Topology Lecture Notes

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This is **not** a complete set of lecture notes for Math 460, Topology. Additional material will be covered in class and discussed in the textbook.

Logic

In this section we give an informal overview of logic and proofs. For a more formal introduction see any logic textbook.

Variables, Expressions, and Statements

Definition *A **set** is a collection of items called the **members** (or **elements**) of the set.*

Remark *An element is either in a set or it is not in a set, it cannot be in a set more than once.*

Definition *An **expression** is an arrangement of symbols which represents an element of a set called the **domain** (or **type**) of the expression.*

Remark *It is not necessary that we know specifically which element of the domain an expression represents, only that it represents some unspecified element in that set.*

Definition *The element of the domain that the expression represents is called a **value** of that expression.*

Definition *A **variable** is an expression consisting of a single symbol.*

Definition *A **constant** is an expression whose domain contains a single element.*

Definition *A **statement** (or **Boolean expression**) is an expression whose domain is* $\{true, false\}$.

Remark *We do not have to know if a statement is true or false, just that it is either true or false.*

Definition *The value of a statement is called its **truth value**.*

Definition *To **solve** a statement is to determine the set of all elements for which the statement is true.*

Remark *More precisely, if a statement contains* n *variables,* $x_1, \ldots x_n,$ *then to solve the statement is to find the set of all* n*-tuples* $(a_1, \ldots, a_n)$ *such that each* a_i *is an element of the domain of* x_i *and the statement becomes true when* $x_1, \ldots, x_n$ *are replaced by* $a_1, \ldots, a_n$ *respectively. Each such* n*-tuple is called a **solution** of the statement.*

Definition *The set of all solutions of a statement is called the **solution set**.*

Definition *An **equation** is a statement of the form* $A = B$ *where* A and B *are expressions.*

Definition *An **inequality** is a statement of the form* $A \star B$ *where* A and B *are expressions and* $\star$ *is one of* $\leq, \geq, >, <$, *or* $\neq$.

Propositional Logic

The Five Logical Operators

Definition *Let* P, Q *be statements. Then the expressions*

1. $\sim P$
2. P and Q
3. P or Q
4. $P \Rightarrow Q$
5. $P \Leftrightarrow Q$

are also statements whose truth values are completely determined by the truth values of P *and* Q *as shown in the following table*

P	Q	$\sim P$	P and Q	P or Q	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Rules of Inference and Proof

Definition *A **rule of inference** is a rule which takes zero or more statements (or other items) as input and returns one or more statements as output.*

Notation *An expression of the form*

$$\begin{array}{l} P_1 \\ \vdots \\ P_k \\ \hline Q_1 \\ \vdots \\ Q_n \end{array}$$

represents a rule of inference whose inputs are $P_1 \ldots P_k$ *and outputs are* $Q_1, \ldots, Q_n$.

Notation *The rule of inference shown above can also be expressed in **recipe notation** as*

Show P_1

$\vdots$

Show P_k

Conclude Q_1

$\vdots$

Conclude Q_n

or equivalently,

To show $Q_1, \ldots, Q_n$

Show P_1

$\vdots$

Show P_k

Definition *A **formal logic system** consists of a set of statements and a set of rules of inference.*

Definition *A **proof** in a formal logic system consists of a finite sequence of statements (and other inputs to the rules of inference) such that each statement follows from the previous statements in the sequence by one or more of the rules of inference.*

Natural Deduction

Definition *The symbol* $\leftarrow$ *is an abbreviation for "end assumption".*

Definition *The rules of inference for propositional logic are shown in Table 1.*

Table 1: Rules of inference for Propositional Logic

and + To show W and V 1. Show W 2. Show V	**and −** To show W 1. Show W and V **and −** To show V 1. Show W and V
$\Rightarrow$ + To show $W \Rightarrow V$ 1. Assume W 2. Show V 3. $\leftarrow$	**$\Rightarrow$ − (modus ponens)** To show V 1. Show W 2. Show $W \Rightarrow V$
$\Leftrightarrow$ + To show $W \Leftrightarrow V$ 1. Show $W \Rightarrow V$ 2. Show $V \Rightarrow W$	**$\Leftrightarrow$ −** To show $W \Rightarrow V$ 1. Show $W \Leftrightarrow V$ **$\Leftrightarrow$ −** To show $V \Rightarrow W$ 1. Show $W \Leftrightarrow V$
***or* +** To show W or V 1. Show W ***or* +** To show W or V 1. Show V	***or* − (proof by cases)** To show U 1. Show W or V 2. Show $W \Rightarrow U$ 3. Show $V \Rightarrow U$
$\sim$ + (proof by contradiction) To show $\sim W$ 1. Assume W 2. Show $\rightarrow\leftarrow$ 3. $\leftarrow$	**$\sim$ − (proof by contradiction)** To show W 1. Assume $\sim W$ 2. Show $\rightarrow\leftarrow$ 3. $\leftarrow$
$\rightarrow\leftarrow$ + To show $\rightarrow\leftarrow$ 1. Show W 2. Show $\sim W$	

Remark *Note that the inputs "Assume -" and "$\leftarrow$" are not themselves statements but rather inputs to rules of inference that may be inserted into a proof at any time. There is no reason however, to insert such statements unless you intend to use one of the rules of inference that* *requires them as inputs.*

Remark *Precedence: In order to eliminate parentheses we give the operators the following precedence (from highest to lowest):*

other math operators ($+, =, \cdot, \cup, -$, *etc*)
$\sim$
and , or
$\Rightarrow$
$\Leftrightarrow$

Example *Use Natural Deduction to prove the following tautologies.*

1. $\sim\sim P \Leftrightarrow P$
2. $\sim(P \text{ and } Q) \Leftrightarrow \sim P \text{ or } \sim Q$ *[Hint: Use P or $\sim P$, proven in the homework]*

Equality

Definition *The equality symbol, $=$, is defined by the two rules of inference given in Table 2.*

Table 2: Rules of Inference for Equality

Reflexive $=$	Substitution
To show $x = x$	To show W with the n^{th} free occurrence of x replaced by y
	1. Show W
	2. Show $x = y$

Remark *Note that in the Reflexive rule there are no inputs, so you can insert a statement of the form $x = x$ into your proof at any time. Note that there is a technical restriction on the Substitution rule that is not listed here (see the Proof Recipes sheet for details). In most situations the restriction is not a concern.*

Example *Use natural deduction to prove that $x = y \Leftrightarrow y = x$.*

Quantifiers

Definition *The symbols $\forall$ and $\exists$ are **quantifiers**. The symbol $\forall$ is called "for all", "for every", or "for each". The symbol $\exists$ is called "for some" or "there exists".*

Definition *If W is a statement and x is any variable then $\forall x, W$ and $\exists x, W$ are both statements. The rules of inference for these quantifiers are given in Table 3.*

Notation *If x is a variable, t an expression, and $W(x)$ a statement then $W(t)$ is the statement obtained by replacing every free occurrence of x in $W(x)$ with (t),*

Table 3: Rules of Inference for Quantifiers

$\forall+$	$\forall-$
To show $\forall x, W(x)$ 1. Let s be arbitrary 2. Show $W(s)$	To show $W(t)$ 1. Show $\forall x, W(x)$
$\exists+$	$\exists-$
To show $\exists x, W(x)$ 1. Show $W(t)$	To show $W(t)$ for some t 1. Show $\exists x, W(x)$

Remark *Note that there are restrictions on the rules of inference for quantifiers which are not listed in Table 3 (see the Proof Recipes sheet for details). In most situations they are not a concern.*

Remark *Precedence: Quantifiers have a lower precedence than $\Leftrightarrow$. Thus they quantify the largest statement to their right possible unless specifically limited by parentheses.*

Example *Prove* $(\sim\exists x, P(x)) \Rightarrow \forall x, \sim P(x)$

Example *Prove* $(\forall x, P(x) \Rightarrow Q(x))$ and $(\forall y, P(y)) \Rightarrow (\forall z, Q(z))$

Definition *Let $W(x)$ be a statement and $W(y)$ the statement obtained by replacing every free occurrence of x in $W(x)$ with y. We define*

$$(\exists! x, W(x)) \Leftrightarrow \exists x, \left(W(x) \text{ and } \forall y, W(y) \Rightarrow y = x\right)$$

The statement $\exists! x, W(x)$ is read "There exists a unique x such that $W(x)$."

Table 4: Rules of Inference for $\exists!$

$\exists!+$	$\exists!-$
To show $\exists! x, W(x)$ 1. Show $W(t)$ 2. Let y be arbitrary 3. Assume $W(y)$ 4. Show $y = t$ 5. $\leftarrow$	To show $\exists x, W(x)$ and $\forall y, W(y) \Rightarrow y = x$ 1. Show $\exists! x, W(x)$

Sets, Functions, Numbers

Some Definitions from Set theory

The symbol $\in$ is formally undefined, but it means "is an element of". Many of the definitions below are informal definitions that are sufficient for our purposes.

Set notation and operations

Finite set notation:	$x \in \{x_1, \ldots, x_n\} \Leftrightarrow x = x_1$ or $\cdots$ or $x = x_n$
Set builder notation:	$x \in \{y : P(y)\} \Leftrightarrow P(x)$
Cardinality (*see below*):	$\#S =$ the number of elements in a finite set S
Subset:	$A \subseteq B \Leftrightarrow \forall x, x \in A \Rightarrow x \in B$
Set equality:	$A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$
Def. of $\notin$:	$x \notin A \Leftrightarrow \sim(x \in A)$
Empty set:	$A = \emptyset \Leftrightarrow \forall x, x \notin A$
Relative Complement:	$x \in B - A \Leftrightarrow x \in B$ and $x \notin A$
Intersection:	$x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$
Union:	$x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$
Power Set:	$x \in 2^A \Leftrightarrow x \subseteq A$
Indexed Intersection:	$x \in \bigcap_{i \in I} A_i \Leftrightarrow \forall i, i \in I \Rightarrow x \in A_i$
Indexed Union:	$x \in \bigcup_{i \in I} A_i \Leftrightarrow \exists i, i \in I$ and $x \in A_i$
Two convenient abbreviations:	$(\forall x \in A, P(x)) \Leftrightarrow \forall x, x \in A \Rightarrow P(x)$ $(\exists x \in A, P(x)) \Leftrightarrow \exists x, x \in A$ and $P(x)$

Some Famous Sets

The Natural Numbers	$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$
The Integers	$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
The Rational Numbers	$\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}, b > 0, \text{ and } \gcd(a,b) = 1\right\}$
The Real Numbers	$\mathbb{R} = \{x : x$ can be expressed as a decimal number$\}$
The Complex Numbers	$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}$ where $i^2 = -1$
The positive real numbers	$\mathbb{R}^+ = \{x : x \in \mathbb{R}$ and $x > 0\}$
The negative real numbers	$\mathbb{R}^- = \{x : x \in \mathbb{R}$ and $x < 0\}$
The positive reals in a set A	$A^+ = A \cap \mathbb{R}^+$
The negative reals in a set A	$A^- = A \cap \mathbb{R}^-$
The first n positive integers	$\mathbb{I}_n = \{1, 2, \ldots, n\}$
The first $n + 1$ natural numbers	$\mathbb{O}_n = \{0, 1, 2, \ldots, n\}$

Cartesian products

Ordered Pairs:	$(x,y) = (u,v) \Leftrightarrow x = u$ and $y = vz$
Ordered n-tuple:	$(x_1,\ldots,x_n) = (y_1,\ldots,y_n) \Leftrightarrow x_1 = y_1$ and $\cdots$ and $x_n = y_n$
Cartesian Product:	$x \in A \times B \Leftrightarrow x = (a,b)$ for some $a \in A$ and $b \in B$
Cartesian Product:	$x \in A_1 \times \cdots \times A_n \Leftrightarrow x = (x_1,\ldots,x_n)$ for some $x_1 \in A_1$ and $\cdots$ and $x_n \in A_n$
Power of a Set	$A^n = A \times A \times \cdots \times A$ where there are n "A's" in the Cartesian product
Product of Sets	$x \in \prod_{i=0}^{\infty} A_i \Leftrightarrow x = (x_0,x_1,x_2,\ldots)$ and $\forall i, x_i \in A_i$ for some $x_0,x_1,\ldots$

Functions and Relations

Def of $\neq$	$x \neq t \Leftrightarrow \sim(x = t)$
Def of relation:	R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$
Def of function:	$f : A \to B \Leftrightarrow f \subseteq A \times B$ and $(\forall x, \exists! y, (x,y) \in f)$
Alt. function notation	$X \xrightarrow{f} Y \Leftrightarrow f : X \to Y$
Def of $f(x)$ notation:	$f(x) = y \Leftrightarrow f : A \to B$ and $(x,y) \in f$
Domain:	$\text{Domain}(f) = A \Leftrightarrow f : A \to B$
Codomain:	$\text{Codomain}(f) = B \Leftrightarrow f : A \to B$
Image (of a set):	$f(S) = \{y : \exists x, x \in S \text{ and } y = f(x)\}$
Range (or Image of f):	$\text{Range}(f) = f(\text{Domain}(f))$
Identity Map:	$id_A : A \to A$ and $\forall x, id_A(x) = x$
Composition:	$f : A \to B$ and $g : B \to C \Rightarrow (g \circ f) : A \to C$ and $\forall x, (g \circ f)(x) = g(f(x))$
Injective (one-to-one):	f is injective $\Leftrightarrow \forall x, \forall y, f(x) = f(y) \Rightarrow x = y$
Surjective (onto):	f is surjective $\Leftrightarrow f : A \to B$ and $(\forall y, y \in B \Rightarrow \exists x, y = f(x))$
Bijective:	f is bijective $\Leftrightarrow f$ is injective and f is surjective
Inverse:	$f^{-1} : B \to A \Leftrightarrow f : A \to B$ and $f \circ f^{-1} = id_B$ and $f^{-1} \circ f = id_A$
Inverse Image:	$f : A \to B$ and $S \subseteq B \Rightarrow f^{-1}(S) = \{x \in A : f(x) \in S\}$
Constant map:	$f : A \to B$ is a constant map $\Leftrightarrow \exists c \in B, \forall x \in A, f(x) = c$
Inclusion map:	$i : A \to B$ is an inclusion map $\Leftrightarrow A \subseteq B$ and $\forall a \in A, i(a) = a$

Example *Prove* $(A - B) \subseteq (A \cup B) - (A \cap B)$

Example *Prove if* $f : A \to B$, $X \subseteq A$, *and* $Y \subseteq B$ *then* $f(X) \subseteq Y \Leftrightarrow X \subseteq f^{-1}(Y)$.

Counting

Definition *Two sets have the same **cardinality** if and only if there is a bijection from one set to the other.*

Definition *A finite set* A *has* n *elements if and only if there is a bijection from* $\{1,2,3,\ldots,n\}$ *to* A.

Remark *If two sets have the same cardinality then they are both infinite, or both finite. If they are finite the have the same number of elements.*

Equivalence Relations

Definition *Let X be a set.*

$$R \text{ is a } \textbf{relation} \text{ on } X \Leftrightarrow R \subseteq X \times X.$$

Definition *Let X be a set and $R \subseteq X \times X$. For any $x,y \in X$,*

$$xRy \Leftrightarrow (x,y) \in R \qquad \text{(infix notation)}$$

and

$$R(x,y) \Leftrightarrow (x,y) \in R \qquad \text{(prefix notation)}$$

Definition *Let X be a set and $R \subseteq X \times X$.*

$$R \text{ is an } \textbf{equivalence relation} \Leftrightarrow \forall x,y,z \in X,$$

$$\text{(0) } xRx \qquad \text{(reflexive)}$$

$$\text{(1) } xRy \Rightarrow yRx \qquad \text{(symmetric)}$$

$$\text{(2) } xRy \text{ and } yRz \Rightarrow xRz \qquad \text{(transitive)}$$

Definition *Let $R \subseteq X \times X$ be an equivalence relation and $a \in X$.*

$$[a]_R = \{x : xRa\}$$

*This is called the **equivalence class** of a (with respect to R).*

Notation *We often abbreviate $[a]_R$ by $[a]$ when the relation R is clear from context.*

Theorem (Fundamental Theorem of Equivalence Relations) *Let $R \subseteq X \times X$ be an equivalence relation and $a,b \in X$. Then*

$$[a] = [b] \Leftrightarrow aRb.$$

Corollary (1) *Let $R \subseteq X \times X$ be an equivalence relation. Then X is a disjoint union of equivalence classes, i.e.*

$$X = \bigcup_{a \in X} [a]$$

and

$$\forall a,b \in X, [a] = [b] \text{ or } [a] \cap [b] = \emptyset.$$

Definition *If X is a set and $P = \{A_i : i \in I\}$ is a set of subsets of X such that*

$$X = \bigcup_{i \in I} A_i$$

and

$$\forall i,j \in I, i \neq j \Rightarrow A_i \cap A_j = \emptyset$$

*we say that P is a **partition** of X.*

Remark *Thus, the set of equivalence classes of an equivalence relation on X is a partition of X.*

Definition *Let $R \subseteq X \times X$ be an equivalence relation. Then the **quotient** of X by the relation R is*

$$X/R = \{[x]_R : x \in X\}$$

In other words X/R is the set of all equivalence classes.

Definition *Let $R \subseteq X \times X$ be an equivalence relation. The **quotient map** is the function $\pi : X \to X/R$ such that for all $x \in X$*

$$\pi(x) = [x]_R$$

Theorem *Every quotient map is onto.*

Composition

Theorem *Composition of functions is associative.*

Theorem *The composition of injective functions is injective and the composition of surjective functions is surjective.*

Theorem (*left cancellation law for injective functions*) *Let $Y \xrightarrow{f} Z$. Then f is injective if and only if for all functions $g, h : X \to Y$*

$$(f \circ g = f \circ h) \Rightarrow g = h$$

Theorem (*right cancellation law for surjective functions*) *Let $X \xrightarrow{f} Y$ and $|Z| > 1$. Then f is surjective if and only if for all functions $g, h : Y \to Z$*

$$(g \circ f = h \circ f) \Rightarrow g = h$$

Inverse Functions

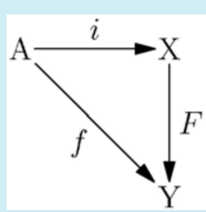
Theorem *A function has an inverse function if and only if it is bijective.*

Theorem *Inverse functions are unique.*

Extensions and Restrictions

Definition *Let $f : A \to Y$, $F : X \to Y$, $A \subseteq X$. If $\forall a \in A, f(a) = F(a)$ then we say that f **is the restriction of** F **to** A and that F **is an extension of** f **to** X. In this situation we write $f = F|A$.*

Remark *In this situation, if $A \xrightarrow{i} X$ is the inclusion map, then $f = F|A = Fi$. In other words the following diagram commutes*



Metric Spaces

Definition *A **metric space** is a pair (X,d) where X is a set and $d : X \times X \to \mathbb{R}$ such that for all $x,y,z \in X$:*

1. $d(x,y) \geq 0$
2. $d(x,y) = 0 \Leftrightarrow x = y$
3. $d(x,y) = d(y,x)$
4. $d(x,y) + d(y,z) \geq d(x,z)$

*In this situation, d is called a **metric** (or distance function) on X, and the elements of X are called the **points** in the metric space. The set X is called the **underlying set** of the metric space.*

Remark *It is quite common to refer to the metric space (X,d) as simply X.*

Examples of Metric Spaces

Example *$(\mathbb{R}, d_{\text{Euc}})$ is a metric space where $d_{\text{Euc}}(x,y) = |x-y|$ for all $x,y \in \mathbb{R}$.*

Notice this is just a special case of the more general theorem:

Theorem *$(\mathbb{R}^n, d_{\text{Euc}})$ is a metric space where*

$$d_{\text{Euc}}((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

*d_{Euc} is called the **Euclidean metric** on $\mathbb{R}^n$.*

Definition *Let $d_{\text{Taxi}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by*

$$d_{\text{Taxi}}((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \sum_{i=1}^{n}|x_i - y_i|$$

*The map d_{Taxi} is called the **lattice metric**, the **Manhattan metric**, or the **taxicab metric**.*

Definition *Let (X,d) be a metric space. Then a **circle** with center $p \in X$ and radius $r \in \mathbb{R}^+$ is*

$$\{x : d(x,p) = r\}$$

Remark *If S is a finite set of real numbers then $\max S$ is the largest number in the set, in other words*

$$m = \max S \Leftrightarrow m \in S \text{ and } \forall n \in S, n \leq m$$

Definition *Let $d_{\max} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by*

$$d_{\max}((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \max\{|x_i - y_i| : i \in \{1,\ldots,n\}\}$$

*The map $d_{\max}$ is called the **maximum metric**.*

Definition *The set of **2-adic integers**, denoted $\mathbb{Z}_2$, is the set of all infinite sequences of 0's and 1's, i.e.*

$$\mathbb{Z}_2 = \{(s_0,s_1,\ldots) : \forall i \in \mathbb{N}, s_i \in \{0,1\}\}$$

or equivalently

$$\mathbb{Z}_2 = \{s : s : \mathbb{N} \to \{0,1\}\}$$

Definition *Let* $d_2 : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{R}$ *by*

$$d_2((s_0,s_1,\dots),(t_0,t_1,\dots)) = \frac{1}{2^k}$$

where $k = \min\{i : s_i \neq t_i\}$ *if* $(s_0,s_1,\dots) \neq (t_0,t_1,\dots)$ *and*

$$d_2((s_0,s_1,\dots),(t_0,t_1,\dots)) = 0$$

if $(s_0,s_1,\dots) = (t_0,t_1,\dots)$. *The map* d_2 *is called the* ***2-adic metric***.

Theorem $(\mathbb{R}^n,\ d_{\text{Taxi}})$, $(\mathbb{R}^n,\ d_{\max})$, *and* $(\mathbb{Z}_2,\ d_2)$ *are metric spaces.*

Remark *It is a fact that* $(\mathbb{Z}_2,\ d_2)$ *cannot be embedded in* $(\mathbb{R}^n, d_{\text{Euc}})$ *for any* n. *The 2-adic metric is simple to compute and work with, but the geometry of* $(\mathbb{Z}_2,\ d_2)$ *is very strange.*

Product metric

Definition *Let* $(X_1,d_1),(X_2,d_2),\dots,(X_n,d_n)$ *be metric spaces and* $X = X_1 \times X_2 \times \cdots \times X_n$. *Define* $d_{\max} : X \times X \to \mathbb{R}$ *by*

$$d_{\max}((x_1,\dots,x_n),(y_1,\dots,y_n)) = \max\{d_i(x_i,y_i) : i \in \{1,\dots,n\}\}$$

where $x = (x_1,x_2,\dots,x_n)$ *and* $y = (y_1,y_2,\dots,y_n)$. *This is called the* ***product metric***.

Theorem *The product metric is a metric.*

Continuity

Maps between metric spaces

Definition *A* ***map between metric spaces*** (X,d) ***and*** (Y,d') *is any ordered tuple* (f,X,d,Y,d') *where* $f : X \to Y$ *and* (X,d) *and* (Y,d') *are metric spaces.*

Notation *We write* $f : (X,d) \to (Y,d')$ *to mean that* (f,X,d,Y,d') *is a map between metric spaces* (X,d) *and* (Y,d').

Continuous maps

Definition *Let* $f : (X,d) \to (Y,d')$. *Then* f *is* ***continuous at*** $a \in X$ *if and only if*

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in X, d(x,a) < \delta \Rightarrow d'(f(x),f(a)) < \varepsilon$$

Definition *Let* $f : (X,d) \to (Y,d')$. *Then* f *is* ***continuous*** *if and only if* f *is continuous at every point* $a \in X$.

Theorem *Every constant map is continuous.*

Theorem *Every identity map from a metric space to itself is continuous.*

Theorem *The identity map* $i : (\mathbb{R}^n,d_{\max}) \to (\mathbb{R}^n,d_{\text{Euc}})$ *and the identity map* $i' : (\mathbb{R}^n,d_{\text{Euc}}) \to (\mathbb{R}^n,d_{\max})$ *are both continuous.*

Theorem *If* $f : (X,d) \to (Y,d')$ *is continuous at* $a \in X$ *and* $g : (Y,d') \to (Z,d'')$ *is continuous at* $f(a)$ *then* $g \circ f : (X,d) \to (Z,d'')$ *is continuous at* a.

Corollary *The composition of continuous functions is continuous.*

Open Balls and Neighborhoods

Definition *Let* (X,d) *be a metric space,* $\delta \in \mathbb{R}^+$, *and* $a \in X$. *Then*

$$B(a;\delta) = \{x \in X \mid d(x,a) < \delta\} \text{ and}$$

$$\overline{B}(a;\delta) = \{x \in X \mid d(x,a) \leq \delta\}$$

$B(a;\delta)$ *is called the* ***open ball of radius*** δ *centered at* a, *and* $\overline{B}(a;\delta)$ *is called the* ***closed ball of radius*** δ *centered at* a.

Remark *This gives us another language for specifying that two elements are close together since*

$$d(x,a) < \delta \Leftrightarrow x \in B(a;\delta)$$

Two useful facts

Lemma (subset) *Let* $f : X \to Y$, $U \subseteq X$, *and* $V \subseteq Y$. *Then*

$$U \subseteq f^{-1}(V) \Leftrightarrow f(U) \subseteq V$$

Lemma (subset) *Let* $f : X \to Y$, $A,B \subseteq X$ *and* $U,V \subseteq Y$. *Then*

$$U \subseteq V \Rightarrow f^{-1}(U) \subseteq f^{-1}(V)$$

and

$$A \subseteq B \Rightarrow f(A) \subseteq f(B)$$

Neighborhoods

Definition *Let* (X,d) *be a metric space,* $a \in X$, *and* $N \subseteq X$. *Then* N *is a* ***neighborhood of*** a *if and only if* $\exists \delta \in \mathbb{R}^+, B(a;\delta) \subseteq N$.

Definition *Let* (X,d) *be a metric space and* $a \in X$. *The set*

$$\mathcal{N}_a = \{N : N \text{ is a neighborhood of } a\}$$

is called the ***complete system of neighborhoods of the point*** a.

Theorem *Every open ball is a neighborhood of all of its points.*

Definition *Let* (X,d) *be a metric space and* $a \in X$. *A set* $\mathcal{B}_a \subseteq \mathcal{N}_a$ *is called a* ***basis for the neighborhood system of*** a *if and only if* $\forall N \in \mathcal{N}_a, \exists B \in \mathcal{B}_a, B \subseteq N$.

Example *The set of all open balls centered at* a *is a basis for the neighborhood system at* a.

Elementary Properties of Neighborhoods and Neighborhood Systems

Theorem *Let* (X,d) *be a metric space* $a \in X$.
N1. a *has a neighborhood.*
N2. a *is an element of each of its neighborhoods.*
N3. Every superset of a neighborhood of a *is a neighborhood of* a.
N4. The intersection of any two neighborhoods of a *is a neighborhood of* a.
N5. Every neighborhood of a *has a subset that is a neighborhood of all of its points.*

Open Balls, Neighborhoods, and Continuity

Theorem *Let* $f : (X,d) \to (Y,d')$ *and* $a \in X$. *The following are equivalent.*

1. f is continuous at a
2. $\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, f(B(a;\delta)) \subseteq B(f(a);\varepsilon)$
3. $\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, B(a;\delta) \subseteq f^{-1}(B(f(a);\varepsilon))$
4. $\forall N \in \mathcal{N}_{f(a)}, f^{-1}(N) \in \mathcal{N}_a$

Open sets and Continuity

Open sets

Definition *Let (X,d) be a metric space and $U \subseteq X$. Then U is **open** if and only if*

$$\forall\ x \in U,\ U \in \mathcal{N}_x$$

Remark *In other words, a set is open if and only if it is a neighborhood of all of its points.*

Definition *Let (X,d) be a metric space and $U \subseteq X$. Then U is **closed** if and only if $X - U$ is open.*

Remark *There are sets which are neither open nor closed.*

An equivalent definition of continuity

Theorem *Let (X,d) and (Y,d') be metric spaces and $f : X \to Y$. Then f is **continuous** with respect to the metrics d and d' if and only if*

$$\forall U \subseteq Y, U \text{ is open in } (Y,d') \Rightarrow f^{-1}(U) \text{ is open in } (X,d).$$

Remark *In other words a function between metric spaces is continuous if and only if the inverse image of every open set is open.*

Properties of the set of all open sets

Theorem *Let (X,d) be a metric space.*
1. *The empty set is open.*
2. *The set X is open.*
3. *The union of any collection of open sets is open.*
4. *The intersection of finitely many open sets is open.*

Topology

Topological Spaces

Definition *Let X be a set and τ a set of subsets of X such that*
1. $\emptyset \in \tau$
2. $X \in \tau$
3. *The union of any collection of elements of τ is an element of τ*
4. *The intersection of finitely elements of τ is an element of τ*

*Then the pair (X,τ) is called a **topological space**, and τ is called a **topology on the set** X. An element of τ is called an **open set**.*

Remark *So τ is by definition the set of open subsets of X.*

Corollary *Let (X,d) be any metric space and τ the set of all open (in the metric space)* *subsets of X. Then (X,τ) is a topological space.*

Definition *The topology τ given in the previous corollary is called the* ***topology induced by the metric*** *d. The topological space (X,τ) is called the* ***associated topological space*** *for the metric space (X,d).*

Remark *Just as we often refer to a metric space (X,d) by X, we also sometimes refer to a topological space (X,τ) by X, and we will often identify a metric space with it's associated topological space.*

Remark *Note that while every metric space has a unique associated topological space, more than one metric space might have the same associated topological space.*

Definition *A topological space that is the associated topological space for some metric space is said to be* ***metrizable****.*

Definition *Let (X,τ) be a topological space. A subset of X is* ***closed*** *if and only if its complement is open.*

Neighborhoods, Interior, Boundary, Closure

Definition *Let (X,τ) be a topological space, $x \in X$, and $N \subseteq X$. Then N is said to be a* ***neighborhood*** *of x if and only if $x \in \mathcal{O} \subseteq N$ for some open set $\mathcal{O} \in \tau$.*

Remark *In other words a neighborhood of a point in topological space is a set that has an open subset that contains the point.*

Definition *Let (X,τ) be a topological space and $A \subseteq X$. Then A is* ***closed*** *if and only if $X - A$ is open.*

Definition *Let (X,τ) be a topological space, $x \in X$, and $A \subseteq X$. Then x* ***is in the closure of*** *A if and only if every neighborhood of x contains an element of A. The set of all points in the closure of A is called the* ***closure of*** *A and is denoted $\overline{A}$.*

Definition *Let (X,τ) be a topological space, $x \in X$, and $A \subseteq X$. Then x* ***is in the interior of*** *A if and only if A is a neighborhood of x. The set of all points in the interior of A is called the* ***interior of*** *A and is denoted $\operatorname{Int}(A)$ or A°.*

Definition *Let (X,τ) be a topological space, $x \in X$, and $A \subseteq X$. Then x* ***is in the boundary of*** *A if and only every neighborhood of x contains an element of A and an element of $X - A$. The set of all points in the boundary of A is called the* ***boundary of*** *A and is denoted $\operatorname{Bdry}(A)$.*

Theorem (Elementary Properties) *Let (X,τ) be a topological space, $x \in X$, and $A \subseteq X$.*

1. *The intersection of any collection of closed sets is closed.*
2. *The union of finitely many closed sets is closed.*
3. $A \subseteq \overline{A}$
4. *The closure of A is the smallest closed set containing A.*
5. $\overline{\overline{A}} = \overline{A}$.
6. $A^\circ \subseteq A$
7. *The interior of a set is the largest open subset of A.*
8. $\operatorname{Bdry}(A) = \overline{A} \cap \overline{X - A}$

9. $\mathrm{Bdry}(A)$ *is closed.*

Applications to metric spaces

Definition *Let* (X,d) *be a metric space,* $x \in X$*, and* $A \subseteq X$*. Then the* ***distance from*** x ***to*** A *is*

$$d(x,A) = \inf\{d(x,a) : a \in A\}$$

Definition *A topological space* (X,τ) *is said to be* ***Hausdorff*** *if and only if for every* $x,y \in X$ *with* $x \neq y$*, there exists neighborhoods* A,B *of* x,y *respectively such that* $A \cap B = \emptyset$*.*

Theorem *Every metrizable topological space is Hausdorff.*

Functions, Continuity, Homeomorphism

Functions

Definition *A* ***map between topological spaces*** (X,τ) ***and*** (Y,τ') *is an ordered tuple* (f,X,τ,Y,τ') *where* $f : X \to Y$ *and* (X,τ) *and* (Y,τ') *are topological spaces.*

Notation *We write* $f : (X,\tau) \to (Y,\tau')$ *to mean that* (f,X,τ,Y,τ') *is a map between topological spaces* (X,τ) *and* (Y,τ')*.*

Continuity

Definition *A map of topological spaces* $f : (X,\tau) \to (Y,\tau')$ *is* ***continuous at*** $a \in X$ *if and only if the inverse image of every neighborhood of* $f(a)$ *in* (Y,τ') *is a neighborhood of* a *in* (X,τ)*, i.e.* $\forall N \in \mathcal{N}_{f(a)}, f^{-1}(N) \in \mathcal{N}_a$*.*

Definition *A map of topological spaces* $f : (X,\tau) \to (Y,\tau')$ *is* ***continuous*** *if and only if the inverse image of every open set is open, i.e.* $\forall \mathcal{O} \in \tau', f^{-1}(\mathcal{O}) \in \tau$*.*

Lemma *A map between topological spaces is continuous if and only if it is continuous at every point.*

Theorem *The composition of continuous maps between topological spaces is continuous.*

Homeomorhisms

Definition *A map of topological spaces* $h : (X,\tau) \to (Y,\tau')$ *is called a* ***homeomorphism*** *if and only if it is a continuous bijection with a continuous inverse.*

Definition *If there exists a homeomorphism between topological spaces* (X,τ) *and* (Y,τ') *we say that these topological spaces are* ***homeomorphic****.*

Remark *Homeomorphic topological spaces are the same topological spaces in disguise!*

Subspaces

Definition *Let* (X,τ) *be a topological space and* $S \subseteq X$*. The* ***subspace topology on*** S *is* $\tau' = \{S \cap \mathcal{O} : \mathcal{O} \in \tau\}$*.*

Theorem *A subspace topology is a topology.*

Definition *Let* (X,τ) *be a topological space,* $S \subseteq X$*, and* τ' *the subspace topology on* S*. We say that* τ' *is the topology on* S ***induced by*** τ*. The topological space* (S,τ') *is called a* ***subspace*** *of* (X,τ)*. An open set* $\mathcal{O}' \in \tau'$ *is said to be* ***relatively open*** *and the neighborhoods* in (S,τ') are said to be ***relative neighborhoods***.

Theorem *Let (S,τ') be a subspace of (X,τ), and $F \subseteq S$. Then F is closed in (S,τ') if and only if $F = S \cap F'$ for some closed set F' in (X,τ).*

Theorem *Let (S,τ') be a subspace of (X,τ), and $x \in N \subseteq S$. Then N is neighborhood of x in (S,τ') if and only if $N = S \cap N'$ for some neighborhood N' of x in (X,τ).*

Theorem *Let (S,τ') be a subspace of (X,τ) and $i : S \to X$ be the inclusion map. Then i is continuous.*

Weak vs Strong topologies

Definition *Let τ and ρ be topologies on X. We say τ* ***is weaker than*** *ρ if and only if $\tau \subseteq \rho$. If τ is weaker than ρ we say ρ is* ***stronger than*** *τ.*

Remark *If a map $f : (X,\tau) \to (Y,\tau')$ is continuous then it will still be continuous if we replace τ with a stronger topology or τ' with a weaker one.*

Theorem
1. Let $f : X \to Y$ and τ' a topology on Y. The is a unique topology τ on X that is the weakest topology for which f is continuous (namely $\tau = \{f^{-1}(\mathcal{O}) : \mathcal{O} \in \tau'\}$).
2. Let $f : X \to Y$ and τ a topology on X. The is a unique topology τ' on Y that is the strongest topology for which f is continuous (namely $\tau' = \{\mathcal{O} \subseteq Y : f^{-1}(\mathcal{O}) \in \tau\}$).

Theorem *The subspace topology is the weakest topology on S for which the inclusion map is continuous.*

Product Topologies

Definition *Given an indexed family of topological spaces $\{(X_i,\tau_i)\}_{i\in I}$ we defined the* ***product topology*** *on $\prod_{i\in I} X_i$ to be the weakest topology such that all of the projection maps $p_i : \prod_{i\in I} X_i \to X_i$ are continuous.*

Remark *Therefore product topology is the smallest topology that contains all sets of the form $p_i^{-1}(\mathcal{O}_i)$ such that $\mathcal{O}_i \in \tau_i$.*

Theorem *The product topology τ on $\prod_{i\in I} X_i$ is the set of all unions of sets which are themselves the intersection of finitely many sets of the form $p_i^{-1}(\mathcal{O}_i)$ where $\mathcal{O}_i \in \tau_i$.*

The Finite Case

Definition *A collection of open subsets $\mathcal{B} = \{\mathcal{O}_i\}_{i\in I}$ of a topological space (X,τ) is a* ***basis*** *for the topology τ, if every open subset of X is a union of elements of $\mathcal{B}$.*

Theorem *Let n be a positive integer and $(X_1,\tau_1),(X_2,\tau_2),\ldots,(X_n,\tau_n)$ topological spaces. Then*

$$\{O_1 \times O_2 \times \cdots \times O_n : O_1 \in \tau_1,\ldots,O_n \in \tau_n\}$$

is a basis for the product topology τ on $X_1 \times X_2 \times \cdots \times X_n$.

Example *Let $(X,\tau),(Y,\tau')$ be topological spaces. Then $\mathcal{O}$ is open in $X \times Y$ (with the product* *topology) if and only if* $\mathcal{O} = \bigcup_{i\in I}(\mathcal{O}_{\alpha_i} \times \mathcal{O}_{\beta_i})$ *for some open sets* $\{\mathcal{O}_{\alpha_i}\}_{i\in I}$ *in* X *and* $\{\mathcal{O}_{\beta_i}\}_{i\in I}$ *in* Y.

Quotient Topology

Definition *Let* (X,τ) *be a topological space and* R *and equivalence relation on* X. *Then the* ***quotient topology*** *(or* ***identification topology****) is the strongest topology on* X/R *for which the quotient map continuous.*

Theorem *Let* (X,τ) *be a topological space and* R *and equivalence relation on* X. *Then the quotient topology on* X/R *is the set*

$$\tau' = \{\mathcal{O} \subseteq X/R : \pi^{-1}(\mathcal{O}) \in \tau\}$$

Example *Let* (X,τ) *be a topological space and* $f : X \to Y$ *any surjective function and let* τ' *be the strongest topology on* Y *for which* f *is continuous. Define an equivalence relation* $\sim_f$ *on* X *by* $a \sim_f b$ *if and only if* $f(a) = f(b)$. *Then* $(X/\sim_f, \tau'')$ *is homeomorphic to* (Y,τ') *where* τ'' *is the quotient topology.*

Remark *Since in the previous example,* (Y,τ') *is homeomorphic to* $(X/R,\tau'')$ *we sometimes refer to* τ' *as a quotient or identification topology as well.*

Connectedness

Definition *A topological space is* ***connected*** *if an only if the only subsets of it that are both open and closed are the empty set and the space itself. A space that is not connected is said to be* ***disconnected****.*

Remark *Hence a subspace of a topological space is connected if and only if the only subsets of it that are both relatively open and relatively closed are the empty set and the subspace itself.*

Theorem *A topological space is disconnected if and only if it is a disjoint union of two nonempty open sets.*

Lemma *Let* X *be a set and* A, B *nonempty subsets of* X. *Then* X *is a disjoint union of* A *and* B *if and only if* $B = A^c$ *(and* $A = B^c$*).*

Theorem *The continuous image of a connected is connected.*

Remark *Here by "continuous image" we mean the image by a continuous function, and to say that the image is connected means that it is a connected topological space when considered as a subspace of the codomain.*

Corollary *A quotient space of a connected space is connected.*

Definition *A property of a topological space is a* ***topological property*** *if and only if it is preserved by homeomorphisms, i.e. homeomorphic spaces either both have the property or both do not have the property.*

Corollary *Connectedness is a topological property.*

Lemma *Let* $Y = \{0,1\}$ *and* τ' *the discrete topology on* Y. *Then* (X,τ) *is connected if and* *only if the only continuous map $f:(X,\tau)\to(Y,\tau')$ is a constant map.*

Theorem *If $(X,\tau),(Y,\tau')$ are connected, then so is $X\times Y$ with the product topology.*

Theorem *In general, the product of connected spaces is connected.*

Applications of Connectedness

Connected Subsets of $\mathbb{R}$

Definition *A subset S of $\mathbb{R}$ is an* ***interval*** *if and only if whenever $a,b\in S$ and $a\le c\le b$ then $c\in S$, i.e. an interval is a set which contains all of the points between any two of its points.*

Theorem *The only connected subsets of $\mathbb{R}$ are intervals.*

Intermediate Value Theorem

Theorem *Let $f:[a\ldots b]\to\mathbb{R}$ be continuous and L any number between $f(a)$ and $f(b)$ inclusive. Then there exists $c\in[a\ldots b]$ such that $f(c)=L$.*

Corollary *If $f:[a\ldots b]\to\mathbb{R}$ is continuous and changes signs in the interval $[a\ldots b]$ then f has a root in $[a\ldots b]$.*

Fixed point theorems

Definition *Let $f:X\to X$ and $a\in X$. Then a is called a* ***fixed point*** *of f if and only if $f(a)=a$.*

Definition *A topological space has the* ***fixed point property*** *if and only if every continuous map from the space to itself has a fixed point.*

Theorem *The fixed point property is a topological property.*

Theorem *The n-disk $D_n=\{z\in\mathbb{R}^n : |z|\le 1\}$ has the fixed point property.*

Example *When $n=1$ this is just a corollary of the intermediate value theorem.*

Theorem *(Borsuk-Ulam) For every continous map $f:S^n\to\mathbb{R}^n$ there exist antipodal points $z,-z\in S^n$ such that $f(z)=f(-z)$.*

Theorem *(Ham Sandwich) Any three subsets of $\mathbb{R}^3$ having finite volume in $\mathbb{R}^3$ can be simultaneously bisected by a single plane.*

Components and Local Connectedness

Connected Components

Definition *Let (X,τ) be a topological space and $a\in X$. Define $\mathrm{Cmp}(a)$ to be the union of all connected subsets of X which contain a, i.e. $\mathrm{Cmp}(a)=\bigcup_{U\in\mathcal{P}}U$ where $\mathcal{P}=\left\{U\subseteq X : a\in U \text{ and } U \text{ is connected}\right\}$. The set $\mathrm{Cmp}(a)$ is called the* ***connected component*** *of X containing a.*

Theorem $\mathrm{Cmp}(a)$ *is connected.*

Remark *In other words,* $\mathrm{Cmp}(a)$ *is the largest connected subset of* X *containing* a.

Lemma $a \in \mathrm{Cmp}(a)$

Lemma $b \in \mathrm{Cmp}(a)$ *if and only if* $\mathrm{Cmp}(b) = \mathrm{Cmp}(a)$.

Theorem *Let* (X, τ) *be a topological space and define* $\sim$ *on* X *by* $a \sim b \Leftrightarrow b \in \mathrm{Cmp}(a)$. *Then* $\sim$ *is an equivalence relation on* X.

Theorem *If* A *is connected then so is* $\overline{A}$.

Theorem *Every connected component of a topological space is closed.*

Remark *But they are not all open!*

Local Connectedness

Definition *A topological space* (X, τ) *is* ***locally connected at*** $a \in X$ *if every neighborhood of* a *contains a connected neighborhood of* a. *The space* X *is* ***locally connected*** *if it is locally connected at every point.*

Theorem *Local connectedness is a topological property.*

(proof is a homework problem)

Theorem *If* (X, τ) *is locally connected then every connected component is open.*

Remark *Is a locally connected space necessarily connected?*

Remark *Is a connected space necessarily locally connected?*

Path Connectedness

Definition *Let* (X, τ) *be a topological space. A continous function* $f : [0..1] \to X$ *is called a* ***path*** *in* X. *The points* $f(0)$ *and* $f(1)$ *are called the* ***initial*** *and* ***terminal*** *points, respectively, of the path.*

Remark *We say that such a path* f ***connects*** *or* ***joins*** *its initial point to its terminal point, or that it is* ***a path from*** *its initial point to its terminal point, or that it is* ***a path between*** *its initial point and its terminal point.*

Definition *A path* f *is called a* ***loop*** *if* $f(0) = f(1)$.

Definition *A topological space is* ***path connected*** *if and only if there exists a path connecting any two of its points.*

Remark *A subspace of a space is path connected if and only if it is path connected as a topological space with the subspace topology.*

Theorem *The continous image of a path connected space is path connected.*

Corollary *Path connectedness is a topological property.*

Corollary *Any quotient space of a path connected space is path connected.*

Theorem *Every path connected space is connected.*

Categories

The Grand Unified Theory of Mathematics!

Definition *A **category** consists of*

*1. a collection of **objects** in the category*

2. for each ordered pair (X,Y) of objects in the category a set $\mathrm{Hom}(X,Y)$

3. there is a rule called $\circ$ which associates to each $f \in \mathrm{Hom}(X,Y)$ and $g \in \mathrm{Hom}(Y,Z)$ an element $g \circ f \in \mathrm{Hom}(X,Z)$

4. $\circ$ is associative

5. for each object X there is an element $1_X \in \mathrm{Hom}(X,X)$

6. for all $f \in \mathrm{Hom}(X,Y)$, $f \circ 1_X = f$ and for all $g \in \mathrm{Hom}(Y,X)$, $1_X \circ g = g$

Definition *In the previous definition, the elements of $\mathrm{Hom}(X,Y)$ are called **maps** (or **morphisms**) from X to Y. The map 1_X is called the **identity map** on X. The operator $\circ$ is called **composition**.*

Remark *These definitions of the terms map, identity map, and composition are new definitions that are unrelated to the definitions given previously for functions between sets. In particular, maps in a category do not have to be ordinary functions, nor do the objects have to be ordinary sets.*

Examples: Most branches of mathematics are examples of categories!

Subject	Objects	Maps
Set Theory	sets	functions
Topology	topological spaces	continuous functions
Metric Space	metric spaces	continuous functions
Linear Algebra	vector spaces	linear transformations
Group Theory	groups	group homomorphisms
Ring Theory	rings	ring homomorphisms
Geometry	underlying space	geometric transformations
Analysis	real numbers	differentiable functions

For those of you who haven't had group theory yet:

Definition *A **group** is a pair $(G,\bullet)$ where G is a set and $\bullet : G \times G \to G$ such that*

1. $\bullet$ is associative

2. there exists $e \in G$ such that for all $g \in G$, $g \bullet e = e \bullet g = g$

3. for all $g \in G$ there exists $h \in G$ such that $g \bullet h = h \bullet g = e$

Remark *e is called the **identity element** of the group, and h is called the **inverse** of g.*

Definition *A **group homomorphism** is a map $f : (G,\bullet) \to (X,*)$ such that for all $g,h \in G$, $f(g \bullet h) = f(g) * f(h)$.*

Example: A single group itself is an entire category if we define $\operatorname{Hom}(G,G)$ to be the elements of G and $\circ$ to be the group operation.

Example: Let the integers in $\mathbb{I}_{12} = \{1,2,\ldots,12\}$ be the objects and for each $A,B \in \mathbb{I}_{12}$ define $\operatorname{Hom}(A,B) = \{(A,B)\}$ if $A \mid B$ and $\emptyset$ otherwise. How can we define composition to turn this into a category? What is 1_5?

Example:

Theorem *In any category if f has a left inverse g and a right inverse g' then $g = g'$.*

Functors

Definition *Let $\mathcal{C},\mathcal{C}'$ be categories and $\mathcal{A},\mathcal{A}'$ their respective collections of objects. A **covariant functor**, $F : \mathcal{C} \to \mathcal{C}'$ is a pair of functions F_1,F_2 such that*

1. $F_1 : \mathcal{A} \to \mathcal{A}'$
2. *for each* $X,Y \in \mathcal{A}$, $F_2 : \operatorname{Hom}(X,Y) \to \operatorname{Hom}(F_1(X),F_1(Y))$ *such that*
   (a). $F_2(1_x) = 1_{F_1(X)}$
   (b). $F_2(g \circ f) = F_2(g) \circ F_2(f)$ *for all* $f \in \operatorname{Hom}(X,Y)$ *and* $g \in \operatorname{Hom}(Y,Z)$

Example *The forgetful functor from $\mathcal{C}_{Top}$ to $\mathcal{C}_{Set}$.*

Example *The associated space functor from $\mathcal{C}_{Met}$ to $\mathcal{C}_{Top}$.*

Homotopy

Definition *Let (X,τ) be a topological space and f,g paths from a to b in X. A **homotopy** between f and g is a continous function $H : [0\ldots1] \times [0\ldots1] \to X$ such that for all $x,t \in [0\ldots1]$*

1. $H(x,0) = f(x)$
2. $H(x,1) = g(x)$
3. $H(0,t) = a$
4. $H(1,t) = b$

*If there exists a homotopy between f and g we say the paths f and g are **homotopic**.*

Definition *Define a relation on the set of paths from a to b in a topological space (X,τ) by $f \cong g$ if and only if f and g are homotopic.*

Theorem *$\cong$ is an equivalence relation.*

Lemma *Let $(X,\tau),(Y,\tau')$ be a topologial spaces and A,B closed subsets of X. Let $f : A \to Y$ and $g : B \to Y$ be continous maps which agree on $A \cap B$. Then the map $h : A \cup B \to Y$ by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{otherwise} \end{cases}$$

is continous.

Remark *As usual, we will denot the equivalence class of a path f as $[f]$.*

The Fundamental Group

Definition *Let (X,τ) be a topological space and $a \in X$. The set of all equivalence classes of paths from a to a (i.e. loops) in X is denoted $\pi(X,a)$.*

Definition *Let (X,τ) be a topological space and f,g paths from a to a in X. The **product** (or **concatenation**) of f and g is the path $f \bullet g$ from a to a in X defined by*

$$f \bullet g(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} < t \leq 1 \end{cases}$$

Theorem *Let (X,τ) be a topologial spaces f,g,f',g' paths from a to a in X. If $f \cong f'$ and $g \cong g'$ then $f \bullet g \cong f' \bullet g'$.*

Definition *Let (X,τ) be a topologial spaces f,g paths from a to a in X. Define a product $\bullet : \pi(X,a) \times \pi(X,a) \to \pi(X,a)$ by $[f] \bullet [g] = [f \bullet g]$.*

Theorem *$(\pi(X,a),\bullet)$ is a group!*

Remark *$\pi(X,a)$ is often denoted $\pi_1(X,a)$. For path connected spaces, the same (isomorphic) group is obtained no matter what base point is selected, so for path connected spaces $\pi(X,a)$ is often abbreviated as $\pi(X)$ or $\pi_1(X)$.*

Theorem *π is a functor from the category of topological spaces with a point to the category of groups.*

Simple Connectedness

Definition *Any one element group is called a **trivial group**.*

Remark *All trivial groups are isomorphic. For example, they are all isomorphic to $(\{0\},+)$ where $+$ is the ordinary addition of integers.*

Definition *A topological space is **simply connected** if and only if its fundamental group is the trivial group at every base point.*

Remark *In other words every loop is homotopic to every other loop at the same point in a simply connected space.*

Theorem *A path connected topological space is simply connected if and only if its fundamental group is the trivial group at some base point.*

For the proof of this we require some notation.

Definition *If $f : [0..1] \to X$ is a path in topological space (X,τ) then $\overleftarrow{f}$ is the path $\overleftarrow{f} : [0..1] \to X$ by $\overleftarrow{f}(t) = f(1-t)$. We will call $\overleftarrow{f}$ the **reverse** of f.*

Remark *The book refers uses f^{-1} to represent $\overleftarrow{f}$, because, hey, you just can't have too many completely different simultaneous definitions for the symbol f^{-1}!*

Definition *If f is a path from a to b in topological space (X,τ), and g is a path from b to b*

then g_f is the path from a to a defined by

$$g_f(t) = \begin{cases} f(3t) & \text{if } 0 \le t \le \frac{1}{3} \\ g(3t-1) & \text{if } \frac{1}{3} \le t \le \frac{2}{3} \\ \overleftarrow{f}(3t-2) & \text{if } \frac{2}{3} \le t \le 1 \end{cases}$$

Definition *If f is a path from a to b in path connected topological space (X,τ), define $\alpha_f : \pi(X,b) \to \pi(X,a)$ by $\alpha_f([g]) = [g_f]$.*

Theorem *α_f is a group isomorphism.*

Compactness

Covers

Definition *Let S be a subset of a set X. An indexed family of sets $\{A_i\}_{i\in I}$ is a **cover** of S if and only if $S \subseteq \cup_{i\in I} A_i$. If I is finite then this cover is said to be a **finite cover** of S. If (X,τ) is a topological space and A_i is an open set for all $i \in I$ then this cover is said to be an **open cover**.*

Definition *A cover $\{B_j\}_{j\in J}$ of S is a **subcover** of $\{A_i\}_{i\in I}$ if and only if $\{B_j : j \in J\} \subseteq \{A_i : i \in I\}$. We say $\{A_i\}_{i\in I}$ **contains** the subcover $\{B_j\}_{j\in J}$ if $J \subseteq I$.*

Definition of Compactness

Definition *A topological space is said to be **compact** if and only if every open cover contains a finite subcover.*

Remark *A subset of a topological space is said compact if it is a compact topological space with the subspace topology. The following shows that for subsets of a topological space we can consider open covers in the larger space instead of those in the subset itself (i.e. an open cover vs a relatively open cover).*

Theorem *A subset S of a topological space is compact if and only if every open cover of S with open sets of X contains a subcover of S with open sets of X.*

Continuity and Compactness

Theorem *The continuous image of a compact set is compact.*

Corollary *Compactness is a topological property.*

Characterizing Compactness

Theorem *A closed subset of a compact space is compact.*

Theorem *Every compact subset of a Hausdorff space is closed.*

Corollary *In a compact Hausdorff space, a subset is compact if and only if it is closed.*

The Heine-Borel Theorem

Definition *A subset of $\mathbb{R}^n$ is **bounded** if and only if it is a subset of some closed ball* *centered at the origin.*

Theorem *A compact subset of $\mathbb{R}^n$ is closed and bounded.*

Theorem *The unit interval $[0\ldots 1]$ is compact.*

Corollary *The closed interval $[a\ldots b]$ is compact.*

Theorem *(Heine-Borel) A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.*

Products of Compact Spaces

Lemma *Let (X,τ) be a topological space, $\mathcal{B}$ a basis for τ, and $S \subseteq X$. If every open cover of S with elements of $\mathcal{B}$ contains a finite subcover, then S is compact.*

Theorem *If (X,τ), (Y,τ') are compact then so is $X \times Y$ (with the product topology).*

Corollary *If $(X_1,\tau_1),(X_2,\tau_2),\ldots(X_n,\tau_n)$ are compact then so is $X_1 \times X_2 \times \cdots \times X_n$ (with the product topology).*

Corollary *The n-dimensional unit hypercube, $[0\ldots 1]^n$ is compact.*

Corollary *(n-dimensional Heine-Borel) A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proofs

Theorem (Fundamental Theorem of Equivalence Relations) *Let $R \subseteq X \times X$ be an equivalence relation and $a,b \in X$. Then*

$$[a] = [b] \Leftrightarrow aRb.$$

Pf.

1.	Let $R \subseteq X \times X$ be an equivalence relation	Given
2.	Let $a,b \in X$	Given
	$\star$ $(\Rightarrow)$	
3.	Assume $[a] = [b]$	-
4.	aRa	reflexive;1,2
5.	$a \in [a]$	def of $[\ \ ]$
6.	$a \in [b]$	substitution;3,5
7.	aRb	def of $[\ \ ]$
8.	$\leftarrow$	-
9.	$[a] = [b] \Rightarrow aRb$	$\Rightarrow +$;3,7,8
	$\star$ $(\Leftarrow)$	
10.	Assume aRb	-
11.	Let $x \in [a]$	-
12.	xRa	def of $[\ \ ]$
13.	xRb	transitivity;1,10,12

14. $x \in [b]$ def of [ ]
15. $[a] \subseteq [b]$ def of $\subseteq$
16. Let $y \in [b]$ -
17. yRb def of [ ]
18. bRa symmetry;1,10
19. yRa transitivity;1,17,18
20. $y \in [a]$ def of [ ]
21. $[b] \subseteq [a]$ def of $\subseteq$
22. $[a] = [b]$ def set $=$
23. $\leftarrow$ -
24. $aRb \Rightarrow [a] = [b]$ $\Rightarrow$ +;
25. $[a] = [b] \Leftrightarrow aRb$ $\Leftrightarrow$ +;

QED

Corollary (1) *Let $R \subseteq X \times X$ be an equivalence relation. Then X is a disjoint union of equivalence classes, i.e.*

$$X = \bigcup_{a \in X} [a]$$

and

$$\forall a, b \in X, [a] = [b] \text{ or } [a] \cap [b] = \emptyset.$$

Pf

1. Let $R \subseteq X \times X$ be an equivalence relation. Given

   ⋆ *show* $X \subseteq \bigcup_{a \in X} [a]$

2. Let $x \in X$
3. xRx reflexive;1,2
4. $x \in [x]$ def of [ ]
5. $x \in [\alpha]$ for some $\alpha \in X$ $\exists$ +;2,4
6. $x \in \bigcup_{a \in X} [a]$ def indexed $\cup$

   ⋆ *show* $\bigcup_{a \in X} [a] \subseteq X$

7. Let $y \in \bigcup_{a \in X} [a]$
8. $y \in [\beta]$ for some $\beta \in X$ def indexed $\cup$
9. yRb def of [ ]
10. $y \in X$ def equiv reln;1,9

    ⋆ *conclude the sets are equal*

11. $X = \bigcup_{a \in X} [a]$ def set $=$;2,6,7,10

    ⋆ *now show* $\forall a, b \in X, [a] = [b]$ or $[a] \cap [b] = \emptyset$

12. Let $a, b \in X$ -

13. aRb or not aRb P or $\sim P$ tautology

⋆ *Case 1:*

14. Assume aRb -

15. $[a] = [b]$ Fund Thm of Equiv Relns;1,12,14

16. $[a] = [b]$ or $[a] \cap [b] = \emptyset$ or +

17. $\leftarrow$ -

⋆ *Case 2:*

18. Assume not aRb -

19. Assume $[a] \cap [b] \neq \emptyset$ -

20. $t \in [a] \cap [b]$ for some t def $\emptyset$

21. $t \in [a]$ and $t \in [b]$ def $\cap$

22. tRa and tRb def $[\ \ ]$

23. aRt symmetry;1,22

24. aRb transitivity;1,22,23

25. $\rightarrow\leftarrow$ $\rightarrow\leftarrow$ +

26. $\leftarrow$

27. $[a] \cap [b] = \emptyset$ pf by contradiction;19,25,26

28. $[a] = [b]$ or $[a] \cap [b] = \emptyset$ or +

29. $\leftarrow$

30. $[a] = [b]$ or $[a] \cap [b] = \emptyset$ pf by cases;13,14,16,18,28

31. $\forall a, b \in X, [a] = [b]$ or $[a] \cap [b] = \emptyset$ $\forall$ +;12,30

QED

Theorem *Every projection map is onto.*

Pf.

1. Let $R \subseteq X \times X$ be an equivalence relation.
2. Let $\pi : X \to X/R$ be the projection map
3. $\forall x \in X, \pi(x) = [x]$ def of projection map
4. Let $q \in X/R$
5. $q = [a]$ for some $a \in X$ def of quotient set
6. $\quad = \pi(a)$ $\forall$ −;3
7. π is onto def of onto
8. Every projection map is onto $\forall$ +;1,2,7

QED

Theorem *Composition of functions is associative.*

Pf.

1. Let $f : A \to B$ and $g : B \to C$ and $h : C \to D$

2. $\text{Domain}((h \circ g) \circ f) = \text{Domain}(f)$ def of $\circ$
3. $= \text{Domain}(g \circ f)$ def of $\circ$
4. $= \text{Domain}(h \circ (g \circ f))$ def of $\circ$
5. $\text{Codomain}((h \circ g) \circ f) = \text{Codomain}(h \circ g)$ def of $\circ$
6. $= \text{Codomain}(h)$ def of $\circ$
7. $= \text{Codomain}(h \circ (g \circ f))$ def of $\circ$
8. Let $x \in A$
9. $((h \circ g) \circ f)(x) = (h \circ g)(f(x))$ def of $\circ$
10. $= h(g(f(x)))$ def of $\circ$
11. $= h((g \circ f)(x))$ def of $\circ$
12. $= (h \circ (g \circ f))(x)$ def of $\circ$
13. $(h \circ g) \circ f = h \circ (g \circ f)$ def function $=$;2-4,5-7,8,9-12
14. Composition of functions is associative. $\forall +$

QED

Theorem (*right cancellation law for surjective functions*) *Let $X \xrightarrow{f} Y$ and $|Z| > 1$. Then f is surjective if and only if for all functions $g, h : Y \to Z$*

$$(g \circ f = h \circ f) \Rightarrow g = h$$

Pf.

1. Let $X \xrightarrow{f} Y$ and $|Z| > 1$ Given

$\star$ $(\Rightarrow)$

2. Assume f is surjective
3. Let $g, h : Y \to Z$
4. Assume $g \circ f = h \circ f$
5. Let $y \in Y$
6. $y = f(x)$ for some $x \in X$ def surjective;1,2,5
7. $g(y) = g(f(x))$ substitution;6
8. $= (g \circ f)(x)$ def $\circ$
9. $= (h \circ f)(x)$ substitution;4
10. $= h(f(x))$ def $\circ$
11. $= h(y)$ substitution;6
12. $g = h$ def function $=$;3,5,7-11
13. $\leftarrow$
14. $(g \circ f = h \circ f) \Rightarrow g = h$ $\Rightarrow +$;4,12,13
15. $\forall g, h : Y \to Z, (g \circ f = h \circ f) \Rightarrow g = h$ $\forall +$;3,14
16. $\leftarrow$
17. f is surjective $\Rightarrow \forall g, h : Y \to Z, (g \circ f = h \circ f) \Rightarrow g = h$ $\Rightarrow +$;2,15

$\star$ $(\Leftarrow)$

18. Assume $(\forall g,h : Y \to Z, (g \circ f = h \circ f) \Rightarrow g = h)$
19. Let $s \in Y$
20. Assume $\sim\exists t \in X, f(t) = s$
21. $\forall t \in X, f(t) \neq s$ DeMorgan
22. Let $g : Y \to Z$ be any function
23. $u \neq g(s)$ for some $u \in Z$ def cardinality;1
24. Define $h : Y \to Z$ by $\forall y \in Y, h(y) = \begin{cases} g(y) & \text{if } y \neq s \\ u & \text{if } y = s \end{cases}$
25. $h(s) = u$ def of h
26. $\neq g(s)$ copy;23
27. $h \neq g$ def function =;25,26
28. $g \circ f : X \to Z$ and $h \circ f : X \to Z$ def $\circ$
29. Let $r \in X$
30. $f(r) \neq s$ $\forall -$;21
31. $(g \circ f)(r) = g(f(r))$ def $\circ$
32. $= h(f(r))$ def of h
33. $= (h \circ f)(r)$ def $\circ$
34. $g \circ f = h \circ f$ def function =;28,31-33
35. $(g \circ f = h \circ f) \Rightarrow g = h$ $\forall -$;18
36. $g = h$ modus ponens
37. $\rightarrow\leftarrow$ $\rightarrow\leftarrow +$;27,36
38. $\leftarrow$
39. $\exists t \in X, f(t) = s$ $\sim -$;20,37
40. f is surjective def surjective;19,39
41. $\leftarrow$
42. $(\forall g,h : Y \to Z, (g \circ f = h \circ f) \Rightarrow g = h) \Rightarrow f$ is surjective $\Rightarrow +$;18,40
43. f is surjective $\Leftrightarrow \forall g,h : Y \to Z, (g \circ f = h \circ f) \Rightarrow g = h$ $\Leftrightarrow +$;

QED

Theorem *A function has an inverse function if and only if it is bijective.*

Pf.

1. Let $f : X \to Y$

   ⋆ $(\Rightarrow)$

2. Assume f has an inverse
3. $\exists g : Y \to X,\ g \circ f = id_X$ and $f \circ g = id_Y$ def inverse function
4. $g : Y \to X$ and $g \circ f = id_X$ and $f \circ g = id_Y$ for some g $\exists -$

   ⋆ *show it is injective*

5. Let $x,y \in X$

6. Assume $f(x) = f(y)$

7. $x = id_X(x)$ — def identity map

8. $= (g \circ f)(x)$ — substitution;4

9. $= g(f(x))$ — def $\circ$

10. $= g(f(y))$ — plug in;6

11. $= (g \circ f)(y)$ — def $\circ$

12. $= id_X(y)$ — substitution;4

13. $= y$ — def identity map

14. $\leftarrow$

15. f is one to one — def one to one;5,6,7-13

⋆ *show it is onto*

16. Let $z \in Y$

17. $g(z) \in X$ — def function;4,16

18. Define $q = g(z)$

19. $f(q) = f(g(z))$ — substitution

20. $= (f \circ g)(z)$ — def $\circ$

21. $= id_Y(z)$ — substitution;4

22. $= z$ — def identity map

23. $\exists q \in X, f(q) = z$ — $\exists$ +;17,19-22

24. f is onto — def onto;16,23

⋆ *so it is bijective*

25. f is bijective — def bijective;15,24

26. $\leftarrow$

27. f has an inverse $\Rightarrow$ f is bijective — $\Rightarrow$ +;2,25

⋆ ($\Leftarrow$)

28. Assume f is bijective

29. f is one to one — def bijective

30. f is onto — def bijective;28

⋆ *it is easier to prove that a relation is a function than to try*
⋆ *to make an inverse function directly, so we switch to ordered pair*
⋆ *notation.*

31. $f \subseteq X \times Y$ — def function;1

32. $\forall x,y \in X, \forall z \in Y, (x,z) \in f$ and $(y,z) \in f \Rightarrow x = y$ — def one to one;29

33. $\forall z \in Y, \exists x \in X, (x,z) \in f$ — def onto;30

⋆ *we define g to be the set of ordered pairs in f with the*
⋆ *coordinates reversed*

34. Define $g = \{(z,x) : (x,z) \in f\}$

⋆ *first we prove g is a function*

⋆ *show its a relation*

35. Let $w \in g$

36.	$w = (z,x)$ and $(x,z) \in f$ for some $x \in X$ and $z \in Y$	def g,f;31,34
37.	$w \in Y \times X$	def $\times$
38.	$g \subseteq Y \times X$	def $\subseteq$;35,37
	⋆ *show it maps everything in the domain to something*	
39.	Let $t \in Y$	
40.	$\exists x \in X, (x,t) \in f$	$\forall -$;33
41.	$(s,t) \in f$ for some $s \in X$	$\exists -$
42.	$(t,s) \in g$	def g;34
43.	$\forall t \in Y, \exists s \in X, (t,s) \in g$	$\forall +, \exists +$;39,42
	⋆ *show that it doesn't map anything to two different places*	
44.	Let $u,v \in X$	
45.	Assume $(t,u) \in g$ and $(t,v) \in g$	
46.	$(u,t) \in f$ and $(v,t) \in f$	def g;34
47.	$u = v$	$\forall -, \Rightarrow -$;32
48.	$\leftarrow$	
49.	$\forall t \in Y, \forall u,v \in X, (t,u) \in g$ and $(t,v) \in g \Rightarrow u = v$	$\Rightarrow +, \forall +$;45,47,44,39
	⋆ *so it's a function*	
50.	$g : Y \to X$	def function;38,43,49
51.	$f \circ g : Y \to Y$ and $g \circ f : X \to X$	def $\circ$;1,50
52.	$id_Y : Y \to Y$ and $id_X : X \to X$	def identity map
	⋆ *now that we know g is a function we can return to*	
	⋆ *using function notation to show it's f^{-1}*	
53.	$(f \circ g)(t) = f(g(t))$	def $\circ$
54.	$= f(s)$	def $f(x)$ notation;42
55.	$= t$	def $f(x)$ notation;41
56.	$= id_Y(t)$	def identity map;39
57.	$f \circ g = id_Y$	def function $=$;51,52,39,53-56
58.	$(u, f(u)) \in f$	def $f(x)$ notation
59.	$(f(u), u) \in g$	def g;34
60.	$g(f(u)) = u$	def $f(x)$ notation
61.	$(g \circ f)(u) = g(f(u))$	def $\circ$
62.	$= u$	substitution;60
63.	$= id_X(u)$	def identity map;44
64.	$g \circ f = id_X$	def function $=$;51,52,44,61-63
65.	$\exists g : Y \to X, g \circ f = id_X$ and $f \circ g = id_Y$	and $+, \exists +$;57,64
66.	f has an inverse	def inverse function
67.	$\leftarrow$	
68.	f is bijective $\Rightarrow f$ has an inverse	$\Rightarrow +$;28,66
69.	f has an inverse $\Leftrightarrow f$ is bijective	$\Leftrightarrow +$;27,68

QED

Theorem *The product metric is a metric.*

Pf

1. Let $(X_1,d_1),(X_2,d_2),\ldots,(X_n,d_n)$ be metric spaces and $X = X_1 \times X_2 \times \cdots \times X_n$.
2. Define $d_{\max} : X \times X \to \mathbb{R}$ by

$$d_{\max}((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \max\{d_i(x_i,y_i) : i \in \{1,\ldots,n\}\}$$

   where $x = (x_1,x_2,\ldots,x_n)$ and $y = (y_1,y_2,\ldots,y_n)$
3. Let $x,y,z \in X$
4. $x = (x_1,x_2,\ldots,x_n)$ for some $x_1 \in X_1$ and $\cdots$ and $x_n \in X_n$ — def $\times$
5. $y = (y_1,y_2,\ldots,y_n)$ for some $y_1 \in X_1$ and $\cdots$ and $y_n \in X_n$ — def $\times$
6. $z = (z_1,z_2,\ldots,z_n)$ for some $z_1 \in X_1$ and $\cdots$ and $z_n \in X_n$ — def $\times$

   ⋆ *show it's nonnegative*

7. $d_{\max}(x,y) = d_{\max}((x_1,x_2,\ldots,x_n),(y_1,y_2,\ldots,y_n))$ — substitution
8. $= \max\{d_i(x_i,y_i) : i \in \{1,\ldots,n\}\}$ — def $d_{\max}$;2
9. $= d_k(x_k,y_k)$ for some $k \in \mathbb{I}_n$ — def max
10. ≥ 0 — def metric;1

    ⋆ *show it's symmetric*

    ⋆ *first show that $d_k(y_k,x_k) = \{d_i(y_i,x_i) : i \in \{1,\ldots,n\}\}$*

11. $d_k(y_k,x_k) \in \{d_i(y_i,x_i) : i \in \{1,\ldots,n\}\}$ — set builder
12. Let $\alpha \in \{d_i(y_i,x_i) : i \in \{1,\ldots,n\}\}$
13. $\alpha = d_j(y_j,x_j)$ for some $j \in \{1,\ldots,n\}$
14. $d_j(y_j,x_j) = d_j(x_j,y_j)$ — def metric
15. $\leq d_k(x_k,y_k)$ — def max;8-9
16. $= d_k(y_k,x_k)$ — def metric;1
17. $d_k(y_k,x_k) = \max\{d_i(y_i,x_i) : i \in \{1,\ldots,n\}\}$ — def max;11,12,14-16
18. $d_{\max}(x,y) = d_k(x_k,y_k)$ — lines 7-9
19. $= d_k(y_k,x_k)$ — def metric;1
20. $= \max\{d_i(y_i,x_i) : i \in \{1,\ldots,n\}\}$ — substitution;17
21. $= d_{\max}(y,x)$ — def $d_{\max}$;2

    ⋆ *prove the triangle inequality*

22. $d_{\max}(x,z) = d_{\max}((x_1,x_2,\ldots,x_n),(z_1,z_2,\ldots,z_n))$ — substitution
23. $= \max\{d_i(x_i,z_i) : i \in \{1,\ldots,n\}\}$ — def $d_{\max}$;2
24. $= d_l(x_l,z_l)$ for some $l \in \mathbb{I}_n$ — def max
25. $d_{\max}(y,z) = d_{\max}((y_1,y_2,\ldots,y_n),(z_1,z_2,\ldots,z_n))$ — substitution
26. $= \max\{d_i(y_i,z_i) : i \in \{1,\ldots,n\}\}$ — def $d_{\max}$;2
27. $= d_m(y_m,z_m)$ for some $m \in \mathbb{I}_n$ — def max
28. $d_{\max}(x,y) + d_{\max}(y,z) = d_k(x_k,y_k) + d_m(y_m,z_m)$ — substitution;7-9,25-27
29. $\geq d_l(x_l,y_l) + d_l(y_l,z_l)$ — def max;8-9,26-27

30. $\geq d_l(x_l,z_l)$ def metric;1

31. $= d_{\max}(x,z)$ substitution;22-24

$\star$ *show* $d(x,x) = 0$

32. $d_{\max}(x,x) = d_{\max}((x_1,x_2,\ldots,x_n),(x_1,x_2,\ldots,x_n))$ substitution

33. $= \max\{d_i(x_i,x_i) : i \in \{1,\ldots,n\}\}$ def $d_{\max}$;2

34. $= \max\{0 : i \in \{1,\ldots,n\}\}$ def metric;1

35. $= 0$ def max

$\star$ *show* $d(x,y) = 0 \Rightarrow x = y$

36. Assume $d_{\max}(x,y) = 0$

37. $d_k(x_k,y_k) = 0$ substitution;7-9

38. Let $i \in \mathbb{I}_n$

39. $0 \leq d_i(x_i,y_i)$ def metric;1

40. $\leq d_k(x_k,y_k)$ def max;8-9

41. $= 0$ substitution;37

42. $d_i(x_i,y_i) = 0$ arithmetic;39-41

43. $x_i = y_i$ def metric;1

44. $\forall i \in \mathbb{N}, x_i = y_i$ $\forall$ +;38,43

45. $(x_1,x_2,\ldots,x_n) = (y_1,y_2,\ldots,y_n)$ def n-tuple

46. $x = y$ substitution;4,5

47. $\leftarrow$

48. $d_{\max}$ is a metric def metric;2,3,7-10,18-21,28-31,32-35,36,46

QED

Note: as we make the transition from semi-formal to informal word-wrapped style proofs we will slowly add additional shortcuts to our proofs. One common shortcut is that in most word wrapped textbook style proofs they do not name the specific rules of logic used for dealing with the five propositional operators and the two quantifiers. Instead they either just say "Hence" or "Thus" or "So" or "Therefore" or "It follows that" as a catch-all phrase to cover all logical rules of inference. Another way they get around that is to say "by (2)" to indicate that the statement they just gave followed from some rule of logic using the line labeled (2) as an input. This is the style we will use in the next proof.

Theorem *Suppose* $f : (X,d) \to (Y,d')$ *is continuous at* $a \in X$ *and* $g : (Y,d') \to (Z,d'')$ *is continuous at* $f(a)$. *Then* $g \circ f : (X,d) \to (Z,d'')$ *is continuous at* a.

Pf.

1. $f : (X,d) \to (Y,d')$ is continuous at $a \in X$ Given
2. $g : (Y,d') \to (Z,d'')$ is continuous at $f(a)$ Given
3. $\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in X, d(x,a) < \delta \Rightarrow d'(f(x),f(a)) < \varepsilon$ def continuous;1
4. $\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall y \in Y, d'(y,f(a)) < \delta \Rightarrow d''(g(y),g(f(a))) < \varepsilon$ def continuous;2
5. Let $\varepsilon \in \mathbb{R}^+$

6. $\exists \delta \in \mathbb{R}^+, \forall y \in Y, d'(y,f(a)) < \delta \Rightarrow d''(g(y),g(f(a))) < \varepsilon$ by (4)
7. $\forall y \in Y, d'(y,f(a)) < \delta_1 \Rightarrow d''(g(y),g(f(a))) < \varepsilon$ for some $\delta_1 \in \mathbb{R}^+$ by (6)
8. $\exists \delta \in \mathbb{R}^+, \forall x \in X, d(x,a) < \delta \Rightarrow d'(f(x),f(a)) < \delta_1$ by (3)
9. $\forall x \in X, d(x,a) < \delta_2 \Rightarrow d'(f(x),f(a)) < \delta_1$ for some $\delta_2 \in \mathbb{R}^+$ by (8)
10. Define $\delta = \delta_2$
11. $\delta \in \mathbb{R}^+$ substitution;10,9
12. Let $x \in X$
13. Assume $d(x,a) < \delta$
14. $= \delta_2$ substitution;10
15. $d'(f(x),f(a)) < \delta_1$ by (9),(13-14)
16. $d''(g(f(x)),g(f(a))) < \varepsilon$ by (7),(16)
17. $d''(g \circ f(x), g \circ f(a)) < \varepsilon$ def $\circ$
18. $\leftarrow$
19. $g \circ f : (X,d) \to (Z,d'')$ is continuous at a def of continuous;5,11,12,13,17

QED

In the following proof we are only numbering lines that are referred to specifically in the reason of some future statement rather than numbering every line in the proof. This is similar to the way proofs in textbooks and articles are numbered... only essential lines that need to be referred to later on in the proof are given equation or line numbers. Because of the lack of line numbers, instead of using the abbreviation "by (n)" for reasons that are rules of logic, we are just giving the name of the rule of logic with no line numbers, the hope being that the reader can determine what lines satisfy the inputs. This is the next step in making a proof that is more like the word wrapped informal proofs found in your book.

Theorem *Let (X,d) and (Y,d') be metric spaces and $f : X \to Y$. Then f is* ***continuous*** *with respect to the metrics d and d' if and only if*

$$\forall U \subseteq Y, U \text{ is open in } (Y,d') \Rightarrow f^{-1}(U) \text{ is open in } (X,d).$$

Pf.

1. Let (X,d) and (Y,d') be metric spaces and $f : X \to Y$.

$\star$ ($\Rightarrow$)

2. Assume f is continuous

Let $U \subseteq Y$

Assume U is open in (Y,d')

Let $a \in f^{-1}(U)$

$f(a) \in U$ def of inverse image

U is a neighborhood of $f(a)$ def of open

$\exists \varepsilon \in \mathbb{R}^+, B(f(a);\varepsilon) \subseteq U$ def of neighborhood

$B(f(a);\varepsilon) \subseteq U$ for some $\varepsilon \in \mathbb{R}^+$

$\exists \delta \in \mathbb{R}^+, \forall x \in X, d(x,a) < \delta \Rightarrow d'(f(x),f(a)) < \varepsilon$ def continuous;1

3. $\forall x \in X, d(x,a) < \delta \Rightarrow d'(f(x),f(a)) < \varepsilon$ for some $\delta \in \mathbb{R}^+$

4. Let $y \in B(a;\delta)$

$y \in X$ and $d(y,a) < \delta$ — def of open ball

$d'(f(y),f(a)) < \varepsilon$ — $\forall -, \Rightarrow -;2$

$f(y) \in B(f(a);\varepsilon)$ — def open ball

5. $y \in f^{-1}(B(f(a);\varepsilon))$ — def inverse image

$B(a;\delta) \subseteq f^{-1}(U)$ — def $\subseteq$;3-4

$\exists \delta \in \mathbb{R}^+, B(a;\delta) \subseteq f^{-1}(U)$ — $\exists +$

$f^{-1}(U) \in \mathcal{N}_a$ — def of neighborhood

$\forall a \in f^{-1}(U), f^{-1}(U) \in \mathcal{N}_a$ — $\forall +$

$f^{-1}(U)$ is open in (X,d) — def of open

$\leftarrow$

U is open in $(Y,d') \Rightarrow f^{-1}(U)$ is open in (X,d) — $\Rightarrow +$

$\forall U \subseteq Y, U$ is open in $(Y,d') \Rightarrow f^{-1}(U)$ is open in (X,d) — $\forall +$

$\star$ ($\Leftarrow$)

Assume $\forall U \subseteq Y, U$ is open in $(Y,d') \Rightarrow f^{-1}(U)$ is open in (X,d)

Let $b \in X$

Let $\varepsilon \in \mathbb{R}^+$

6. Define $\mathcal{U} = B(f(b);\varepsilon)$

$\mathcal{U} \subseteq Y$ — def of open ball

$\mathcal{U}$ is open in (Y,d') — Thm: open balls are open;5

7. $f^{-1}(\mathcal{U})$ is open in (X,d) — $\forall -; \Rightarrow -$

$f(b) \in \mathcal{U}$ — Lemma: every open ball contains its center;5

$b \in f^{-1}(\mathcal{U})$ — def inverse image

$f^{-1}(\mathcal{U})$ is a neighborhood of b — def open set;6

$\exists \delta \in \mathbb{R}^+, B(b;\delta) \subseteq f^{-1}(\mathcal{U})$ — def neighborhood

8. $B(b;\delta) \subseteq f^{-1}(\mathcal{U})$ for some $\delta \in \mathbb{R}^+$ — $\exists -$

Let $z \in X$

Assume $d(z,b) < \delta$

$z \in B(b;\delta)$ — def open ball

$z \in f^{-1}(\mathcal{U})$ — def $\subseteq$;7

$f(z) \in \mathcal{U}$ — def inverse image

$f(z) \in B(f(b);\varepsilon)$ — substitution;5

$d'(f(z),f(b)) < \varepsilon$ — def open ball

$\leftarrow$

$d(z,b) < \delta \Rightarrow d'(f(z),f(b)) < \varepsilon$ — $\Rightarrow +$

$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in X, d(x,b) < \delta \Rightarrow d'(f(x),f(b)) < \varepsilon$ — $\forall +, \exists +, \forall +$

f is continuous at b — def continuous at a point

$\forall x \in X, f$ is continuous at x — $\forall +$

f is continuous — def continuous

QED

Theorem *Let (X,d) be a metric space.*
1. The empty set is open.
2. The set X is open.
3. The union of any collection of open sets is open.
4. The intersection of finitely many open sets is open.

Pf

1. Let (X,d) be a metric space.
Let x be arbitrary

⋆ *Show (1)*

Assume $x \in \emptyset$	
$x \notin \emptyset$	by def of empty set
$\rightarrow\leftarrow$	$\rightarrow\leftarrow +$
$\emptyset$ is a neighborhood of x	Thm:$\rightarrow\leftarrow\Rightarrow$anything
$\leftarrow$	
$x \in \emptyset \Rightarrow \emptyset$ is a neighborhood of x	$\Rightarrow +$
$\forall x, x \in \emptyset \Rightarrow \emptyset$ is a neighborhood of x	$\forall +$
$\emptyset$ is open	def of open

⋆ *Show (2)*

Assume $x \in X$	
$B(x;\pi) \subseteq X$	def open ball
$\pi \in \mathbb{R}^+$	arithmetic
$\exists \delta \in \mathbb{R}^+, B(x;\delta) \subseteq X$	$\exists +$
X is a neighborhood of x	def neighborhood
$\leftarrow$	
$x \in X \Rightarrow X$ is a neighborhood of x	$\Rightarrow +$
$\forall x, x \in X \Rightarrow X$ is a neighborhood of x	$\forall +$
X is open	def of open

⋆ *Show (3)*

2. Let I be a set and $\{O_i\}_{i \in I}$ an indexed family of open subsets of X
Define $U = \cup_{i \in I} O_i$

Assume $x \in U$	
$x \in O_k$ for some $k \in I$	def union
O_k is open	by (1)
$\forall y \in O_k, O_k$ is a neighborhood of y	def open

O_k is a neighborhood of x — $\forall -$

$O_k \subseteq U$ — Exercise 1.4.1.a

U is a neighborhood of x — Thm N3

$\leftarrow$

$x \in U \Rightarrow U$ is a neighborhood of x — $\Rightarrow +$

$\forall x, x \in U \Rightarrow U$ is a neighborhood of x — $\forall +$

U is open — def of open

⋆ *Show (4)*

3. Let n be a positive integer and $V_1, V_2, \ldots, V_n$ by open subsets of X

Define $V = V_1 \cap V_2 \cap \ldots \cap V_n$

Assume $x \in V$

$\forall k \in \mathbb{I}_n, x \in V_k$ — def intersection

$\forall k \in \mathbb{I}_n, \exists \delta_k \in \mathbb{R}^+, B(x;\delta_k) \subseteq V_k$ — def open;2

$B(x;\delta_k) \subseteq V_k$ for some $\delta_1, \delta_2, \ldots, \delta_n \in \mathbb{R}^+$ — $\exists -$

Define $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_n\}$

Let $k \in \mathbb{I}_n$

Let $z \in B(x;\delta)$

$d(z,x) < \delta$ — def open ball

$\leq \delta_k$ — def min

$z \in B(x;\delta_k)$ — def open ball

$B(x;\delta) \subseteq B(x;\delta_k)$ — def $\subseteq$

$\subseteq V_k$ — def open ball

$\forall k \in \mathbb{I}_n, B(x;\delta) \subseteq V_k$ — $\forall +$

$B(x;\delta) \subseteq \bigcap_{k \in \mathbb{I}_n} V_k$ — def intersection

$= V$ — substitution

V is a neighborhood of x — def neighborhood

$\leftarrow$

$x \in V \Rightarrow V$ is a neighborhood of x — $\Rightarrow +$

$\forall x, x \in V \Rightarrow V$ is a neighborhood of x — $\forall +$

V is open — def of open

QED

Lemma *A subset of a topological space is open if and only if it is a neighborhood of each of its points.*

Pf.

Let (X,τ) be a topological space and $U \subseteq X$.

⋆ *($\Rightarrow$)*

Assume U is open

$U \subseteq U$ — from page 3

Let $u \in U$

U is a neighborhood of u — def neighborhood

U is a neighborhood of each of its points — $\forall +$

$\leftarrow$

$\star$ ($\Leftarrow$)

Assume U is a neighborhood of each of its point

Let $x \in U$

U is a neighborhood of x — $\forall -$

$x \in \mathcal{O}_x \subseteq U$ for some open set $\mathcal{O}_x$ — def neighborhood

$\forall x \in U, \exists \mathcal{O}_x \in \tau, x \in \mathcal{O}_x \subseteq U$ — $\forall +$

Let $y \in U$

$y \in \mathcal{O}_y \subseteq U$ for some open set $\mathcal{O}_y$ — def neighborhood

$y \in \bigcup_{x \in U} \mathcal{O}_x$ — def indexed union

$U \subseteq \bigcup_{x \in U} \mathcal{O}_x$

Let $z \in \bigcup_{x \in U} \mathcal{O}_x$

$z \in \mathcal{O}_t$ for some $t \in U$ — def indexed union

$\mathcal{O}_t \subseteq U$ — def $\mathcal{O}_x$ above

$z \in U$ — def $\subseteq$

$\bigcup_{x \in U} \mathcal{O}_x \subseteq U$ — def $\subseteq$

$U = \bigcup_{x \in U} \mathcal{O}_x$ — def set $=$

$\bigcup_{x \in U} \mathcal{O}_x$ is open — def topology

U is open — substitution

$\leftarrow$

QED

Lemma *Let $f : X \to Y$ and $A, B \subseteq Y$. If $A \subseteq B$ then $f^{-1}(A) \subseteq f^{-1}(B)$.*

Pf.

Let $f : X \to Y$ and $A, B \subseteq Y$

Assume $A \subseteq B$

Let $x \in f^{-1}(A)$

$f(x) \in A$ — def inverse image

$f(x) \in B$ — def $\subseteq$

$x \in f^{-1}(B)$ — def inverse image

$f^{-1}(A) \subseteq f^{-1}(B)$ — def $\subseteq$

$\leftarrow$

QED

Lemma *A map between topological spaces is continuous if and only if it is continuous at every point.*

Pf.

Let $f : (X, \tau) \to (Y, \tau')$ be a map between topological spaces.

$\star$ ($\Rightarrow$)

Assume f is continuous

Let $a \in X$
Let N be a neighborhood of $f(a)$
$f(a) \in \mathcal{O} \subseteq N$ for some open set $\mathcal{O} \in \tau'$ — def of neighborhood
$f^{-1}(\mathcal{O}) \in \tau$, i.e. its open!! Yay! — def of continuous
$a \in f^{-1}(\mathcal{O})$ — def inverse image
$f^{-1}(\mathcal{O}) \subseteq f^{-1}(N)$ — by Lemma above
$f^{-1}(N)$ is a neighborhood of a — def of neighborhood
f is continuous at a — def continuous at a point
f is continuous at every point — $\forall +$
$\leftarrow$

$\star$ $(\Leftarrow)$
Assume f is continuous at every point
Let $U \in \tau'$ be an open subset of Y
Let $a \in f^{-1}(U)$
$f(a) \in U$ — def inverse image
$U \subseteq U$ — pg 3
U is a neighborhood of $f(a)$ — def of neighborhood
f is continuous at a — $\forall -$
$f^{-1}(U)$ is a neighborhood of a — def continuous at a point
$f^{-1}(U)$ is a neighborhood of each of its points — $\forall +$
$f^{-1}(U)$ is open — by the Lemma above
$\forall U \in \tau', f^{-1}(U)$ is open — $\forall +$
f is continuous — def continuous
$\leftarrow$
QED

Theorem *Let (X,τ) be a topological space and R and equivalence relation on X. Then the quotient topology on X/R is the set*

$$\tau' = \{\mathcal{O} \subseteq X/R : \pi^{-1}(\mathcal{O}) \in \tau\}$$

Pf.
Let (X,τ) be a topological space and R and equivalence relation on X.
Let $\pi : X \to X/R$ be the quotient map.
Define $\tau' = \{\mathcal{O} \subseteq X/R : \pi^{-1}(\mathcal{O}) \in \tau\}$
$\pi^{-1}(\emptyset) = \emptyset$ by def of inverse image.
$\quad \in \tau$ by def of topology.
$\emptyset \in \tau'$ by def of τ'.
$\pi^{-1}(X/R) = X$ by def of inverse image.
$\quad \in \tau$ by def of topology.
$X/R \in \tau'$ by def of τ'.
Let $\{\mathcal{O}_i\}_{i \in I}$ be an indexed family of elements of τ'.
$\forall i \in I, \pi^{-1}(\mathcal{O}_i) \in \tau$ by def of τ'.
$\pi^{-1}\left(\bigcup_{i \in I} \mathcal{O}_i\right) = \bigcup_{i \in I} \pi^{-1}(\mathcal{O}_i)$ by some result in chapter 1.

$\in \tau$ by def of topology.
$\bigcup_{i\in I} \mathcal{O}_i \in \tau'$ by def of τ'.
Let $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_n \in \tau'$.
$\forall i \in \mathbb{I}_n, \pi^{-1}(\mathcal{O}_i) \in \tau$ by def of τ'.
$\pi^{-1}(\mathcal{O}_1 \cap \mathcal{O}_2 \cap \ldots \cap \mathcal{O}_n) = \pi^{-1}(\mathcal{O}_1) \cap \pi^{-1}(\mathcal{O}_2) \cap \ldots \cap \pi^{-1}(\mathcal{O}_n)$ by some result in chapter 1.
$\in \tau$ by def of topology.
$\mathcal{O}_1 \cap \mathcal{O}_2 \cap \ldots \cap \mathcal{O}_n \in \tau'$ by def of τ'.
τ' is a topology.
Let T be a topology on X/R such that $\pi : (X,\tau) \to (X/R, T)$ is continuous.
Let $U \in T$.
$\pi^{-1}(U) \in \tau$ by definition of continuous.
$U \in \tau'$ by def of τ'.
$T \subseteq \tau'$ by def of $\subseteq$.
τ' is stronger than T by def of strong.
τ' is stronger than every topology on X/R such that the quotient map is continuous (by for all plus!).
τ' is the quotient topology!
QED

Theorem *Let $(X,\tau),(Y,\tau')$ be topological spaces and ρ the product topology on $X \times Y$. Let $y_0 \in Y$ and $S = \{(x,y_0) : x \in X\}$. Then (X,τ) is homeomorphic to (S,ρ') where ρ' is the subspace topology on S.*

Pf.
Let $(X,\tau),(Y,\tau')$ be topological spaces and ρ the product topology on $X \times Y$.
Let $y_0 \in Y$, $S = \{(x,y_0) : x \in X\}$, and ρ' the subspace topology on S.
Define $h : X \to S$ by $h(x) = (x,y_0)$ for all $x \in X$.
⋆ *We will show h is a homeomorphism*

⋆ *First we show it is injective*
Let $a,b \in X$.
Assume $h(a) = h(b)$.
$(a,y_0) = (b,y_0)$ by def of h.
$a = b$ by def of ordered pair.
$\leftarrow$
h is injective.

⋆ *Now we show it is surjective*
Let $s \in S$.
$s = (t,y_0)$ for some $t \in X$ by def of S.
$= h(t)$ by def of h.
h is surjective.

⋆ *so it is both injective and surjective*
h is bijective.

⋆ *now we show that h is continuous by showing the inverse image of an arbitrary open set is open*

Let $\mathcal{O}$ be an open subset of (S, ρ').
$\mathcal{O} = \mathcal{U}' \cap S$ for some open set $\mathcal{U}'$ in $(X \times Y, \rho)$ by def of subspace topology.
$\mathcal{U}' = \cup_{i \in I'} A_i \times B_i$ for some open sets $\{A_i\}_{i \in I'}$ of X and open sets $\{B_i\}_{i \in I'}$ of Y by the definition of product topology.

⋆ *in order for this to work, we need to "trim" our set $\mathcal{U}'$ a little by throwing away any of the basis elements which do not intersect S.*

Let $I = \{i \in I' : y_0 \in B_i\}$.
Define $\mathcal{U} = \cup_{i \in I} A_i \times B_i$.
$\mathcal{U} \subseteq \mathcal{U}'$ by problem 1.3.b.
Let $r \in \mathcal{U} \cap S$
$r \in \mathcal{U}$ and $r \in S$ by def of $\cap$.
$r \in \mathcal{U}'$ and $r \in S$ by def subset.
$r \in \mathcal{U}' \cap S$ by def of $\cap$.
$\mathcal{U} \cap S \subseteq \mathcal{U}' \cap S$ by def of subset.
$= \mathcal{O}$ by substitution.
Let $q \in \mathcal{O}$
$= \mathcal{U}' \cap S$ by substitution.
$q \in \mathcal{U}'$ and $q \in S$ by def of $\cap$.
$q \in \cup_{i \in I'} A_i \times B_i$ by substitution.
$q \in A_\iota \times B_\iota$ for some $\iota \in I'$ by def of union.
$q = (q_1, q_2)$ for some $q_1 \in A_\iota$ and $q_2 \in B_\iota$ by def of Cartesian product.
$q = (a_0, y_0)$ for some $a_0 \in X$ by def of S.
$(q_1, q_2) = (a_0, y_0)$ by substitution.
$q_1 = a_0$ and $q_2 = y_0$ by def of ordered pair.
$y_0 \in B_\iota$ by substitution.
$\iota \in I$ by definition of I.
$q \in \cup_{i \in I} A_i \times B_i$ by definition of union.
$= \mathcal{U}$ by substitution.
$q \in \mathcal{U} \cap S$ by def $\cap$.
$\mathcal{O} \subseteq \mathcal{U} \cap S$ by def of subset.
$\mathcal{O} = \mathcal{U} \cap S$ by def of set equality.

⋆ *we will now show that $h^{-1}(\mathcal{O}) = \cup_{i \in I} A_i$ and therefore is open in X*
⋆ *to do this we have to show that two sets are equal*
⋆ *first we show $h^{-1}(\mathcal{O}) \subseteq \cup_{i \in I} A_i$*

Let $x \in h^{-1}(\mathcal{O})$
$= h^{-1}(\mathcal{U} \cap S)$ by substitution,
$= h^{-1}((\cup_{i \in I} A_i \times B_i) \cap S)$ by substitution.
$h(x) \in (\cup_{i \in I} A_i \times B_i) \cap S$ by def of inverse image.
$h(x) \in (\cup_{i \in I} A_i \times B_i)$ and $h(x) \in S$ by def $\cap$.
$h(x) \in A_\alpha \times B_\alpha$ for some $\alpha \in I$ by def of union.
$h(x) = (a_\alpha, b_\alpha)$ for some $a_\alpha \in A_\alpha, b_\alpha \in B_\alpha$ by def of Cartesian product.
$h(x) = (x, y_0)$ by def of h.
$(x, y_0) = (a_\alpha, b_\alpha)$ by substitution.

$x = a_\alpha$ and $b_\alpha = y_0$ by def of ordered pair.
$x \in A_\alpha$ by substitution.
$x \in \bigcup_{i\in I} A_i$ by def of union.
$h^{-1}(\mathcal{O}) \subseteq \bigcup_{i\in I} A_i$ by def of subset.

⋆ *now we show* $\bigcup_{i\in I} A_i \subseteq h^{-1}(\mathcal{O})$

Let $z \in \bigcup_{i\in I} A_i$.
$z \in A_\gamma$ for some $\gamma \in I$ by def of union.
$h(z) = (z, y_0)$ by def of h.
$\in S$ by def of S.
$h(z) \in A_\gamma \times B_\gamma$ by def of I.
$= \bigcup_{i\in I} A_i \times B_i$ by def of union
$= \mathcal{U}$ by substitution.
$h(z) \in \mathcal{U} \cap S$ by def of $\cap$.
$= \mathcal{O}$ by substitution.
$z \in h^{-1}(\mathcal{O})$
$\bigcup_{i\in I} A_i \subseteq h^{-1}(\mathcal{O})$ by def of subset.

$h^{-1}(\mathcal{O}) = \bigcup_{i\in I} A_i$ by def of set equality.

$\bigcup_{i\in I} A_i$ is open in X by def of topology since all A_i are open.
$h^{-1}(\mathcal{O})$ is open by substitution.
h is continuous by def of continuous.

⋆ *now we define the inverse function of* h

Let $g : S \to X$ by $g(x, y_0) = x$.
Let $w \in S$.
$w = (w_1, y_0)$ for some $w_1 \in X$ by def of S.
$(h \circ g)(w) = (h \circ g)(w_1, y_0)$ by substitution.
$= h(g(w_1, y_0))$ by def of $\circ$.
$= h(w_1)$ by def of g.
$= (w_1, y_0)$ by def of h.
$= w$ by substitution.
Let $v \in X$.
$(g \circ h)(v) = g(h(v))$ by def of $\circ$.
$= g(v, y_0)$ by def of h.
$= v$ by def of g.
So g and h are inverse functions.

⋆ *and finally we show the inverse function is continuous... but not by brute force like we did above*

Let $i : S \to X \times Y$ be the inclusion map and $p_1 : X \times Y \to X$ the projection map onto the first component.
i is continuous by Thm 6.6.
p_1 is continuous by the definition of product topology.
$p_1 \circ i$ is continuous by the corollary to Thm 5.6.
Let $u \in S$.
$u = (u_1, y_0)$ for some $u_1 \in X$ by def of S.

$(p_1 \circ i)(u) = (p_1 \circ i)(u_1, y_0)$ by substitution.
$= p_1(i(u_1, y_0))$ by def of $\circ$
$= p_1(u_1, y_0)$ by def of i
$= u_1$ by def of p_1
$= g(u_1, y_0)$ by def of g
$= g(u)$ by substitution.
$g = p_1 \circ i$ by def of function equality.
g is continuous by substitution.

h is a homeomorphism by def of homeomorphism (it is a continuous bijection with a continuous inverse).

(X, τ) is homeomorphic to (S, ρ') by def of homeomorphic.
QED

Remark *Note that in this proof we showed that the projection map restricted to a subset of its domain is still continuous by composing it with the inclusion map. This proof works in general, namely, if we have any continuous function* $f : X \to Y$ *and* $A \subseteq X$, *then the function obtained by restricting the domain of* f *to* A *is still continuous (with the subspace topology on* A *from* X*).*