\angle ABD| \neq 90$ by substitution.
There exists a unique perpendicular $\overleftrightarrow{MC}$ through M to m meeting m at E by the uniqueness of perpendiculars theorem.
There exists a unique perpendicular $\overleftrightarrow{MD}$ through M to n meeting n at F by the uniqueness of perpendiculars theorem.
$\triangle MEA \equiv \triangle MFB$ by AAS.
$\angle AME \equiv \angle BMF$ since CPOCTAC.
E, M, F are collinear by the converse of the vertical angles theorem.
$\overleftrightarrow{EF}$ is a common perpendicular to m and n.
$m \parallel n$ by the common perpendicular theorem.
$\leftarrow$
$m \parallel n$ in both cases.

$\leftarrow$
($\Rightarrow$)
Assume $m \parallel n$.
There exists a point D' on n such that $B \neq D'$ since every line has infinitely many points.
There exists C' on the opposite side of $\overleftrightarrow{AB}$ from D such that $|\angle BAC'| = |D'BA|$ by the angle construction axiom (S8).
$\overleftrightarrow{AC'} \parallel n$ by the ($\Leftarrow$) direction of this theorem proven above.
$\overleftrightarrow{AC'}$ and m are both parallel to n and contain A
($\star$Note that the following line is the first place we are using the parallel axiom, so this is the first SMSG theorem in our list that does not apply to both Euclidean and hyperbolic geometry, but rather just to Euclidean.$\star$)
There exists a unique line through A parallel to n by the parallel axiom (S12).
$m = \overleftrightarrow{AC'}$ by the definition of unique.
m makes congruent alternate interior angles with $\overleftrightarrow{AB}$ by substitution.
$\leftarrow$
QED

Theorem *($\triangle$ sum) The sum of the measures of the angles in a triangle is 180.*

Proof:
Let $\triangle ABC$ be a triangle.
There exists a unique line l through A which is parallel to $\overleftrightarrow{BC}$ by the parallel axiom (S12).
There exist P, Q on l with $P.A.Q$ by the point plotting theorem.
$\angle QAC, \angle PAC$ are a linear pair.
$|\angle QAC| + |\angle PAC| = 180$ by the supplement axiom (S10)..
($\star$WLOG we can assume P is on same side of $\overleftrightarrow{AC}$ from B $\star$)
$\angle PAB \equiv \angle B$ and $\angle QAC \equiv \angle C$ by the alternate interior angle theorem.
$|\angle PAB| = |\angle B|$ and $|\angle QAC| = |\angle C|$ by the definition of $\equiv$ angles.
B is on the same side of l as C by the separation axiom since BC doesn't intersect l.
B is on the same side of $\overleftrightarrow{AC}$ as P from above.
So B is in the interior of $\angle PAC$.
$|\angle PAC| = |\angle PAB| + |\angle BAC|$ by the angle addition axiom (S9).
$|\angle PAB| + |\angle BAC| + |\angle QAC| = 180$ by substitution.
$|\angle B| + |\angle BAC| + |\angle C| = 180$ by substitution.
QED

Lemma *(intersect one-intersect all) If one line intersects a second dstinct line it must intersect every line in the parallel class of the second line as well.*

Proof:
$\mathbb{E}$ is an affine plance by SMSG Thm 2.
In any affine plane, if one line intersects a second distinct line it must intersect every line in the parallel class of the second line as well by the theorem proved for homework in Affine Plane Part II #1.

QED

Theorem *(parallel projection) Let A, B, C, D be distinct points on line l with $AB \equiv CD$ and A', B', C', D' points on m such that $AA' \parallel BB' \parallel CC' \parallel DD'$ then $A'B' \equiv C'D'$.*

Proof:
Let A, B, C, D be distinct points on line l with $AB \equiv CD$.
Let A', B', C', D' points on m such that $AA' \parallel BB' \parallel CC' \parallel DD'$.
There exists a unique line q through A parallel to m by the parallel axiom (S12).
There exists a unique line r through C parallel to m by the parallel axiom (S12).
The Euclidean place is an affine plane by SMSG Thm 2.
q intersects BB' at some unique point Q and r intersects DD' at some unique point R by the intersect one-intersect all lemma.
$AQB'A'$, $CRD'C'$ are parallelograms.
$AQ \equiv A'B'$ and $CR \equiv C'D'$ by the parallelogram theorem (SMSG Thm 47).
$l \parallel m$ or $l \nparallel m$.

(case 1)
Assume $l \parallel m$
$l = q = r$ and $Q = B$ and $R = D$ by definition of unique.
$AB \equiv A'B'$ and $CD \equiv C'D'$ by substitution.
$A'B' \equiv C'D'$ by transitivity of $\equiv$.
$\leftarrow$

(case 2)
Assume $l \nparallel m$.

Assume $q = r$.
A is on r by substitution.
$r = \overleftrightarrow{AC} (= l)$ because two points determine a line (S1).
$l \parallel m$ by substitution.
$\rightarrow\leftarrow$
$\leftarrow$

$q \neq r$ by proof by contradiction.
$q \parallel r$ by SMSG Thm 42.
$\angle QAB \equiv \angle RCD$ by the corresponding angles theorem.
$\angle ABQ \equiv \angle CDR$ by the corresponding angles theorem.
$\triangle ABQ \equiv \triangle CDR$ by ASA.
$AQ \equiv CR$ since CPOCTAC.
$A'B' \equiv C'D'$ by transitivity of $\equiv$.
$\leftarrow$

$A'B' \equiv C'D'$ in both cases.
QED

Theorem *(area of a triangle) The area of a triangle is half of the product of an altitude and its corresponding base.*

Proof:

($\star$We first prove this for right triangles$\star$)
Let $\triangle PQR$ be a right triangle with right angle at P.
There exists a unique perpendicular l to $\overleftrightarrow{PR}$ through R by the perpendiculars theorem.
$l \parallel \overleftrightarrow{PQ}$ by the common perpendicular theorem.
There exists a unique line m parallel to $\overleftrightarrow{PR}$ through Q by the parallel axiom.
m intersects l at some point S by the intersect one-intersect all lemma.
$PQRS$ is a parallelogram by definition of parallelogram.
$PQRS$ is a rectangle by the rectangle theorem.
$\triangle PQR \equiv \triangle SRQ$ by the parallelogram theorem.
$|\triangle PQR| = |\triangle SRQ|$ by the congruence preserves areas axiom (S14).
$|PR||PQ| = |PQRS|$ by the area of a rectangle axiom (S16).
$= |\triangle PQR| + |\triangle SRQ|$ by the area addition axiom (S15).
$= |\triangle PQR| + |\triangle PQR|$ by substitution.
$= 2|\triangle PQR|$ by substitution.
$|\triangle PQR| = \frac{1}{2}|PR||PQ|$ by algebra.
Thus, every right triangle has area equal to half the product of an altitude and its corresponding base.

($\star$Now we can use this for general triangles$\star$)
Let $\triangle ABC$ be a triangle.
There exists a line through A perpendicular to $\overleftrightarrow{BC}$ meeting BC at some point H by the perpendicular theorem.
$H = B$ or $H = C$ or $B.H.C$ or $H.B.C$ or $B.C.H$ by SMSG Thm 8.

(case 1)
Assume $H = B$ or $H = C$.
$\triangle ABC$ is a right triangle.
$|\triangle ABC| = \frac{1}{2}|AH||BC|$ by the result proven above.
$\leftarrow$

(case 2)
Assume $B.H.C$
$\triangle ABC$ is a right triangle.
$|\triangle ABC| = |\triangle AHB| + |\triangle AHC|$ by the area addition axiom (S15).
$= \frac{1}{2}|AH||HB| + \frac{1}{2}|AH||HC|$ by the result proven above.
$= \frac{1}{2}|AH|(|BH| + |HC|)$ by algebra.
$= \frac{1}{2}|AH||BC|$ by the definition of between.
$\leftarrow$

(case 3)
Assume $H.B.C$.
$\triangle AHB$ and $\triangle AHC$ are right triangles.
$|\triangle AHC| = |\triangle AHB| + |\triangle ABC|$ by the area addition axiom (S15).
$\frac{1}{2}|AH||HC| = \frac{1}{2}|AH||BC| + |\triangle ABC|$ by the result proven above.
$\frac{1}{2}|AH|(|HB| + |HC|) = \frac{1}{2}|AH||BC| + |\triangle ABC|$ by the definition of between.
$|\triangle ABC| = \frac{1}{2}|AH||BC|$ by algebra.
$\leftarrow$

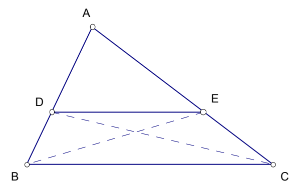
(case 4 is similar to case 3 and is omitted)

$|\triangle ABC| = \frac{1}{2}|AH||BC|$ in every case.
QED

Theorem *(Pythagorean Theorem) In any right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.*

Proof: (See page 9 of Baragar's book for a nice proof. Notice that we can fill in the missing details he mentions after the proof using the SMSG theorems we have proven. For example, squares exist because we can construct them in a manner similar to the rectangle construction in the previous proof above. The area of a square is it's side length squared by the area of a rectangle axiom (S16).)

Theorem *(basic proportionality) A segment connecting points on two sides of a triangle is parallel to the third side if and only if the segments it cuts off are proportional to the sides.*



Proof:
Let $\triangle ABC$ be a triangle, D on AB, and E on AC.
($\Rightarrow$)

Assume $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$.

There exist altitudes DH and EH' of triangles $\triangle DBC$ and ECB respectively by the perpendiculars theorem.

DH and EH' are perpendicular to DE by the common perpendicular theorem.

$DEH'H$ is a rectangle.

$|DH| = |EH'|$ by the parallelogram theorem.

$|\triangle DBC| = \frac{1}{2}|BC||DH|$ by the area of a triangle theorem.

$= \frac{1}{2}|BC||DH'|$ by substitution.

$= |\triangle EBC|$ by the area of a triangle theorem.

$|\triangle DAC| + |\triangle DBC| = |\triangle ABC|$ by the area addition axiom (S15).

$= |\triangle EBA| + |\triangle EBC|$ by the area addition axiom (S15).

$= |\triangle EBA| + |\triangle DBC|$ by substitution.

$|\triangle DAC| = |\triangle EBA|$ by algebra.

Define h to be the length of the altitude of $\triangle ABC$ through C.

Define j to be the length of the altitude of $\triangle ABC$ through B.

$\frac{1}{2}h|AD| = \frac{1}{2}j|AE|$ by the area of a triangle theorem.

$\frac{h}{j} = \frac{|AE|}{|AD|}$ by algebra.

$\frac{1}{2}h|AB| = |\triangle ABC|$ by the area of a triangle theorem.

$= \frac{1}{2}j|AC|$ by the area of a triangle theorem.

$\frac{h}{j} = \frac{|AC|}{|AB|}$ by algebra.

$\frac{|AE|}{|AD|} = \frac{|AC|}{|AB|}$ by substitution.
QED

Theorem *(similarity SSS) If* $\frac{|AB|}{|DE|} = \frac{|AC|}{|DF|} = \frac{|BC|}{|EF|}$ *then* $\triangle ABC \sim \triangle DEF$.

Proof:
Let $\triangle ABC, \triangle DEF$ be triangles with $\frac{|AB|}{|DE|} = \frac{|AC|}{|DF|} = \frac{|BC|}{|EF|} \leq 1$ (WLOG).
There exist X, Y on DE and DF respectively with $|DX| = |AB|$ and $|DY| = |AC|$ by the point plotting theorem.
$\frac{|DX|}{|DE|} = \frac{|DY|}{|DF|}$ by substitution.
$\triangle DXY \sim \triangle ABC$ by the basic proportionality theorem.
$\frac{|XY|}{|EF|} = \frac{|DY|}{|DF|}$ by CPOST.
$\quad = \frac{|AC|}{|DF|}$ by substitution.
$\quad = \frac{|BC|}{|EF|}$ by substitution.
$|XY| = |BC|$ by algebra.
$XY \equiv BC, DX \equiv AB, DY \equiv AC$ by definition of $\equiv$ segments.
$\triangle DXY \equiv \triangle ABC$ by SSS.
$\angle D \equiv \angle A$ since CPOCTAC.
$\triangle ABC \sim \triangle DEF$ by similarity SAS.
QED

Theorem *(fundamental theorem for circles) Let l be a line, $\odot OA$ a circle, and F the foot of the perpendicular to l through O. Then either*
(a) Every point of l is outside the circle, or
(b) F is on the circle and every other point of l is outside the circle (and thus l is a tangent line), or
(c) F is inside the circle and l intersects the circle in two points which are equidistant from F.

Proof:
Let l be a line, $\odot OA$ a circle, and F the foot of the perpendicular to l through O.
Define $r = |OA|$.
$|OF| > r$ or $|OF| = r$ or $|OF| < r$
(case 1)
Assume $|OF| > r$.
Let $X \in l$.
$|OX| \geq |OF|$ by the point-to-line distance theorem.
$\quad > r$ from above.
X is outside $\odot OA$ by definition of circle exterior.
Every point of l is outside the circle since X was arbitrary.
$\leftarrow$
(case 2)
Assume $|OF| = r$.

$F \in \odot OA$ by definition of circle.
Let $X \in l$ and $X \neq F$.
$\triangle OFX$ is a right triangle by definition of perpendicular.
$|OX|^2 = |FX|^2 + r^2$ by the Pythagorean theorem.
$> r^2$ by algebra.
$|OX| > r$ by algebra.
X is outside $\odot OA$ by definition of circle exterior.
Every point of l other than F is outside the circle since X was arbitrary.
F is on the circle and every other point of l is outside the circle from above.
$\leftarrow$
(case 3)
Assume $|OF| < r$.
F is inside the circle by definition of circle interior.
There exist points P, Q on both sides of F on l such that $|FP| = |FQ| = \sqrt{r^2 - |OF|^2}$ by the point plotting theorem.
$\triangle PFO, \triangle QFO$ are right triangles by definition of perpendicular.
$|OP| = |OQ| = r$ by the Pythagorean theorem and algebra.
P, Q are on $\odot OA$ by the definition of circle.
F is inside the circle and l intersects the circle in two points which are equidistant from F from above.
$\leftarrow$
QED

Theorem *(Two Circle Theorem) If two circles having radii a and b have centers that are a distance c apart, and if each of a, b, c is less than the sum of the other two, then the two circles intersect at exactly two points, one on each side of the line through their centers.*

Proof:
Let $\odot AA'$ and $\odot BB'$ be circles with radii a and b respectively and $|AB| = c$ where any of a, b, c is less than the sum of the other two and wlog assume $a \geq b$.
$a^2 < b^2 + c^2$ or $a^2 = b^2 + c^2$ or $a^2 > b^2 + c^2$ by trichotomy.
(case 1)
Assume $a^2 < b^2 + c^2$
Define $x = \frac{b^2+c^2-a^2}{2c}$.
$b - x = b - \frac{b^2+c^2-a^2}{2c}$ by substitution.
$= \frac{2bc-b^2-c^2+a^2}{2c}$ by algebra.
$= \frac{a^2-(b-c)^2}{2c}$ by algebra.
$= \frac{(a+c-b)(a+b-c)}{2c}$ by algebra.
> 0 by algebra since $a + c > b$ and $a + c > b$ from above.
So $x < b$ by algebra.
$c - x = c - \frac{b^2+c^2-a^2}{2c}$ by substitution.
$= \frac{2c^2-b^2-c^2+a^2}{2c}$ by algebra.
$= \frac{a^2+c^2-b^2}{2c}$ by algebra.
$> \frac{a^2+(a^2-b^2)-b^2}{2c}$ since $c^2 > a^2 - b^2$ by the assumption above.

$= \frac{a^2-b^2}{c}$ by algebra.
$= \frac{(a-b)(a+b)}{c}$ by algebra.
> 0 since $a \geq b$ from above.
So $x < c$ by algebra.
$a - (c - x) = a - c + \frac{b^2+c^2-a^2}{2c}$ by substitution.
$= \frac{2ac-2c^2+b^2+c^2-a^2}{2c}$ by algebra.
$= \frac{b^2-c^2+2ac-a^2}{2c}$ by algebra.
$= \frac{b^2-(c-a)^2}{2c}$ by algebra.
$= \frac{(b-c+a)(b+c-a)}{2c}$ by algebra.
> 0 by algebra since $a + b > c$ and $b + c > a$ from above.
So $c - x < a$.
There exists a point P on AB with $|BP| = x$ by the point plotting theorem.
P is inside $\odot BB'$ by definition of interior since $|BP| < b$.
There exists Ψ on $\overleftrightarrow{AB}$ with $\Psi(B) = 0$ and $\Psi(A) > 0$ by the ruler placement axiom (S4).
$\Psi(P) = x$ and $\Psi(A) = c$ by the coordinate axiom (S3).
$|AP| = |\Psi(A) - \Psi(P)|$ by the coodinate aciom (S3).
$= |c - x|$ by substitution.
$= c - x$ by algebra since $x < c$.
$< a$ from above.
So P is inside $\odot AA'$ by definition of interior since $|AP| < a$.

There exists a unique line l through P that is perpendicular to $\overleftrightarrow{AB}$ by the perpendiculars theorem.

l intersects $\odot BB'$ at exactly two points C, C' on opposite sides of $\overleftrightarrow{AB}$ by the fundamental theorem for circles.

l intersects $\odot AA'$ at exactly two points S, S' on opposite sides of $\overleftrightarrow{AB}$ by the fundamental theorem for circles (wlog we can assume S is on the same side as C).
$\left(a^2 - (c-x)^2\right) = a^2 - c^2 + 2xc - x^2$ by algebra.
$= a^2 - c^2 + 2\left(\frac{b^2+c^2-a^2}{2c}\right)c - x^2$ by substitution.
$= a^2 - c^2 + b^2 + c^2 - a^2 - x^2$ by algebra.
$= b^2 - x^2$ by algebra.
$|PS| = \sqrt{a^2 - (c-x)^2}$ by the Pythagorean theorem applied to $\triangle APS$.
$= \sqrt{b^2 - x^2}$ by substitution.
$= |PC|$ by the Pythagorean theorem applied to $\triangle BPS$.

There exists a unique point on $\overrightarrow{PS}$ whose distance from P is $|PS|$ by the point plotting theorem.
$S = C$ by the definition unique.
$S' = C'$ by a similar argument.
$\odot AA', \odot BB'$ intersect at two points, C, C' on each side of $\overleftrightarrow{AB}$ by the definition of intersect.
Let X be any point of intersection of these circles (wlog on the same side of $\overleftrightarrow{AB}$ as C).
$|AX| = a$ and $|BX| = b$ by definition of circle.
$\triangle ABX \equiv \triangle ABC$ by SSS.
$\angle BAX \equiv \angle BAC$ since CPOCTAC.
$\overrightarrow{AX} = \overrightarrow{AC}$ by the angle construction axiom (S8).

X, C are the unique point on $\overrightarrow{AC}$ at distance a from A by the point plotting theorem.
$X = C$ by the definition of unique.
$\odot AA', \odot BB'$ intersect at exactly two points, C, C' on each side of $\overleftrightarrow{AB}$.
$\leftarrow$
(case 2 and case 3 are similar proofs using $x = 0$ and $x = -\frac{b^2+c^2-a^2}{2c}$ respectively and are omitted)
QED

Theorem *(triangle existance) For any positive real numbers a, b, c such that the sum of any two is greater than the third there is a triangle $\triangle ABC$ having side lengths a, b, c.*

Proof:
Let a, b, c be positive real numbers such that the sum of any two is greater than the third.
There exists distinct points A, B' by the point existance axiom (S5).
There exists B on $\overleftrightarrow{AB'}$ such that $|AB| = c$ by the point plotting theorem.
Define $\odot B$ to be the circle of radius a centered at B.
Define $\odot A$ to be the circle of radius b centered at A.
$\odot A$ intersects $\odot B$ at a point C not on $\overleftrightarrow{AB}$ by the Two Circle Theorem.
$|BC| = a$ and $|AC| = b$ by definition of circle.
$\triangle ABC$ has side lengths a, b, c from above.
QED

Further Study of Euclidean Geometry

Lemma *(fun with fractions) Let a, b, x, y, r be real numbers. Then*

$$\frac{x}{y} = \frac{a}{b} = r \Rightarrow \frac{x-a}{y-b} = \frac{x+a}{y+b} = r$$

Pf:
Let a, b, x, y, r be real numbers.

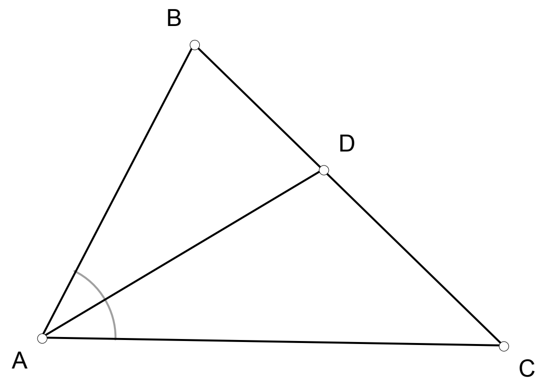
Assume $\frac{x}{y} = \frac{a}{b} = r$.	
$ay = bx$	by algebra
$xy \pm ay = xy \pm bx$	substitution
$(x \pm a)y = (y \pm b)x$	by algebra
$\frac{(x\pm a)}{(y\pm b)} = \frac{x}{y}$	by algebra
$= r$	substitution

QED

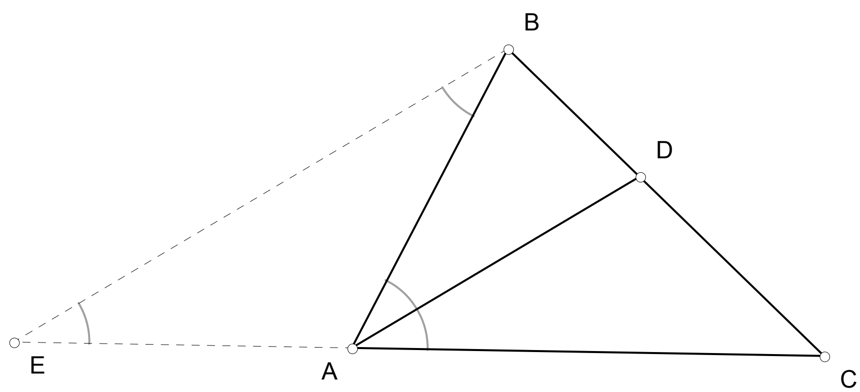
Theorem *(Angle Bisector Theorem) If D is the point where the angle bisector of $\angle A$ in $\triangle ABC$ meets BC then*

$$\frac{|BD|}{|BA|} = \frac{|CD|}{|CA|}$$

Proof:



Let D be the point where the angle bisector of $\angle A$ in $\triangle ABC$ meets BC (which exists by the crossbar theorem).
There is a unique line through B parallel to AD by the parallel axiom.
This line intersects $\overleftrightarrow{AC}$ at some point E by the intersect one, intersect all lemma in the proof section below.



$\angle EBA \equiv \angle BAD$ by the alternate interior angles theorem.
$\angle AEB \equiv \angle DAC$ by the corresponding angles theorem.
$\angle BAD \equiv \angle DAC$ by the definition of angle bisector.
So $\angle EBA \equiv \angle AEB$ by transitivity of $\equiv$.
Thus $EA \equiv BA$ by the isosceles triangle theorem,
and $|EA| = |BA|$ by the definition of congruent segments.
$\frac{|CD|}{|CB|} = \frac{|CA|}{|CE|}$ by the basic proportionality theorem.
$\frac{|CD|}{|CA|} = \frac{|CB|}{|CE|}$ by algebra.
$\frac{|CD|}{|CA|} = \frac{|CB|-|CD|}{|CE|-|CA|}$ by the fun with fractions lemma.
$= \frac{|BD|}{|EA|}$ by the definition of between (segment addition).
$= \frac{|BD|}{|BA|}$ by substitution.
QED