Coach Monks's High School Playbook

This playbook is meant as a training reference for high school math competitions. Students should be familiar with all material in Coach Monks's MathCounts Playbook (a 6-8th grade level contest) as a prerequisite to learning this material. Learn the items marked with a ⋆ first. Then once you have mastered them try to learn the other topics.

Arithmetic!

- In addition to the values memorized for MathCounts, the following facts can also be useful.

n	n^2
21	441
22	484
23	529
24	576
25	625
26	676
27	729
28	784
29	841
30	900

n	$n!$
1	1
2	2
3	6
4	24
5	120
6	720
7	5040
8	40320
9	362880
10	3628800

$\sqrt{n}$
$\sqrt{1} = 1$
$\sqrt{2} \cong 1.414$
$\sqrt{3} \cong 1.732$
$\sqrt{4} = 2$
$\sqrt{5} \cong 2.236$
$\sqrt{6} \cong 2.449$
$\sqrt{7} \cong 2.646^*$
$\sqrt{8} \cong 2.828$
$\sqrt{9} = 3$
$\sqrt{10} \cong 3.162$

$\log n$
$\log 1 = 0$
$\log 2 \cong 0.301$
$\log 3 \cong 0.477$
$\log 4 \cong 0.602$
$\log 5 \cong 0.699$
$\log 6 \cong 0.778$
$\log 7 \cong 0.845$
$\log 8 \cong 0.903$
$\log 9 \cong 0.954$
$\log 10 = 1$

*this is the only value in these tables that was rounded up when rounded to the nearest thousandth

- ⋆ $\pi \cong 3.14159265358979$ and $e \cong 2.7182818284590452$
- Prime factorizations of recent, current, and upcoming years:
  - $2002 = 2 \cdot 7 \cdot 11 \cdot 13$
  - 2003 is prime
  - $2004 = 2^2 \cdot 3 \cdot 167$
  - $2005 = 5 \cdot 401$
  - $2006 = 2 \cdot 17 \cdot 59$
  - $2007 = 3^2 \cdot 223$
  - $2008 = 2^3 \cdot 251$
  - $2009 = 7^2 \cdot 41$
  - $2010 = 2 \cdot 3 \cdot 5 \cdot 67$

Combinatorics and Probability

1. **Binomial Coefficient Identities**

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$	⋆ factorial expansion	$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$	⋆ binomial theorem
$\binom{n}{k} = \binom{n}{n-k}$	⋆ symmetry	$\binom{n}{k} = \sum_{m=0}^{k} \binom{n-1-m}{k-m}$	hockey stick
$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$	absorption	$\binom{n}{k} = \sum_{m=0}^{n-k} \binom{n-1-m}{k-1}$	hockey stick
$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$	⋆ recursion	$\binom{n}{k} = \sum_{m=0}^{k} \binom{n-s}{k-m}\binom{s}{m}$	Vandermonde convolution
$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$	trinomial revision		

a. **Generalized Binomial Coefficients**: $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}$ is a well defined polynomial in α and therefore well defined for real (or even complex) values of α.

b. Generalized Binomial Theorem: $(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$

2. ⋆ **Multinomial Coefficients**: $\binom{n}{k_1,k_2,\ldots,k_m}$ is the number of ways of putting n distinct objects into m categories so that the i^{th} category contains k_i objects

a. $\binom{n}{k_1,k_2,\ldots,k_m} = \frac{n!}{k_1!k_2!\cdots k_m!}$ where $k_1 + k_2 + \ldots + k_m = n$

b. $\binom{n}{k_1,k_2,\ldots,k_m} = \binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3}\cdots\binom{n-k_1-k_2-\cdots-k_{m-1}}{k_m}$

c. $(x_1+x_2+\cdots+x_m)^n = \sum_{k_1+\cdots+k_m=n} \binom{n}{k_1,\ldots,k_m} x_1^{k_1}x_2^{k_2}\cdots x_m^{k_m}$ (the multinomial theorem)

3. Catalan Numbers: C_n is the number of triangulations of a convex $(n+2)$-gon having no internal vertices. C_n is also the number of ways to parenthesize $x_1x_2\cdots x_n$ completely into binary products.

a. $C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-2}C_1 + C_{n-1}C_0$ for $n \geq 1$ and $C_0 = 1$

b. $C_n = \frac{1}{n+1}\binom{2n}{n}$ for $n \geq 0$

c. The first few values are $1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots$

4. Stirling Numbers of the first kind: $\left[{n \atop k} \right]$ is the number of permutations of a set with n elements having exactly k distinct cycles.

a. $\left[{n \atop 0} \right] = 0$, $\left[{n \atop 1} \right] = (n-1)!$, $\left[{n \atop n-1} \right] = \binom{n}{2}$, $\left[{n \atop n} \right] = 1$

b. $\left[{n \atop k} \right] = (n-1)\left[{n-1 \atop k} \right] + \left[{n-1 \atop k-1} \right]$ for $n > 1$ and $1 < k < n$

c. The first few values of $\left[{n \atop k} \right]$:

$n \backslash k$	1	2	3	4	5	6	7
1	1						
2	1	1					
3	2	3	1				
4	6	11	6	1			
5	24	50	35	10	1		
6	120	274	225	85	15	1	
7	720	1764	1624	735	175	21	1

5. Stirling Numbers of the second kind: $\left\{ {n \atop k} \right\}$ is the number of partitions of a set with n elements into k non-empty subsets.

a. $\left\{ {n \atop 1} \right\} = 1$, $\left\{ {n \atop 2} \right\} = 2^{n-1} - 1$, $\left\{ {n \atop n-1} \right\} = \binom{n}{2}$, $\left\{ {n \atop n} \right\} = 1$

b. $\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\}$ for $n > 1$ and $1 < k < n$

c. The first few values of $\left\{ {n \atop k} \right\}$:

$n \backslash k$	1	2	3	4	5	6	7	8
1	1							
2	1	1						
3	1	3	1					
4	1	7	6	1				
5	1	15	25	10	1			
6	1	31	90	65	15	1		
7	1	63	301	350	140	21	1	
8	1	127	966	1701	1050	266	28	1

d. $\left\{ {n \atop k} \right\} = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{n}{k}(k-i)^n$

6. Partition Formula: Let $P(n,k)$ = the number of partitions of n having largest summand k. Then

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$P(n,1) = P(n,n) = 1$ and

$$P(n,k) = P(n-k,k) + P(n-1,k-1)$$

This recursion produces the Pascal-like triangle:

n \ k	1	2	3	4	5	6	7	8	9
1	1								
2	1	1							
3	1	1	1						
4	1	2	1	1					
5	1	2	2	1	1				
6	1	3	3	2	1	1			
7	1	3	4	3	2	1	1		
8	1	4	5	5	3	2	1	1	
9	1	4	7	6	5	3	2	1	1

7. ⋆ **Pigeonhole Principle**: If you have n pigeons in k holes some hole contains at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons and some hole contains at most $\left\lfloor \frac{n}{k} \right\rfloor$ pigeons.
8. ⋆ **Inclusion-Exclusion Principle**: Given finite sets $A_1, A_2, \ldots, A_n$ and let $S_1 = \sum_i |A_i|$, $S_2 = \sum_{i<j} |A_i \cap A_j|, \ldots, S_n = |A_1 \cap A_2 \cap \cdots \cap A_n|$. Then
$$|A_1 \cup A_2 \cup \cdots \cup A_n| = S_1 - S_2 + S_3 - S_4 + \cdots + (-1)^{n+1} S_n$$
9. ⋆ **Expected Value**: Given a sample space S and a function $f : S \to \mathbb{R}$ the expected value of f on this sample space is $\sum_{x \in S} P(x) f(x)$ where $P(x)$ is the probability of x.
10. **Van der Waerden's Theorem**: Let n and k be positive integers. Then there exists a positive integer N such that if the numbers $1, 2, \ldots, N$ are colored in k colors, one color always contains an arithmetic progression of length n.

Graph Theory

1. ⋆ **Euler paths**: traverse every edge in a graph exactly once.
   a. A connected graph has an Euler path if and only if the number of odd degree vertices is zero or two. If it is zero then the path is a cycle, and if it is two the path must begin and end at the odd degree vertices.
2. A **simple** graph has no edges from a node to itself.
3. ⋆ **Hamiltonian paths**: traverse every vertex in a graph exactly once.
   a. **Dirac's Theorem**: If every vertex in a simple graph with v vertices has degree at least $v/2$ then the graph has a Hamiltonian cycle.
   b. **Ore's Theorem**: A simple graph with n nodes has an Hamiltonian cycle if whenever two nodes are not connected by an edge the sum of their degrees is at least n.
4. **Ramsey's Theorem**: Let $N(a,b)$ be the smallest number such that any group of $N(a,b)$ people must contain either a mutual friends or b mutual strangers.
   a. $N(a,b) = N(b,a)$
   b. $N(a,2) = a$
   c. $N(a,b) \leq N(a-1,b) + N(a,b-1)$
5. **Turán's Theorem**: Let G be a graph with n nodes which contains no complete subgraph of k nodes. Let $t(n,k)$ be the maximum number of edges of such a graph. Then
$$t(n,k) \leq \left(\frac{k-2}{k-1}\right)\frac{n^2}{2}$$
   a. The graph having $t(n,k)$ edges, n nodes, and no complete k-subgraph is the complete $(k-1)$-partite graph which has the most evenly divided arrangement of nodes, i.e. in which the numbers of nodes each pair of the $k-1$ groups differ by at most 1.

Sequences and Series

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1. ⋆ **Some Sums**
   a. $1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$
   b. $1^3 + 2^3 + 3^3 + \ldots + (n-1)^3 + n^3 = \left(\frac{n(n+1)}{2}\right)^2$
   c. $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! + \ldots + k \cdot k! = (k+1)! - 1$
2. ⋆ **Infinite geometric series**:

$$1 + r + r^2 + r^3 + \ldots = \frac{1}{1-r} \quad \text{for } -1 < r < 1$$

   Many other useful series can be derived from this by substitution, differentiation, etc.
3. **Generating Functions**: Let $a_0, a_1, a_2, \ldots$ be a sequence of numbers. The generating function for this sequence is $f(x) = \sum_{n=0}^{\infty} a_n x^n$. It can be used to solve recurrences explicitly. Variations such as $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$ are sometimes useful.
   a. **Properties**:
      i. Addition and multiplication of generating functions are commutative and associative.
      ii. The constant functions 0 and 1 are additive and multiplicative identities, respectively.
      iii. Every generating function has an additive inverse.
      iv. A generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$ has a multiplicative inverse $B(x) = \sum_{n=0}^{\infty} b_n x^n$ if and only if $a_0 \neq 0$.

         In this case, $B(x)$ is given by the recursion $b_0 = \frac{1}{a_0}$ and $b_k = \frac{-1}{a_0} \sum_{i=1}^{k} a_i b_{k-i}$.
   b. **Manipulating ordinary generating functions**: Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$
      i. $A(x) = B(x)$ if and only if $a_n = b_n$ for all n.
      ii. $\frac{A(x)}{(1-x)} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k \right) x^n$
      iii. $A(x)B(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n$
      iv. $xA'(x) = \sum_{n=0}^{\infty} n a_n x^n$
      v. ${}^{\backprime}A(x) = C + \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} x^n$
   c. **Manipulating exponential generating functions**: Let $A(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$ and $B(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n$
      i. $A(x)B(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \binom{n}{k} \frac{a_k b_{n-k}}{n!} \right) x^n$
      ii. $A'(x) = \sum_{n=0}^{\infty} \frac{a_{n+1}}{n!} x^n$

Number Theory

Distribution of Primes

1. **The Prime Number Theorem**: The approximate number of primes less than or equal to a positive integer x converges to $\frac{x}{\ln x}$ as $x \to \infty$.
2. **Bertrand's Postulate**: For any $n > 1$, there is always at least one prime between n and $2n$.
3. **Dirichlet's Theorem**: For any relatively prime natural numbers a, b, the arithmetic sequence $a, a+b, a+2b, \ldots$ contains infinitely many primes.

Continued Fractions

1. **Continued Fraction Representation**: Every rational number q can be represented uniquely by the sequences $\langle a_0, a_1, \ldots, a_n \rangle$ and $\langle a_0, a_1, \ldots, a_n - 1, 1 \rangle$ where

$$q = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}$$

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and $a_0 = \lfloor q \rfloor$ and the a_i are positive integers determined by converting the reciprocal of the remainder to a mixed number iteratively. Irrational numbers have a unique infinite representation of this form.

2. **Self-similar expressions**: Let $X = \frac{r}{s+\frac{r}{s+\frac{r}{s+\cdots}}}$. Then $X = \frac{r}{s+X}$ which can be solved for X.

Diophantine Equations

1. ⋆ **Linear Equations**: For integers $a_1, \ldots, a_n, c$, the equation $a_1x_1 + \ldots + a_nx_n = c$ has integer solutions if and only if $\gcd(a_1, \ldots, a_n) \mid c$. (*See also:* ***GCDst Theorem*** *in Number Theory - Modular Arithmetic - Other Applications - #1b below*)
2. **Sums of Squares**
   a. For any integer n, the equation $x^2 + y^2 = n$ has integer solutions iff any prime factor of n that is congruent to $3 \pmod 4$ occurs to an even power in the prime factorization of n.
   b. An odd prime p can be written as the sum of two squares iff $p \equiv 1 \pmod 4$
   c. A positive integer n can be written as the sum of three squares iff n cannot be written in the form $(8k+7)4^m$ for any nonnegative integers k, m.
   d. All positive integers can be written as the sum of four squares.
3. **Pell's Equation**: For any non-square positive integer D, the equation $x^2 - Dy^2 = 1$ has infinitely many integer solutions, each of which is of the form $(\pm x_n, \pm y_n)$ given by
$$x_n + y_n\sqrt{D} = \left(x_1 + y_1\sqrt{D}\right)^n$$
where (x_1, y_1) is the solution with $x_1, y_1 \geq 0$ and y_1 minimal, and n is a nonnegative integer.

Modular Arithmetic

Euler's ϕ Function (A.K.A. Euler's totient function)

⋆ Let $\phi(n) =$ the number of positive integers less than the positive integer $n > 1$ which are relatively prime to n.

1. ⋆ $\phi(p^k) = p^{k-1}(p-1)$
2. ⋆ $\phi(ab) = \phi(a) \cdot \phi(b)$ for relatively prime a, b.
3. ⋆ **Euler's Theorem**: Let a, n be relatively prime integers with $n > 1$. Then $a^{\phi(n)} \equiv 1 \pmod n$.
4. ⋆ **Fermat's Little Theorem**: For any nonzero integer a and positive prime p, $a^{p-1} \equiv 1 \pmod p$ and $a^p \equiv a \pmod p$.
5. **Order**: If $\gcd(a, n) = 1$ then the order of a modulo n is the smallest positive integer k such that $a^k \underset{n}{\equiv} 1$. It has the following properties:
   a. k divides $\phi(n)$
   b. $a^i \underset{n}{\equiv} a^j$ if and only if $i \underset{k}{\equiv} j$
6. **Primitive roots**: If the order of a modulo n is $\phi(a)$ then a is called a primitive root of n. It has the following properties:
   a. if b is relatively prime to n then $b = a^k$ for some k
   b. n has a primitive root if and only if $n = 2, 4, p^k$, or $2p^k$ for some odd prime p and $k > 0$.

Other applications

1. **GCD**: Let $d = \gcd(a, b)$.
   a. ⋆ **Euclidean Algorithm**: For all a, b, $\gcd(a, b) = \gcd(b, a \bmod b)$. Iterating this formula computes $\gcd(a, b)$ by reducing it to $\gcd(d, 0)$.
   b. ⋆ **GCDst Theorem**: For all $a, b \in \mathbb{Z}$ there exists integers s, t such that $\gcd(a, b) = sa + tb$. One such pair (s, t) can be found by the Euclidean algorithm as follows:

$s_ia + t_ib$	s_i	t_i
$a_0 = a$	1	0
$a_1 = b$	0	1
a_2	s_2	t_2
a_3	s_3	t_3
$\vdots$	$\vdots$	$\vdots$
$\gcd(a, b)$	s	t

where $a_i = a_{i-2} \bmod a_{i-1}$, $s_i = s_{i-2} - \lfloor a_{i-2}/a_{i-1} \rfloor s_{i-1}$, and $t_i = t_{i-2} - \lfloor a_{i-2}/a_{i-1} \rfloor t_{i-1}$. The pair (S, T) is a solution of $Sa + Tb = d$ if and only if $(S, T) = \left(s - \frac{b}{d}k, t + \frac{a}{d}k\right)$ for some $k \in \mathbb{Z}$. (A common situation is when

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$d = 1$).

2. $\star$ **Base Conversion**: To convert a whole number n to a base b let $f(x) = \frac{x-(x \bmod b)}{b}$. Then $n, f(n), f(f(n))\dots$ taken $\bmod\ b$, are the digits of n in base b in reverse order.
3. $\star$ **Chinese Remainder Theorem**: If $b_0, b_1, \dots, b_n$ are pairwise relatively prime and $B = b_0 b_1 \cdots b_n$ then the system of congruences:

$$x \underset{b_0}{\equiv} a_0,\ x \underset{b_1}{\equiv} a_1,\ \dots,\ x \underset{b_n}{\equiv} a_n$$

has unique solution $\bmod B$:

$$\sum_{i=0}^{n} a_i u_i \left(\frac{B}{b_i}\right)$$

where $u_i \underset{b_i}{\equiv} \left(\frac{B}{b_i}\right)^{-1}$.

4. **Wilson's Theorem**: For any integer n greater than one, n is prime if and only if $(n-1)! \underset{n}{\equiv} -1$.
5. **Wolstenholme's Theorem**: For any prime $p \geq 5$ and any nonnegative integers a and b, $p^3 \mid \binom{ap}{bp} - \binom{a}{b}$.
6. **Roots of Unity mod p^n**: Let p be an odd prime, $n \in \mathbb{Z}^+$. Then $x^p \equiv 1 \left(\bmod p^n\right)$ iff $x \equiv 1 \left(\bmod p^{n-1}\right)$.
7. **Quadratic Reciprocity**: In the following let p be an odd prime and $\gcd(a,p) = \gcd(b,p) = 1$.
   a. **Quadratic residue**: a is a quadratic residue $\bmod p$ if and only if there is an integer x such that $x^2 \underset{p}{\equiv} a$
   b. **Legendre Symbol**: of a on p is $\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue } \bmod p \\ -1 & \text{otherwise} \end{cases}$.
   c. **Euler's Criterion**: $\left(\frac{a}{p}\right) \underset{p}{\equiv} a^{(p-1)/2}$
   d. **Properties of the Legendre symbol**:
      i. if $a \underset{p}{\equiv} b$ then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$
      ii. $\left(\frac{a^2}{p}\right) = 1$
      iii. $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$
      iv. $\left(\frac{1}{p}\right) = 1$ and $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$
      v. $\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \underset{8}{\equiv} 1 \text{ or } p \underset{8}{\equiv} -1 \\ -1 & \text{otherwise} \end{cases}$
   e. **Law of Quadratic Reciprocity**: If p, q are distinct odd primes then

$$\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \left(\frac{q}{p}\right)$$

      i. Corollary: $\left(\frac{p}{q}\right) = \begin{cases} \left(\frac{q}{p}\right) & \text{if } p \underset{4}{\equiv} 1 \text{ or } q \underset{4}{\equiv} 1 \\ -\left(\frac{q}{p}\right) & \text{if } p \underset{4}{\equiv} q \underset{4}{\equiv} 3 \end{cases}$

Algebra

Basics

1. **Absolute value**:
   a. $\star$ **Geometric interpretation**: $|x|$ is the distance x is from the origin.
   b. $\star$ **Algebraic interpretation**: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.
   c. $\star$ $\sqrt{x^2} = |x|$
2. $\star$ **Adding Proportions**: Let a, b, x, y, r be real numbers. Then

$$\frac{x}{y} = \frac{a}{b} = r \Rightarrow \frac{x-a}{y-b} = \frac{x+a}{y+b} = r$$

3. **Useful Factorizations**:
   a. For any positive integer n, $(x^n - y^n) = (x-y)(x^{n-1} + x^{n-2}y + x^{n-2}y^2 + \cdots + xy^{n-2} + y^{n-1})$

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b. For odd positive integers n, $(x^n + y^n) = (x+y)\left(x^{n-1} - x^{n-2}y + x^{n-2}y^2 + \cdots + (-1)^{n-1}y^{n-1}\right)$

c. $(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2 = (ac+bd)^2 + (ad-bc)^2$

- Thus the product of two sums of squares is a sum of squares.

d. $x^4 + 4y^4 = (x^2+2xy+2y^2)(x^2-2xy+2y^2)$

- This partially alleviates the problem of not being able to factor sums of squares.

4. Simplifying Nested Radicals: It is sometimes possible to simplify nested radicals with the *denesting equation*

$$\sqrt{X \pm Y} = \sqrt{\frac{X+\sqrt{X^2-Y^2}}{2}} \pm \sqrt{\frac{X-\sqrt{X^2-Y^2}}{2}}$$

Number systems

1. Fields: $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ and $\mathbb{Z}_p$ for a prime p are all examples of fields. These sets with their usual operations of $+, \times$ satisfy the properties: $+, \times$ are commutative and associative, $+$ is distributive over $\times$, there is an additive and multiplicative identity, every element has an additive inverse and every nonzero element has a multiplicative inverse.

Linear Algebra

1. $\star$ **Vectors**: a real vector is finite sequence of real numbers denoted $(a_1, a_2, \ldots, a_n)$. The set of all such vectors is denoted $\mathbb{R}^n$.

a. $\star$ **Vector addition**: $(a_1,a_2,\ldots,a_n) + (b_1,b_2,\ldots,b_n) = (a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$

b. $\star$ **Scalar multiplication**: If r is a real number then $r(a_1,a_2,\ldots,a_n) = (ra_1, ra_2, \ldots, ra_n)$

c. $\star$ **Dot Product**: $(a_1,a_2,\ldots,a_n) \cdot (b_1,b_2,\ldots,b_n) = a_1b_1 + a_2b_2 + \cdots + a_nb_n$

d. $\star$ **Cross Product**: $(a_1,a_2,a_3) \times (b_1,b_2,b_3) = (e,f,g)$ where $e = \det\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix}$, $f = -\det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix}$, and $g = \det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$.

e. (*See also:* ***Analytic Geometry*** *below*)

2. $\star$ **Matrix**: An array M consisting of m rows and n columns of complex numbers is called an $m \times n$ matrix. The entry in the i^{th} row and j^{th} column is denoted $M\langle i,j\rangle$

3. $\star$ **Matrix multiplication**: If M is an $m \times n$ matrix and N is an $n \times p$ matrix the MN is the $m \times p$ matrix such that $MN\langle i,j\rangle$ is the dot product of the i^{th} row of M with the j^{th} column of N.

4. $\star$ **Determinant**: $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ and

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b\det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c\det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

$$= aei + bfg + cdh - afh - bdi - ceg$$

5. $\star$ **Identity matrix**: is an nxn matrix I_n having $I_n\langle i,j\rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$. For any $n \times n$ matrix M we have $MI_n = I_nM = M$.

6. $\star$ **Inverse matrix**: Two $n \times n$ matrices M, N are inverses if and only if $MN = NM = I_n$. A matrix M has an inverse if and only if $\det(M) \neq 0$.

a. Inverse of a 2×2 **Matrix**: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

7. $\star$ **Systems of linear equations**: the system

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$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

has a unique solution (x,y,z) if and only if $\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \neq 0$.

8. **Cramer's Rule**: If the system of linear equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

has solution (x,y,z) then

$$x = \frac{\det\begin{bmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{bmatrix}}{D}, y = \frac{\det\begin{bmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{bmatrix}}{D}, z = \frac{\det\begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix}}{D},$$

where $D = \det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$. This generalizes to any number of equations and unknowns.

Polynomials

1. ⋆ **Polynomial**: Let F be either a field or $\mathbb{Z}$, and $a_0, a_1, \ldots, a_n \in F$. Then $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ with $a_n \neq 0$ is called a polynomial of degree n with coefficients in F. The set of all such polynomials is denoted $F[x]$.

Quotients, Remainders, and Factorization

1. ⋆ **Division algorithm**: Let F be a field and $f(x), g(x) \in F[x]$ with $g(x) \neq 0$. Then there exist unique polynomials $q(x), r(x)$ such that

$$f(x) = q(x)g(x) + r(x) \text{ and } \left(r(x) = 0 \text{ or } \deg(r(x)) < \deg(g(x))\right).$$

As with integers $q(x)$ is called the quotient and $r(x)$ the remainder when $f(x)$ is divided by $g(x)$.

2. ⋆ **Euclidean algorithm**: the Euclidean algorithm and GCDst theorem can be applied to two polynomials with real coefficients (*see Number Theory - Modular Arithmetic - Other Applications - #1 above*)
3. ⋆ **Remainder Theorem**: Let F be a field, $p(x) \in F[x]$, and $a \in F$. Then there exists $q(x) \in F[x]$ such that

$$p(x) = (x-a)q(x) + p(a)$$

i.e. the remainder when $p(x)$ is divided by $x - a$ is $p(a)$.

4. ⋆ **Factor Theorem**: Let F be a field, $p(x) \in F[x]$, $p(x) \neq 0$, and $a \in F$. Then $(x-a)$ is a factor of $p(x)$ if and only if $p(a) = 0$.
5. ⋆ **Fundamental Theorem of Algebra**: Let $p(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0 \in \mathbb{C}[x]$ with $a_n \neq 0$. Then $p(x)$ factors uniquely (up to reordering the factors) as:

$$p(x) = a_n(x-r_1)(x-r_2)\cdots(x-r_n)$$

for some $r_1, r_2, \ldots, r_n \in \mathbb{C}$.

6. ⋆ **Irreducible Polynomials**: Every polynomial with real coefficients factors as a product of irreducible linear and quadratic polyomials.
7. **Gauss's Theorem**: If $p(x) \in \mathbb{Z}[x]$ and $p(x)$ can be factored over the rationals, then it can be factored over the integers.

Synthetic Division and Substitution

1. ⋆ **Synthetic Division/Substitution**: To compute the quotient and remainder when $p(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0$ is divided by $(x - r)$ we can use synthetic substitution:

$$\begin{array}{c|ccccc} r & a_n & a_{n-1} & a_{n-2} & \cdots & a_0 \\ & a_n & (a_nr + a_{n-1}) & (a_nr + a_{n-1})r + a_{n-2} & \cdots & p(r) \end{array}$$

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2. **Upper and Lower Bounds Theorem**: If the numbers in the second row of the synthetic division all have the same sign or are zero then r is an upper bound for the roots of p. If the numbers in the second row of the synthetic division have alternating signs then r is a lower bound for the roots of p.

Symmetric Polyonomials

1. ⋆ **Coefficients vs. Roots**: Let $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0 \in \mathbb{C}[x]$ and $r_1, r_2, \ldots, r_n$ its (not necessarily distinct) roots. Then for all $0 \le i < n$

$$a_i = \sum_{1 \le k_1 < \ldots < k_i \le n} (-1)^{n-i} r_{k_1} r_{k_2} \cdots r_{k_i}$$

In particular, $a_0 = (-1)^n r_1 r_2 \cdots r_n$ and $a_{n-1} = -(r_1 + r_2 + \cdots + r_n)$.

2. **Reduction Algorithm for Symmetric Polynomials**: Define $s_p(x_1, \ldots, x_n) = \sum_{1 \le k_1 < \ldots < k_p \le n} x_{k_1} x_{k_2} \cdots x_{k_p}$ to be the elementary symmetric polynomial in n variables of degree p (and define $s_p(x_1, \ldots, x_n) = 0$ if $p > n$). These form a basis for the algebra of all symmetric polynomials. Define the height of a monomial $x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$ to be $e_1 + 2e_2 + \cdots + ne_n$ and the height of a polynomial to be the maximum height of any of its monomial terms and zero for the zero polynomial. If f is a symmetric polynomial whose maximal height term is $cx_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$, then the polynomial $g = f - cs_1^{e_n - e_{n-1}} s_2^{e_{n-1} - e_{n-2}} \cdots s_{n-1}^{e_2 - e_1} s_n^{e_1}$ has strictly lower height than f so that iterating gives an expression for f as a polynomial in the elementary symmetric polynomials.

3. **Newton-Girard Identities**: Define

$$N_p(s_1, \ldots, s_p) = \det \begin{bmatrix} s_1 & 1 & 0 & 0 & \cdots & 0 \\ 2s_2 & s_1 & 1 & 0 & \cdots & 0 \\ 3s_3 & s_2 & s_1 & 1 & \cdots & 0 \\ 4s_4 & s_3 & s_2 & s_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ ps_p & s_{p-1} & s_{p-2} & s_{p-3} & \cdots & s_1 \end{bmatrix}$$

where s_i are the elemetary symmetric polynomials in $x_1, x_2, \ldots, x_n$ then expanding N gives

$$N_p = x_1^p + x_2^p + \cdots + x_n^p$$

The first few values of N_p are

$$\begin{aligned} N_1 &= s_1 \\ N_2 &= s_1^2 - 2s_2 \\ N_3 &= s_1^3 - 3s_1 s_2 + 3s_3 \\ N_4 &= s_1^4 - 4s_1^2 s_2 + 2s_2^2 + 4s_1 s_3 - 4s_4 \end{aligned}$$

These satisfy the recurrence

$$N_n = s_1 N_{n-1} - s_2 N_{n-2} + s_3 N_{n-3} - \cdots + (-1)^{n+1} s_n N_0$$

Roots

1. ⋆ **Descartes's Rule of Signs**: If $p(x) \in \mathbb{R}[x]$ then the number of positive roots of $p(x)$ is equal to $N - 2k$ for some $k \in \mathbb{Z}$, where N is the number of sign changes in the sequence $a_0, a_1, \ldots, a_n$. The number of negative roots of $p(x)$ equals the number of positive roots of $p(-x)$.
2. ⋆ **Rational Root Theorem**: If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$ and $\frac{r}{s}$ (in reduced form) is a rational root of $p(x)$, then r divides a_0 and s divides a_n.
3. ⋆ **Complex Conjugate Roots**: If $p(x) \in \mathbb{R}[x]$ and $a + bi$ is a complex root of p then so is $a - bi$.
4. ⋆ **Irrational Conjugate Roots**: If $p(x) \in \mathbb{Q}[x]$ and $a + b\sqrt{c}$ is a root of p where $a, b, c \in \mathbb{Q}$ and $\sqrt{c} \in \mathbb{R} - \mathbb{Q}$, then $a - b\sqrt{c}$ is also a root of p.
5. **Eisenstein's Irreducibility Criterion**: If $p(x) \in \mathbb{Z}[x]$ and if there exists a prime q that divides each of the coefficients except a_n and q^2 does not divide a_0, then $p(x)$ is irreducible over the rationals.
6. **Lagrange Interpolation**: Let $\{(x_i, y_i)\}_{i=0}^n$ be a set of points. Then the n^{th}-degree polynomial

$$p(x) = \sum_{i=0}^{n} y_i \prod_{j \ne i} \frac{(x - x_j)}{(x_i - x_j)}$$

is the unique polynomial of degree at most n passing through each of the points.

Partial Fractions

1. ⋆ **Equality of Polynomial functions**: If $p(x), q(x) \in \mathbb{R}[x]$ then the functions p and q are equal if and only if the

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polynomials $p(x)$ and $q(x)$ have the same degree and their corresponding coefficients are equal.

2. **Partial Fraction Decomposition**: If $p(x) \in \mathbb{R}[x]$ has degree less than $k+2m$ and $l_1(x)\cdots l_k(x) \in \mathbb{R}[x]$ are irreducible linear polynomials and $q_1(x)\cdots q_m(x) \in \mathbb{R}[x]$ are irreducible quadratic polynomials then there exist real numbers $A_1,\ldots,A_k,B_1,\ldots,B_k,C_1,\ldots,C_k$ such that
$$\frac{p(x)}{l_1(x)\cdots l_k(x)q_1(x)\cdots q_m(x)} = \frac{A_1}{l_1(x)} + \cdots + \frac{A_k}{l_k(x)} + \frac{B_1x+C_1}{q_1(x)} + \cdots + \frac{B_mx+C_m}{q_m(x)}$$

Synthetic Geometry

Terminology and Common Notation

1. **Definitions**: The following notation is somewhat standard and will be used in this document.
   a. ⋆ **Cevian**: Any segment from a vertex of a triangle to a point on the opposite side
   b. ⋆ **Median**: A cevian which bisects the opposite side
   c. ⋆ **Altitude**: The perpendicular from a vertex to the opposite side of a triangle
   d. ⋆ **Centroid** (G): The point of intersection of the medians of a triangle (they are concurrent)
   e. ⋆ **Orthocenter** (H): The point of intersection of the altitudes of a triangle (they are concurrent)
   f. ⋆ **Circumcircle**, **Circumcenter** (O), **Circumradius** (R): Every triangle can be circumscribed by a unique circle whose center is the intersection of the perpendicular bisectors of the three sides.
   g. ⋆ **Incircle**, **Incenter** (I), **Inradius** (r): Every triangle circumscribes a unique circle whose center is the intersection of the angle bisectors.
   h. ⋆ **Excircles**, **Excenters** (I_A,I_B,I_C), **Exradii** (r_A,r_B,r_C): Any of the three centers of the excircles (tangent to one side and the extensions of the other two) of a triangle; also the intersection of the external bisectors.
   i. ⋆ **Semiperimeter** (s): half of the perimeter
   j. ⋆ **Medial Triangle**: The triangle whose vertices are the midpoints of the sides of a given triangle. It subdivides the triangle into four congruent sub-triangles.
   k. ⋆ **Orthic Triangle**: The triangle whose vertices are the feet of the altitudes of a given triangle. It is the triangle with minimum perimeter of all triangles whose vertices are on the three sides.
   l. **Euler Line**: the line containing the orthocenter, centroid, and circumcenter of a triangle ($\overleftrightarrow{HGO}$). In any triangle $|HG| = 2|GO|$ and $9(OH)^2 = a^2+b^2+c^2$ where a, b, c are the sides of the triangle.
   m. **Gergonne Point**: the point of intersection of the cevians through the points of tangency of the incircle to the sides of a triangle
   n. **Nagle Point**: the point of intersection of the cevians through the points of tangency of the excircles to the sides of a triangle
   o. **Fermat Point**: the point F in an acute triangle $\triangle ABC$ for which $|FA|+|FB|+|FC|$ is minimal. In any acute triangle $\angle AFB = \angle BFC = \angle CFA = 120°$.

Triangles

⋆ In $\triangle ABC$ we define $a = |BC|,\ b = |AC|,\ c = |AB|,$ and abbreviate the three angles as $\angle A,\ \angle B,$ and $\angle C$.

1. ⋆ **Pythagorean triples**: A right triangle has relatively prime integer length sides if and only if it has legs $2uv$ and (u^2-v^2) and hypotenuse (u^2+v^2) for some relatively prime, opposite parity, positive integers u,v with $u>v$.
2. ⋆ **Area**: Let $|\triangle ABC|$ denote the area of $\triangle ABC$. Then
$$\begin{aligned}|\triangle ABC| &= rs\\ &= r_A(s-a) = r_B(s-b) = r_C(s-c)\\ &= \frac{abc}{4R}\\ &= \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B\\ &= \sqrt{s(s-a)(s-b)(s-c)}\end{aligned}$$
   a. **Area from coordinates**: Suppose A,B,C are vectors in $\mathbb{R}^2$. Define $\mathbf{u} = (u_1,u_2) = A-C$ and $\mathbf{v} = (v_1,v_2) = B-C$. Then

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$$|\triangle ABC| = \frac{1}{2}\left|\det\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}\right|$$
$$= \frac{1}{2}|\mathbf{u} \times \mathbf{v}|$$

i. ⋆ **Shoelace Theorem**: If $(x_1, y_1), \ldots, (x_n, y_n)$ are the vertices of an n-gon, the area of the n-gon is

$$Area = \frac{1}{2}\left\|\begin{matrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_{n-1} & y_{n-1} \\ x_n & y_n \end{matrix}\right\|$$
$$= \frac{1}{2}\left|\left(x_n y_1 + \sum_{k=1}^{n-1} x_k y_{k+1}\right) - \left(x_1 y_n + \sum_{k=1}^{n-1} x_{k+1} y_k\right)\right|$$

b. (*See also:* ***Pick's Theorem*** *in Geometry - Polygons - #1 below*)

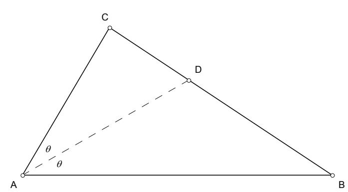
3. **Euler's Theorem**: Let d be the distance between the incenter I and circumcenter O of triangle ABC
$$d^2 = R(R - 2r)$$
   a. Corollary: $d^2 - R^2 = -2rR$ is the power of the point I with respect to the circumcircle (*see also Geometry - Circles - #2 below*)
   b. **Euler's Inequality**: In any triangle $R \geq 2r$.

Similarity

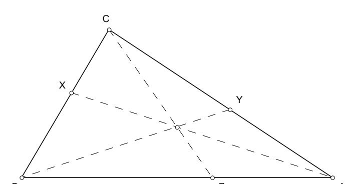
1. ⋆ **Basic proportionality**: A segment connecting points on two sides of a triangle is parallel to the third side if and only if the segments it cuts off are proportional to the sides. (*see also Algebra - Basics - #2 above*)

Cevians

1. ⋆ **Angle Bisector Theorem**: If D is the point where the angle bisector of $\angle A$ in $\triangle ABC$ meets BC then
$$\frac{|BD|}{|BA|} = \frac{|CD|}{|CA|}$$



2. **Ceva's Theorem**: Three cevians AX, BY, CZ of $\triangle ABC$ are concurrent if and only if
$$\frac{|BX|}{|XC|}\frac{|CY|}{|YA|}\frac{|AZ|}{|ZB|} = 1$$



3. **Menelaus Theorem**: Let D, E, F be three points on, respectively, the lines $\overleftrightarrow{BC}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{AB}$ containing the sides of $\triangle ABC$. Then D, E, F are collinear if and only if
$$\frac{\|AF\|}{\|BF\|}\frac{\|BD\|}{\|CD\|}\frac{\|CE\|}{\|AE\|} = -1$$
where $\|XY\|$ is the signed length of the directed segment XY.

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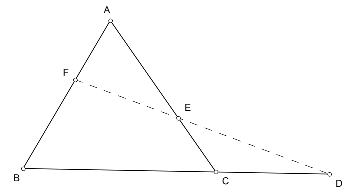
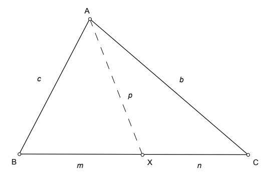
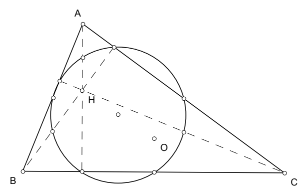


4. **Stewart's Theorem**: Let ABC be a triangle with cevian AX of length p and let $m = XB$ and $n = XC$. Then $a(p^2 + mn) = b^2m + c^2n$.



5. **Inradius in terms of altitudes**: Let h_a, h_b, h_c be the lengths of the altitudes and r be the inradius of $\triangle ABC$. Then
$$\frac{1}{r} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}$$

6. **Nine Point Circle**: The circle whose center is the midpoint of the Euler Line (N) of $\triangle ABC$ with radius $\frac{R}{2}$ passes through the feet of the altitudes, the midpoints of the sides, and the midpoints of HA, HB, and HC.



7. **Feuerbach's Theorem**: The nine point circle of a triangle is tangent to the incircle and to the three excircles.
8. **Brocard Points**: There is exactly one point P in $\triangle ABC$ such that $\omega = |\angle PAB| = |\angle PBC| = |\angle PCA|$ which is the point where the circle through A tangent to BC at B intersects the circle through C tangent to AB at A. This point and its isogonal conjugate P' (the point making $\omega' = |\angle PBA| = |\angle PCB| = |\angle PAC|$) are called the Brocard points of the triangle. Since ω is always equal to ω', this angle is called the Brocard angle of the triangle. It is given by the formula:
$$\cot\omega = \cot A + \cot B + \cot C$$

Trigonometry

1. **Triangle solvers**: Assume a triangle with angles A, B, and C, that respectively intercept sides a, b, and c, having circumradius R.
   a. ⋆ **Extended Law of Sines**:
   $$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} = 2R$$
   b. ⋆ **Law of Cosines**:
   $$a^2 = b^2 + c^2 - 2bc\cos(A)$$
   $$b^2 = a^2 + c^2 - 2ac\cos(B)$$
   $$c^2 = a^2 + b^2 - 2ab\cos(C)$$
   c. **Law of Tangents**:
   $$\frac{a-b}{\tan\frac{A-B}{2}} = \frac{a+b}{\tan\frac{A+B}{2}}$$
2. **Identities**:
   a. ⋆ **Pythagorean Identity**: $\sin^2(x) + \cos^2(x) = 1$
   b. ⋆ **Angle Addition**:
      i. $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
      ii. $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

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iii. $\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$

c. ⋆ **Double Angle**:

i. $\sin(2x) = 2\sin(x)\cos(x)$

ii. $\cos(2x) = \cos^2(x) - \sin^2(x)$

iii. $\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$

d. ⋆ **Triple Angle**:

i. $\sin(3x) = -(4\sin^3(x) - 3\sin(x))$

ii. $\cos(3x) = 4\cos^3(x) - 3\cos(x)$

e. ⋆ **Half Angle**:

i. $\sin^2(\frac{x}{2}) = \frac{1-\cos(x)}{2}$

ii. $\cos^2(\frac{x}{2}) = \frac{1+\cos(x)}{2}$

iii. $\tan^2(\frac{x}{2}) = \frac{1-\cos(x)}{1+\cos(x)}$

iv. $\tan(\frac{x}{2}) = \frac{\sin(x)}{1+\cos(x)} = \frac{1-\cos(x)}{\sin(x)}$

f. **Sum to Product**:

i. $\sin(x) \pm \sin(y) = 2\sin\left(\frac{x\pm y}{2}\right)\cos\left(\frac{x\mp y}{2}\right)$

ii. $\cos(x) + \cos(y) = 2\cos(\frac{x+y}{2})\cos(\frac{x-y}{2})$

iii. $\cos(x) - \cos(y) = -2\sin(\frac{x+y}{2})\sin(\frac{x-y}{2})$

iv. $\tan(x) \pm \tan(y) = \frac{\sin(x\pm y)}{\cos(x)\cos(y)}$

g. ⋆ **Product to Sum**:

i. $\sin(x)\cos(y) = \frac{\sin(x+y)+\sin(x-y)}{2}$

ii. $\cos(x)\cos(y) = \frac{\cos(x+y)+\cos(x-y)}{2}$

iii. $\sin(x)\sin(y) = -\left(\frac{\cos(x+y)-\cos(x-y)}{2}\right)$

iv. $\tan(x)\tan(y) = \frac{\cos(x-y)-\cos(x+y)}{\cos(x-y)+\cos(x+y)}$

h. **In** $\triangle ABC$:

i. $\tan(\frac{A}{2}) = \frac{|\triangle ABC|}{s(s-a)} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

ii. $\tan(A) + \tan(B) + \tan(C) = \tan(A)\tan(B)\tan(C)$

iii. $c = a\cos(B) + b\cos(A)$

i. **Miscellaneous**:

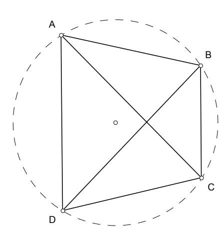
i. **Difference of two squares for sine**: $\sin^2(x) - \sin^2(y) = \sin(x+y)\sin(x-y)$

ii. $\cos(x) + \sin(x) = \sqrt{2}\cos(\frac{\pi}{4} - x)$

iii. $\sin(15^\circ) = \frac{1}{4}\left(\sqrt{6} - \sqrt{2}\right)$ and $\cos(15^\circ) = \frac{1}{4}\left(\sqrt{6} + \sqrt{2}\right)$

Quadrilaterals

1. ⋆ **Ptolemy's Theorem**



In any cyclic quadrilateral $ABCD$, $AB \cdot CD + BC \cdot AD = AC \cdot BD$.

i.e. The sum of the products of the opposite sides of a cyclic quadrilateral is equal to the product of the diagonals.

2. **Ptolemy's inequality**: In any quadrilateral $ABCD$, $AB \cdot CD + BC \cdot AD \geq AC \cdot BD$.

i.e. The sum of the products of the opposite sides of any quadrilateral is greater than or equal to the product of the diagonals.

3. **Inscribed circle**: A quadrilateral $ABCD$ has an inscribed circle if and only if $AB + CD = AC + BD$.

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4. **Midline of diagonals**: In a quadrilateral with side lengths a, b, c, and d and diagonals e and f, let X and Y be the midpoints of the diagonals. Then

$$4|XY|^2 = a^2 + b^2 + c^2 + d^2 - e^2 - f^2$$

   a. Corollary: In a parallelogram, the sum of the squares of the sides is equal to the sum of the squares of the diagonals.

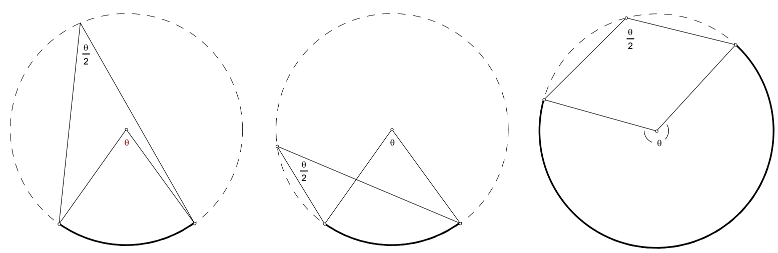
5. **Ways to prove a quadrilateral is cyclic**:
   a. ⋆ The converse of Ptolemy’s Theorem is true.
   b. ⋆ If a pair of opposite angles of a quadrilateral are supplementary, the quadrilateral is cyclic.
   c. ⋆ If one side of the quadrilateral subtends equal angles with the other two vertices, the quadrilateral is cyclic. (*see Circles, 1b*)

Polygons

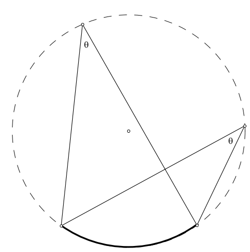
1. ⋆ **Pick’s Theorem**: The area of any closed polygon whose vertices have integer coordinates is $i + b/2 - 1$ where i is the number of points with integer coordinates in the interior of the figure and b is the number of points on the boundary of the figure.

Circles

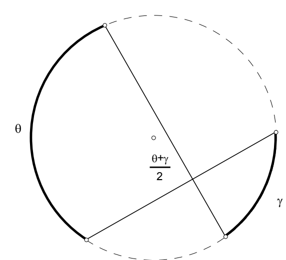
1. **Angles on a circle**
   a. ⋆ **Star Trek Lemma**: An inscribed angle has one half as many degrees as the intercepted arc (Cor: Any angle that intercepts a diameter is a right angle).



   b. ⋆ Different inscribed angles intercepting the same arcs are equal.



   c. ⋆ An angle formed by two chords intersecting within a circle has one-half as many degrees as the sum of the intercepted arcs.



   d. ⋆ Any angle formed by two secants, a secant and a tangent, or two tangents is equal to half the difference of the intercepted arcs.

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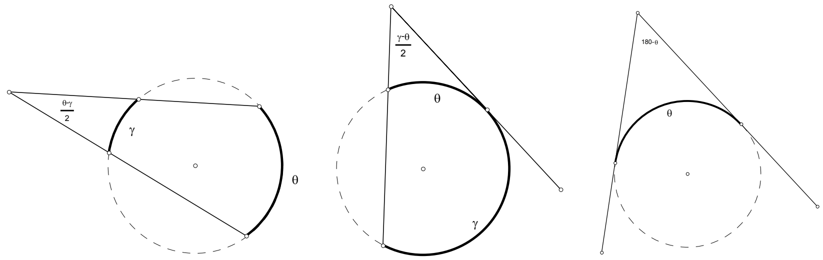
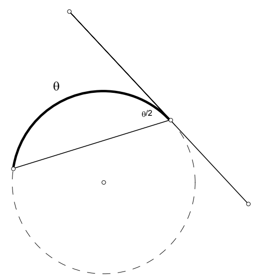
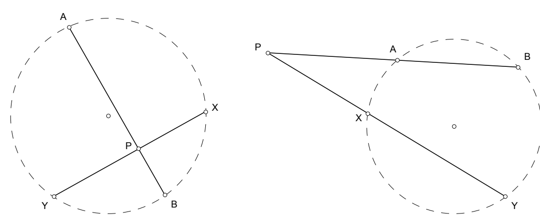


e. ⋆ An angle formed by a chord and a tangent to a circle has one-half as many degrees as the intercepted arc. ($\frac{1}{2}\,\text{arc}\,AT = \angle ATB$)

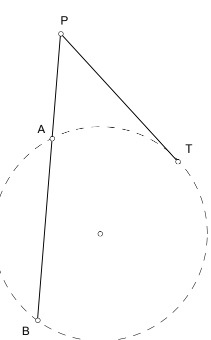


2. Power of a Point:

a. ⋆ In both of the following: $PA \cdot PB = PX \cdot PY$



b. ⋆ Assuming PA is a secant and PT is a tangent, $PA \cdot PB = PT \cdot PT$



c. Using Euler's Theorem, we find that the power of the incenter I of a triangle with respect to the circumcircle is $-2rR$

3. The Radical Axis

a. The Radical Axis of two circles is the locus of points with equal power to both circles.

b. The Radical Axis is a straight line, perpendicular to the line connecting the centers of the circles. If the circles intersect, it passes through the two points of intersection. If the circles are tangent to each other, the axis is their common tangent.

c. Given three circles, either the three radical axes between each pair are parallel, or they are concurrent.

4. Other circle facts:

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a. **The Butterfly**: Given a circle and a chord AB of the circle whose midpoint is M, let XY and ZW be two chords passing through M, and let XW and YZ intersect AB at P and Q, respectively. Then M is also the midpoint of PQ.

b. The formula for the graph of a circle centered at (h,k) with radius r is:

$$(x-h)^2 + (y-k)^2 = r^2$$

c. **Brahmagupta's Formula**: The area of a quadrilateral inscribed in a circle with side lengths a,b,c,d is

$$\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where $s = \frac{a+b+c+d}{2}$.

d. The coordinates of the center of a circle that is inscribed in a triangle whose legs are on the positive x and y axes are (a,a) where $a = \frac{x+y-z}{2}$ given that x and y are the lengths of the legs and z is the length of the hypothenuse.

Transformational Geometry

(*See also: **Complex Transformations** in Complex Numbers*)

Inversive

1. ⋆ **Definitions and Notation**:
   a. **Inversive plane**: $\mathbb{P}^2 = \mathbb{R}^2 \cup \{P_\infty\}$ where $P_\infty \notin \mathbb{R}^2$ is called the **point at infinity**.
   b. **Figure**: in the inversive plane is a set of points $F \subseteq \mathbb{P}^2$.
   c. **Cline**: A figure that is either a circle in $\mathbb{R}^2$ or a figure $l \cup \{P_\infty\}$ where l is a line in $\mathbb{R}^2$.
   d. **Inverse**: Given any point A and any circle ω with center O and radius k, the **inverse** of A with respect to ω is the point B on ray OA satisfying $OA \cdot OB = k^2$. The inverse of A with respect to a line l is the reflection of A about l.
   e. The inverse of a point A is denoted A'. Similarly, the inverse image of a figure F is denoted F'.
2. ⋆ **Properties of Inversion**:
   a. $O' = P_\infty$ and $P'_\infty = O$, where O is the center of the circle of inversion.
   b. For any point or figure A, $(A')' = A$.
   c. **Clines invert to clines**
      i. $\omega = \omega'$, where ω is the circle of inversion.
      ii. For any circle α passing through O, α' is the radical axis of α and ω.
      iii. For any circle α not passing through O, α' is a circle.
      iv. For any line l passing through O, $l' = l$.
      v. For any line l not passing through O, l' is a circle passing through O.
   d. **Conformal (angle-preserving)**: If two clines α and β intersect at an angle θ, then α' and β' intersect at an angle θ.
3. **Inversive Distance Formula**: Let A, B be two points in the inversive plane. Then

$$A'B' = \frac{k^2 \cdot AB}{OA \cdot OB}$$

Projective

1. **Definitions**:
   a. **Pencil**
      i. A **pencil of parallel lines** is the set of all lines in $\mathbb{R}^2$ parallel to a given line, together with the line itself.
      ii. A **pencil of concurrent lines** is the set of all lines passing through a given point.
   b. **Projective plane**: $\mathbb{R}^2 \cup l_\infty$ where l_∞ consists of an infinite set of points, one for each pencil of parallel lines in $\mathbb{R}^2$.
   c. **Perspective from a point**: Triangles $A_1A_2A_3$ and $B_1B_2B_3$ are perspective from point C iff the lines A_1B_1, A_2B_2, A_3B_3 are concurrent at C.
   d. **Perspective from a line**: Triangles $A_1A_2A_3$ and $B_1B_2B_3$ are perspective from line l iff the points $A_1A_2 \cap B_1B_2$, $A_2A_3 \cap B_2B_3$, and $A_3A_1 \cap B_3B_1$ lie on l.
2. **Duality of projective theorems**: If P is a theorem about the projective plane, then the **dual** of P is the statement obtained by interchanging "point" with "line", "collinear" with "concurrent", etc. The dual of P is always a theorem as well.

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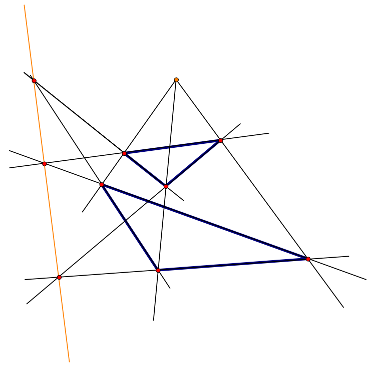
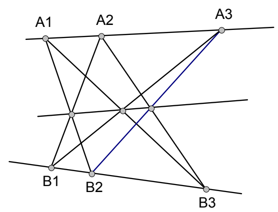
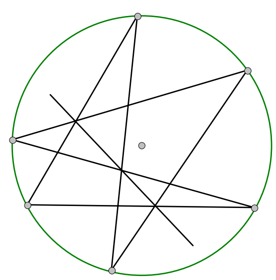
3. **Desargues's Theorem**: If two triangles are perspective from a point, then they are perspective from a line.



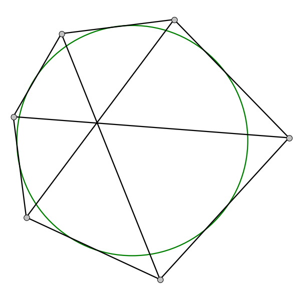
4. **Pappus's Theorem**: If A_1, A_2, and A_3 are collinear and B_1, B_2, and B_3 are collinear, then $A_1B_1 \cap A_3B_2$, $A_2B_1 \cap A_3B_3$, and $A_2B_2 \cap A_1B_3$ are collinear.



5. **Pascal's Theorem** (Dual of Brianchon): If a hexagon is inscribed in a conic, the points of intersection of pairs of opposite sides are collinear.



6. **Brianchon's Theorem** (Dual of Pascal): If a hexagon is circumscribed about a conic, its three diagonals (joining pairs of opposite vertices) are concurrent.



Analytic Geometry

Basics

1. **Distance Formula**: If $P_1 = (x_1, x_2, \ldots, x_n)$ and $P_2 = (y_1, y_2, \ldots, y_n)$ then

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$$|P_1P_2| = \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2}$$

Lines

1. ⋆ **Midpoint Formula**: The midpoint of a segment whose endpoints are $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ is given by $\left(\frac{(x_1+y_1)}{2}, \frac{(x_2+y_2)}{2}, \ldots, \frac{(x_n+y_n)}{2}\right)$.
2. Forms of Lines:
   a. ⋆ **Point-Slope**: $y - y_1 = m(x - x_1)$
   b. ⋆ **Slope-Intercept**: $y = mx + b$
   c. ⋆ **General**: $Ax + By + C = 0$, where $A^2 + B^2 \neq 0$
3. ⋆ **Slope Formula**: $m = \frac{y_2 - y_1}{x_2 - x_1}$
4. ⋆ **Perpendicular lines**: Two nonvertical lines are perpendicular if and only if the product of their slopes is -1.
5. **Distance from a Point to a Line**: the distance from point (p, q) to the line $ax + by = c$ where $a^2 + b^2 = 1$ (which can always be obtained by dividing both sides of the equation by $\sqrt{a^2 + b^2}$ if necessary) is

$$|ap + bq - c|$$

Conic Sections

Standard Equations

1. ⋆ **Circle**: a circle is the set of all points a fixed distance r from a point (a, b) called the center.

$$(x - a)^2 + (y - b)^2 = r^2$$

2. ⋆ **Parabolas**: a parabola is the set of points equidistant from a given point (the focus) and a given line (the directrix). For focus $(0, p)$ and directrix $y = -p$

$$x^2 = 4py$$

3. ⋆ **Ellipses**: an ellipse is the set of all points such that the sum of the distances to two fixed foci is constant. For foci $(-c, 0)$ and $(c, 0)$ and semimajor axis a and semiminor axis b with $a > b$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

   Note that $b^2 + c^2 = a^2$.
   a. The area of an ellipse is πab, where a and b are the semimajor and semiminor axes.
4. ⋆ **Hyperbolas**: a hyperbola is the set of all points such that the difference of the distances to two fixed foci is constant. For foci $(-c, 0)$ and $(c, 0)$ and x-intercepts a and $-a$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

   where $a^2 + b^2 = c^2$.

General Equations

1. **General Equations for conic sections**: can be obtained from the standard equations by applying the appropriate transformations
   a. **Translation by** (h, k): $x' = x - h$ and $y' = y - k$
   b. **Rotation about the origin by angle** θ: $x' = x\cos\theta - y\sin\theta$ and $y' = x\sin\theta + y\cos\theta$
   c. **Reflection across the** x**-Axes**: $x' = x$ and $y' = -y$
   d. (*See also:* ***Transformation*** *in Complex Numbers - Complex Transformations - #1a,b,c)*
2. **Removal of xy-term**: The xy-term can be removed from $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,\ B \neq 0$ by a rotation of the axes if θ is selected so that $\cot 2\theta = \frac{A-C}{B}$.

Other Representations

1. ⋆ **Polar-Rectangular Relations**:

$$x = r\cos\theta, \quad y = r\sin\theta$$
$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$

2. ⋆ **Parametric Equations**: The set of equations $x = f(t),\ y = g(t)$ are parametric equations of the relation

$$\left\{(x, y) : x = f(t) \text{ and } y = g(t) \text{ and } t \text{ is in the intersection of the domains of } f \text{ and } g\right\}.$$

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Vector Geometry

In the following if $\vec{\mathbf{v}} \in \mathbb{R}^2$ then $\vec{\mathbf{v}} = (v_1, v_2)$.

1. **Equal Vectors**: two vectors are equal if and only if their corresponding coordinates are equal.
2. **Vector Addition**: $\vec{\mathbf{a}} + \vec{\mathbf{b}} = (a_1 + b_1, a_2 + b_2)$
3. **Length**: $\left|\vec{\mathbf{v}}\right| = \sqrt{v_1^2 + v_2^2}$ (a **unit vector** has length one)
4. **Direction of Vector**: the unit vector in the direction of $\vec{\mathbf{v}}$ is
$$\vec{\mathbf{u}} = \frac{\vec{\mathbf{v}}}{\left|\vec{\mathbf{v}}\right|} = \left(\frac{v_1}{\left|\vec{\mathbf{v}}\right|}, \frac{v_2}{\left|\vec{\mathbf{v}}\right|}\right) = (\cos\theta, \sin\theta)$$
where $\theta = \arctan(\frac{v_2}{v_1})$ is the angle between $\vec{\mathbf{u}}$ and the positive x-axis in standard position
5. **Multiplication by Scalar**: $k\vec{\mathbf{v}} = (kv_1, kv_2)$ for any $k \in \mathbb{R}$
6. **Dot Product**: $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_1b_1 + a_2b_2 = \left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{b}}\right|\cos(\alpha - \beta)$ where α, β are the direction angles for $\vec{\mathbf{a}}, \vec{\mathbf{b}}$ respectively.
    a. two vectors are perpendicular if and only if their dot product is zero
    b. $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ is the length of the projection of $\vec{\mathbf{a}}$ onto the line containing $\vec{\mathbf{b}}$ if $\vec{\mathbf{b}}$ is a unit vector

Analysis

Real Analysis

Inequalities

1. **The Arithmetic-Geometric-Harmonic Mean Theorem**
    a. ⋆ For all nonnegative real numbers $x_1, \ldots, x_n$,
$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}} \le \sqrt[n]{x_1 x_2 \ldots x_n} \le \frac{x_1 + x_2 + \ldots + x_n}{n}$$
    $\frac{x_1+x_2+\ldots+x_n}{n}$ is called the Arithmetic Mean, or average, of $x_1, \ldots, x_n$.
    $\sqrt[n]{x_1 x_2 \ldots x_n}$ is called the Geometric Mean of $x_1, \ldots, x_n$.
    $\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}}$ is called the Harmonic Mean of $x_1, \ldots, x_n$.
    b. **(Weighted) Power Mean Theorem**: For all nonnegative real numbers $x_1, \ldots, x_n$, $\lambda_1, \ldots, \lambda_n$ with $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1$, and all real numbers $p \neq 0$ define $m_p = \left(\sum_{i=1}^{n} \lambda_i x_i^p\right)^{1/p}$ and $m_0 = \lim_{p \to 0} m_p$. Then
        i. For all real numbers r, s with $r < s$ we have $m_r \le m_s$ (this is the generalization of the AM-GM-HM theorem)
        ii. An important special case is $\lambda_1 = \lambda_2 = \ldots = \lambda_n = \frac{1}{n}$; in this case m_0 is the geometric mean, m_1 is the arithmetic mean, and m_{-1} is the harmonic mean. In general, m_p is called the pth power mean.
        iii. $m_0 = x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n}$ is the weighted geometric mean.
2. ⋆ **Triangle Inequality**: For all real or complex numbers $x_1, \ldots, x_n$,
$$|x_1 + x_2 + \ldots + x_n| \le |x_1| + |x_2| + \ldots + |x_n|$$
    a. **Minkowski's Inequality**: is the generalization of the triangle inequality to higher dimensions. Given real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$
$$\sqrt{\sum_{i=1}^{n} (a_i + b_i)^2} \le \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}$$
3. ⋆ **Geometric Mean Machine**: If AC is a diameter of a circle through point B, and D is the foot of the perpendicular through B to AC then BD is the geometric mean of AD and DC.

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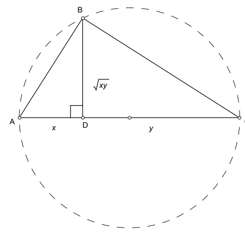

- **a.** **Symmetry-Product Principle**: As the distance between two positive numbers decreases their product increases if their sum remains constant.

4. ⋆ **Cauchy-Schwartz Inequality**: For any two sequences of real numbers $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$,

$$(x_1y_1 + x_2y_2 + \ldots + x_ny_n)^2 \leq (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2)$$

5. **Rearrangement**: For any sequences of real numbers $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ and any permutation π of $\{1,2,\ldots,n\}$,

$$a_1b_n + a_2b_{n-1} + \ldots + a_nb_1 \leq a_1b_{\pi(1)} + a_2b_{\pi(2)} + \ldots + a_nb_{\pi(n)} \leq a_1b_1 + a_2b_2 + \ldots + a_nb_n$$

6. **Chebyshev's Inequality**: For any sequences of real numbers $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$,

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_kb_{n-k+1}\right) \leq \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right) \leq \left(\frac{1}{n}\sum_{k=1}^{n} a_kb_k\right)$$

7. **Jensen's Inequality**: If f is a continuous real valued function that is concave upwards on the closed interval $[a..b]$ (e.g. $f(x) = x^2$) then for all $\lambda_1, \lambda_2, \ldots, \lambda_n$ in $[0..1]$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$ and for all $x_1, x_2, \ldots, x_n \in [a..b]$

$$f(\lambda_1x_1 + \lambda_2x_2 + \cdots + \lambda_nx_n) \leq \lambda_1f(x_1) + \lambda_2f(x_2) + \cdots + \lambda_nf(x_n)$$

If the function is concave downwards the inequality is reversed. An important special case is where each $\lambda_k = \frac{1}{n}$.

8. **Hölder's Inequality**

- **a.** Let $(a_{11}, \ldots, a_{1n}), (a_{21}, \ldots, a_{2n}), \ldots, (a_{k1}, \ldots, a_{kn})$ be sequences of nonnegative real numbers and $\lambda_1, \ldots, \lambda_k$ nonnegative reals satisfying $\lambda_1 + \cdots + \lambda_k = 1$. Then

$$\left(a_{11}^{\lambda_1}a_{21}^{\lambda_2}\cdots a_{k1}^{\lambda_k}\right) + \left(a_{12}^{\lambda_1}a_{22}^{\lambda_2}\cdots a_{k2}^{\lambda_k}\right) + \cdots + \left(a_{1n}^{\lambda_1}a_{2n}^{\lambda_2}\cdots a_{kn}^{\lambda_k}\right) \leq (a_{11} + \cdots + a_{1n})^{\lambda_1}(a_{21} + \cdots + a_{2n})^{\lambda_2}\cdots(a_{k1} + \cdots + a_{kn})^{\lambda_k}$$

  i.e. given a matrix of nonnegative real numbers

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix}$$

  the arithmetic mean of the (weighted) geometric means of the columns is less than or equal to the (weighted) geometric mean of the arithmetic means of the rows.

- **b.** (Generalized Minkowski's and Hölder's) In the matrix above, for any reals $r < s$, the sth power mean of the rth power means of the columns is less than or equal to the rth power means of of the sth power means of the rows.

9. **Bernoulli's Inequality**: For any nonzero real number $x > -1$ and integer $n > 1$

$$1 + nx < (1+x)^n$$

10. **Nesbitt's Inequality**: For all positive reals a, b, c

$$\frac{3}{2} \leq \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}$$

11. **Schur's Inequality**: Given positive real numbers a, b, c and any real number r

$$0 \leq a^r(a-b)(a-c) + b^r(b-a)(b-c) + c^r(c-a)(c-b)$$

12. **Muirhead's Inequality**: Let $0 \leq s_1 \leq \cdots \leq s_n$ and $0 \leq t_1 \leq \ldots \leq t_n$ be real numbers such that $\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} t_i$ and $\sum_{i=1}^{k} s_i \leq \sum_{i=1}^{k} t_i \quad (k = 1, \ldots, n-1)$. Then for any nonnegative numbers $x_1, \ldots, x_n$,

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$$\sum_{\sigma} x_1^{t_{\sigma(1)}} \ldots x_n^{t_{\sigma(n)}} \leq \sum_{\sigma} x_1^{s_{\sigma(1)}} \ldots x_n^{s_{\sigma(n)}}$$

where the sums run over all permutations σ of $\{1,2,\ldots,n\}$.

13. (*See also:* ***Euler's Inequality*** *in Geometry - Triangles - #3b;* ***Ptolemy's Inequality*** *in Geometry - Quadrilaterals - #2*)

Logarithms

If $b > 0$, $b \neq 1$, and $x > 0$ then

1. ⋆ $\log_b(x) + \log_b(y) = \log_b(xy)$ for $y > 0$
2. ⋆ $\log_b(x^y) = y\log_b(x)$ for all y
3. ⋆ $\log_y(x) = \frac{\log_b(x)}{\log_b(y)}$ for $y > 0$ and $y \neq 1$
4. For any functions f and g, $f^{\ln(g)} = g^{\ln(f)}$

Note: $\ln = \log_e$ where $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Cauchy's Functional Equations

1. If f is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ then
   a. If $f(x+y) = f(x) + f(y)$ for every x,y then $f(x) = mx$ for some m.

Complex Numbers

Basics

Let $\mathbb{C} = \mathbb{R}^2$. For each $(x,y) \in \mathbb{C}$ we formally write $(x,y) = x + yi$.
Let $x + yi, a + bi \in \mathbb{C}$, then:

1. ⋆ **Complex conjugate**: $\overline{x+yi} = x - yi$
2. ⋆ **Complex norm**: $|x+yi| = \sqrt{x^2+y^2}$
3. ⋆ **Argument**: $\operatorname{Arg}(x+yi) =$ the angle in $[0\ldots 2\pi)$ of (x,y) in polar form (not defined for $x = y = 0$)
4. ⋆ **Real part**: $\operatorname{Re}(x+yi) = x$
5. ⋆ **Imaginary part**: $\operatorname{Im}(x+yi) = y$
6. ⋆ **Addition**: $(x+yi) + (a+bi) = (x+a) + (y+b)i$
7. ⋆ **Multiplication**: $(x+yi)(a+bi) = (xa - yb) + (ya + xb)i$
8. ⋆ **Complex exponential**: Let $\theta \in \mathbb{R}$. Then $e^{i\theta} = \cos\theta + i\sin\theta$
9. ⋆ **Standard polar form**: of $x + yi$ is $re^{i\theta}$ where $r = |x+yi|$ and $\theta = \operatorname{Arg}(x+yi)$
10. ⋆ **Distance**: between complex numbers z, w is $|z - w|$

Properties

1. ⋆ $e^{i\pi} + 1 = 0$
2. ⋆ Let $\theta, \gamma \in \mathbb{R}$
   a. $e^{i\theta}e^{i\gamma} = e^{i(\theta+\gamma)}$
   b. $|e^{i\theta}| = 1$
   c. $\overline{e^{i\theta}} = e^{i(-\theta)}$
3. ⋆ Let $z, z_1, z_2 \in \mathbb{C}$. Then:
   a. $|z_1 z_2| = |z_1||z_2|$
   b. $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$
   c. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
   d. $z\,\overline{z} = |z|^2$
   e. $|z| = |\overline{z}|$
   f. If $z = re^{i\theta}$ in polar form, then $\overline{z} = re^{i(-\theta)}$

Complex Transformations

1. **Transformation**: of a set S is a bijection from S to S. (a.k.a. a **permutation** of S)
   Let $w \in \mathbb{C}$ and $\theta, k \in \mathbb{R}$.
   a. **Translation by** w: $T(z) = z + w$
   b. **Rotation by** θ **radians counterclockwise about the origin**: $T(z) = e^{i\theta}z$

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c. **Reflection across the x-axis**: $T(z) = \overline{z}$
d. **Homothety by positive factor k with respect to the origin**: $T(z) = kz$
e. **Inversion* with respect to the unit circle**: $T(z) = \frac{1}{\overline{z}}$ (*Inversion is a transformation of the extended complex plane $\mathbb{C}^+ = \mathbb{C} \cup \{\infty\}$ with $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.)

Strategies & Tactics

General

Paul Zietz, in his book *The Art and Craft of Problem Solving* suggests the following strategies and tactics for approaching any problem.

1. Get oriented.
2. Consider the penultimate step.
3. Get your hands dirty.
4. Impose or look for symmetry.
5. Use wishful thinking.
6. Consider a simpler case.
7. Use peripheral vision.
8. Consider the extreme cases.
9. Find an invariant.
10. Draw a picture.

Arithmetic

1. Be careful!

Combinatorics and Probability

1. Use combinatorial arguments to solve binomial coefficient identities.
2. Make a bijection and count something easier, or count the complement.
3. Try recursion - to solve the recursion explicitly:
   a. Conjecture and induct
   b. Use generating functions
   c. Algebraic manipulation
4. Ways to think of binomial coefficients
   a. Coefficients of $(x + y)^n$
   b. The number of ways to choose k things from n things
   c. The elements of Pascal's triangle
5. Use inclusion-exclusion.

Number Theory

Melanie Wood suggested the following ways to approach a number theory problem in her 2005 MOP lectures and notes.

1. Plug in simple values. (*See also Strategies & Tactics - General #3 and #6*)
2. Check values modulo m, where m is carefully chosen.
3. Divide the problem into multiple cases.
4. Consider the orders of values modulo some integer.
5. Don't be afraid to use the quadratic formula.
6. Use infinite descent.
7. Build large numbers with the properties you want, for example using the Chinese Remainder Theorem.
8. Keep in mind that
   a. Consecutive numbers are relatively prime.
   b. If $p \mid a$ and $p \mid b$, then $p \mid (a + b)$.
   c. $a \equiv b \pmod{n}$ if and only if $n \mid a - b$.
9. Ways to tell that a number is an integer:
   a. It is rational and the root of a monic polynomial with integer coefficients.
   b. It is the answer to a counting problem.

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c. It is a term in a recursive sequence with integer initial values and a recursion that is closed for integer values.

Algebra

1. Factor!
2. Make a substitution.
3. Find a telescoping sum.
4. Every polynomial can be thought of
   a. as a function.
   b. in terms of its coefficients.
   c. in terms of its roots.

Geometry

1. Draw very accurate and large diagrams.
2. Make cyclic quadrilaterals and parallel lines.
3. Melanie Wood suggested in her 2005 MOP notes that a problem solver should consider using inversion when the problem contains:
   a. circles.
   b. a busy point (with many circles and lines passing through it).
   c. weird angle conditions.
   d. products of lengths.
   e. reciprocals of lengths.
   f. tangencies and orthogonalities.
4. Use trigonometry for both angle-chasing and side-chasing.
5. Find a useful transformation.
6. Consider area.
7. Collinearity and Concurrency:
   a. Find the special point of intersection or line of concurrence (e.g. orthocenter, Euler line, etc.)
   b. Proof by contradiction.
   c. Try to use Menelaus, Ceva, all projective theorems.

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