Problem Solving - Nonlecture Notes

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Contemplation within activity is a million times better than contemplation within stillness. - Hakuin

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The Way of Problem Solving

- **Art**: Problem solving is an art. Like any art it requires proper attitude, practice, creativity, and passion to master. Like any artist the problem solver creates works of wonder and surprise and sublime aesthetic value.
- **Beauty**: A correct solution is better than no solution. A massive straightforward slog (a.k.a. *dumb-assing*) that gets the correct answer is better than no solution at all. But a *clever* correct solution is better than a straightforward or obvious solution. All else being equal, the shorter the solution, the better. A solution that does not require a calculator or computer is better than one that does. A solution that does not require algebra is better than one that does.
- **Fellowship**: As with any art form, we can benefit from interacting with other artists. By aspiring to learn from those who are more experienced, by cooperating with our peers, and by assisting those who are less experienced, everyone benefits. Several minds can produce several perspectives on the same problem. As with any group of artisans, problem solvers naturally bond together into a community of people who share a common interest.

The life of Zen attainment is not like standing on a riverbank watching the current and appreciating the water or landscape as a witness; it is jumping into the current and becoming one with it. - Leggett

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Problems vs Exercises

Zeitz distinguishes between a *problem* and an *exercise.*

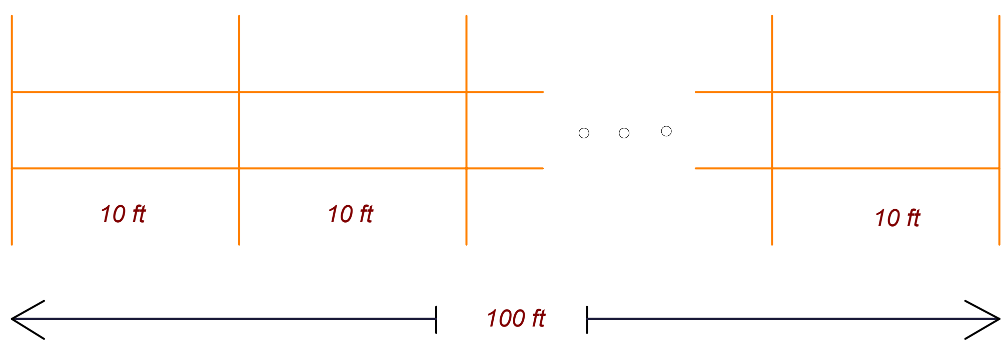
Some Characteristics of a Good Problem

- Your first impression is “This is impossible!”.
- You are surprised, or delighted by the question itself.
- It is simple to state, but hard to answer.
- You’ve never seen a question just like it before.

- You don't know immediately how to solve it.
- It is addictive. You want to know the answer.
- It has an obvious, straightforward, ugly, messy, lengthy solution, but also a clever, ingenious, short, elegant solution.
- It only requires only very elementary mathematics, but is quite challenging nonetheless.
- It has some symmetry, or a story, or a pattern or picture that is aesthetically pleasing.

Some Problem Solving Terminology and Folklore

- **crux move**: Zeitz calls a key insight or key step in the solution to a problem a *crux move*. It is the key realization that allows you to solve the problem, all other parts of the solution being more or less straightforward. A problem can have more than one crux move. It refers to the most difficult, tricky, or creative step or steps in a solution.
- **fence post error**: Probably the number one killer of otherwise perfectly good solutions is being off by one when counting something. This gets its name from a problem similar to this one: A fence is constructed of ten foot sections consisting of two horizontal bars supported by



posts at each end. How many fence posts are needed to construct a fence of this type that is 100 ft long. If your answer is ten, then you have committed the fence post error. We say a proposed solution is "off by a fence post" to mean that the solution is wrong because something was counted incorrectly and the count came out to be either one to high or one two low. As you solve many problems you will learn to have a healthy hatred of fence posts.
- **pwn**: refers to dominating of an opponent, or something great or ingenious applied to methods or objects. This term is used by problem solvers to indicate a great ingenious solution to a problem, one that dominates and completely solves the problem in the best possible manner.
- **spoilers**: it is traditional in problem solving, just as with movies or puzzles, to respect other's rights and desire to enjoy a problem to it's fullest. Therefore it is important not to provide a "spoiler" to someone who is working hard on a problem by revealing the solution or giving them a big hint unless they ask you for it first. Also it is much more rewarding to solve a problem if you only have a little hint than if someone just tells you the solution. A good problem solver respects their fellow problem solvers need to enjoy the solution and will devote some thought to giving "just the right hint" if someone

requests it so as not to spoil their fun.

Why Problem Solving?

- *For Pure Mathematicians*: the closest activity in mathematics to problem solving is mathematical research. In both research and problem solving the mathematician must learn how to solve problems whose solution is not immediately apparent. All of the same strategies, tactics, tools, skills, and attitudes that are used by the problem solver can also be used effectively by the researcher. Problem solving provides an excellent training ground for research in a more well-defined environment in which the problems are known to have a solution and are meant to be solved in only a few hours rather than over several months or years.
- *For Applied Mathematicians*: the applied mathematician or scientist benefits from a problem solving background by practicing to be accurate, careful, creative, and confident when faced with a problem. The applied mathematician is often faced with a complicated or ill defined problem and must use his skills as a problem solver to come to a deeper understanding of the problem and solve it. Just as in problem solving, an efficient solution to a problem is often substantially more valuable than an inefficient one.
- *For Math Teachers*: problem solving has its roots in mathematics competitions starting at the 4th grade level and continuing on up through the Putnam exam. Mathematics teachers who have a problem solving background gain a substantially deeper understanding of the topics they must teach and gain the ability to coach students who wish to participate in mathematical competitions.

The Problem Solving Mindset

There are several attitudes or psychological perspectives that are needed to be a successful problem solver.

- **concentration**: it is easy to get distracted or frustrated by a difficult problem. Problem solving requires sometimes lengthy, intense, focused concentration on a single topic.
- **confidence**: it is important to believe that you will eventually be able to solve a problem, even if you have no idea how to do it at first. Even if you are a beginner at problem solving, you should approach a problem with a confident attitude. Don't worry that you might not remember a key theorem or an important fact. Every problem has to be solved with what you already know.
- **creativity**: a problem solver must always remain open to all and any ideas that may come to mind and always on the lookout for new ways to approach a problem. A change of perspective, a reinterpretation of the question, a nonstandard approach to a otherwise familiar situation can have tremendous benefits. It can also be a dead end. But if even one idea in ten is fruitful, that may be the only one you need to solve the problem.
- **peripheral vision**: when looking at the night sky we can see fainter objects by not looking directly at them. The receptors on the sides of our eyes are more sensitive to faint light than those in the center. Similarly, when solving a problem, we should not always think

about solving the problem itself directly, but rather allow ourselves to ponder things that are perhaps only vaguely related to the problem. This is similar to a smell or gut instinct or intuition that leads you in a certain direction without being 100% certain why you think you ought to go that way. The more you practice, the more reliable your instincts will become.

- **thinking on your feet**: problem solvers strive to develop the ability to think on their feet with the minimal amount of assistance possible. A solution that does not require a calculator or computer is better than one that does. A solution that does not require a pencil and paper is better than one that does. A short elementary solution that does not require any advanced theorems or previously proven results is better than one that does. The problem solver solves problems in the shower, while lying in bed before going to sleep or right after waking up, while running or biking or hiking or driving in the car. The problem solver may actually look forward to time in the waiting room at the doctor's office or dentist as it provides uninterrupted time to work on their problems.
- **stay loose**: The mind is a more flexible and fluid canvass than pencil and blank paper. We can manipulate ideas freely in our mind. Putting something down on paper tends to make it more concrete and cast in stone. The more you practice, the better you will become at not needing paper and pencil to do mathematics. As you do you will sometimes find that you have more success solving difficult problems if you don't use paper than if you do! Especially at the beginning, when you first approach a problem, it is important to stay loose and flexible. Working mostly in your head is often the best way to do that. Once you have an epiphany and see the crux move, it may then be time to break out the paper or calculators.
- **be careful**: without accuracy and care, stupid mistakes can easily turn an otherwise correct solution into an incorrect one. Also some problems may be easy to solve if you do them correctly but a hideous nightmare if you make a small mistake. The problem solver must also strive to be sure that every case has been considered and that there is no omission in the solution that could catastrophic.

If you would be a real seeker after truth, it is necessary that at least once in your life you doubt, as far as possible, all things. - DESCARTES

Master Zeitz's Threefold Path

The experienced problem solver operates on three different levels:

1. **Strategy**: mathematical and psychological ideas for starting and pursuing problems.
2. **Tactics**: Diverse mathematical methods that work in many different settings.
3. **Tools**: Narrowly focused techniques and "tricks" for specific situations.

If you do not follow the right path, you will be lost. - THE BUDDHA

Strategies

Zeitz identifies the following strategies.

Get Oriented

Get Oriented: Take time to understand exactly what the question is asking. Notice *every word* and make a mental inventory of everything you are given, and exactly what you are asked. Words and phrases like “positive” or “at most” or “integer” or “unique” can be crucial. Also be aware of what the question does *not* say. Don’t assume anything that isn’t stated in the question and don’t ignore anything that is.

Common pitfalls: It is very easy to interpret a question the wrong way by skipping a single word, or incorrectly identifying it as another similar question that you are more familiar with.

Example: Suppose $12a + 10b = 1020$. Find $\frac{a}{5} + \frac{b}{6}$.

Discussion: In a question like this it is as important to realize what it is *not* asking as what it *is* asking. It is not asking you to determine the values of a and b. How can that help?

Read the question! - Monks

Get Your Hands Dirty

Get Your Hands Dirty: Try some sample computations. Do some experiments. Draw some pictures. Build models. Play with the “toys” that are given to you in the question. Muck around. If the question asks you to prove something for all natural numbers n, try it for $n = 0, 1, 2, 3, 4, 5$. Playing and computing and doing sample calculations and experimenting can build insight into what is actually going on. If you are very lucky, sometimes a few sample computations are all that is needed to solve the problem.

Example: Find all prime numbers that are the sum of four consecutive prime numbers.

Discussion: How can we get our hands dirty in this problem?

Example: Suppose $12a + 10b = 1020$. Find $\frac{a}{5} + \frac{b}{6}$.

Discussion: How can we get our hands dirty in such a problem? Can it be helpful?

Practice until concepts have become so obvious, so intuitive, that you could handle them without thinking - in your sleep. You must see them in your eye, have them right in your fingers. - Benoit Mandelbrot

Consider the Penultimate Step

Consider the Penultimate Step: It is often helpful to consider what the next to last step in the solution could be in order to solve the question. This is "working" backwards from the desired goal. This can be generalized by considering the step before the penultimate step, and so on, working backwards from the goal and forwards from the hypotheses in the hope of meeting up somewhere in the middle.

Common Pitfalls: Note that often there is more than one penultimate step possible, and you should remain open to all possibilities rather than committing yourself to the first plan of attack that comes to mind, which may inevitably prove to be impossible or unwieldy.

Example: In triangle $\triangle ABC$, point D on BC is equidistant from the vertices $A, B,$ and C. Prove that $|AB|^2 + |AC|^2 = |BC|^2$.

Discussion: Where have we seen this kind of equation before? What penultimate step would suffice to prove such an equation?

Consider a Simpler Problem

Consider a Simpler Problem: Another way to gain an insight into a difficult problem is to try solving a simpler problem that is similar to the difficult one. This may involve solving the same problem with fewer variables, or smaller numbers.

Example: How many ordered triples of positive integers sum to 20?

Discussion: What is problems can you think of that are similar to this one, but seem to be simpler? Can solving the simpler problem help?

If the given problem is too hard, solve an easier one. - ZEITZ

Wishful Thinking

Wishful Thinking: It is sometimes helpful to consider something that is blatantly false that you wish were true because it would make the problem much easier to solve. In addition to giving some glimpses of how the problem might be solved, understanding why the thing you wish to be true is false is often essential to understanding the key difficulty in the problem.

Example: The product of five consecutive integers is 2441880. What is the largest of the five integers?

Discussion: How can we attack this with wishful thinking? Can you solve it cleverly without a calculator and with the minimum amount of arithmetic (i.e. without dumb-assing it)?

Good, obedient boys and girls solve fewer problems than naughty and mischievous ones. - ZEITZ

Common Strategies for Proofs

Informally, a **proof** is just an explanation that logically guarantees the correctness of a statement. More formally, a proof is a sequence of statements that are either (a) a definition (b) a previously proven statement or (c) follow from previous statements in the proof by logical **rules of inference**. A simple example of formal proofs are the statement-reason proofs that you did in high school geometry.

In a problem solving, most proofs are not written in that statement-reason two column format but rather are written informally as careful detailed explanation that leads the reader inexorably to the desired conclusion with no ambiguity. But even though the style of informally written proofs differs from that of a formal proof, the content is exactly the same: every claim in your proof must follow from previous statements by the rules of logic or must be a previously proven result, definition, or a fact given in the statement of the problem. Thus, you should strive to give reasons for every claim you make in a proof and show why it follows from what you have said previously.
There are several very common arguments used in proofs that every problem solver must be familiar with.

1. **Proving a Conditional statement**: To prove a statement of the form "If P then Q" where P and Q are statements, you should assume P is true and then prove that Q is true under that assumption.
2. **Proving a Contrapositive**: To prove a conditional statement like "If P then Q" it is sometimes easier to prove the equivalent statement "If not Q then not P" by assuming that Q is false and showing that P is false under that assumption.
3. **Proof by Contradiction**: To prove a statement P sometime it is useful to assume P is false and then show that that assumption leads to a contradiction, i.e. that you can prove both some statement and its negation.
4. **Proof by Cases**: If you know that either P or Q is true, and want to show R, you can prove R by considering separate cases. In the first case, assume P and prove R. In the second case assume Q and show R. Since one of P or Q is known to be true, R must be true as well. This method also generalizes to situations where you have more than two cases to consider.
5. **Proof by Induction**: Let $P(n)$ be a statement about an unspecified natural number n. To prove that P is true for all natural number values of n, show $P(0)$ is true, then let k be a natural number and assuming $P(k)$ is true, show that $P(k+1)$ is also true.
6. **Proof of Univerality**: To prove that a statement is true about every element in some set, let x be an arbitrary, unspecified element of that set and prove that the statement is true

about x.

7. **Proof of Existence**: To prove that a mathematical object with certain properties exists, either make an example of such an object (i.e. construct one), or show that if it didn't exist there would be a contradiction (i.e. use proof by contradiction).

If it is a Miracle, any sort of evidence will answer, but if it is a Fact, proof is necessary.
- MARK TWAIN

Tactics

Symmetry

A mathematical object (shape, expression, system of equations, etc.) is **symmetric** with respect to some action or operation if it is unchanged by the action or operation. The actions that do this are called the **symmetries** of the object.

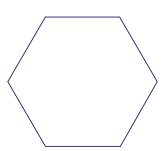
***Tactic**: When a problem has symmetry, try to use it. Try to maintain that symmetry while solving the problem. If a problem doesn't have symmetry, but you wish it did, try to introduce symmetry into the situation if possible.*

Example: 2002 Workout 8, number 8.

A **group** is a set together with an associative binary operator on that set that has an identity element and inverses for every element in the set. The set of symmetries of an object often forms a group.

Example: The set $\{1,2,3,\ldots,n\}$ is unchanged by permuting its elements. The set of all permutations of $\{1,2,3,\ldots,n\}$ is called the **symmetric group** S_n.

Example: Plane geometric figures which are unchanged by reflection across a line, or rotation through a certain angle, or inversion in a circle, or translation by a fixed vector, etc. are said to be symmetric with respect to that line, rotation, inversion, translation, etc.



The set of all isometries (bijections from the plane to itself that preserve distance) which map a given figure to itself is called the **symmetry group** for that figure.

Example: A function f is called a **symmetric function** in n variables if

$$f(x_1,x_2,\ldots,x_n) = f(x_{\sigma(1)},x_{\sigma(2)},\ldots,x_{\sigma(n)})$$

for permutation σ of $\{1,2,3,\ldots,n\}$ and any values of $x_1,x_2,\ldots,x_n$. If $f(x_1,x_2,\ldots,x_n)$ is a polynomial in $x_1,x_2,\ldots,x_n$ we say $f(x_1,x_2,\ldots,x_n)$ is a **symmetric polynomial**. e.g. here are some symmetric polynomials in two variables:

$$x+y$$
$$x^2+y^2$$
$$x^2y+xy^2$$
$$x^3+x^2y+xy^2+y^3$$

Example: A system of equations in n variables is a symmetric with respect to its variables if permuting the variables leaves the system unchanged. For example,

$$x+y=z$$
$$x+z=y$$
$$y+z=x$$

In such situations it is usually in your best interest to try to maintain the symmetry in whatever algebraic operations you do, i.e. do the same thing to all three equations at once rather than operating on one equation at a time. For example, find $x+y+z$ if $x,y,$ and z satisfy the above system of equations. Similar situations can arise with inequalities.

Example: Triangle problem from 1989 friendly competition.

Look for harmony and beauty, whenever you investigate a problem. If you can do something that makes things more harmonious or more beautiful, even if you have no idea how to define these two terms, then you are often on the right track. - ZEITZ

The Extreme Principle

Definition *A **poset** (or **partially ordered set**) is a pair (A,R) where A is a set and R is a relation on A that is reflexive, transitive, and antisymmetric:*
1. (reflexive) $\forall a\in A,\ aRa$
2. (transitive) $\forall a,b,c\in A,\ aRb$ and $bRc\Rightarrow aRc$
3. (antisymmetric) $\forall a,b\in A,\ aRb$ and $bRa\Rightarrow a=b$

Note that a poset might not have a largest or smallest element.

Definition *A poset (A,R) is **well-ordered** if and only if it satisfies these two conditions:*
1. (totally ordered) $\forall a,b\in A, aRb$ or bRa
2. Every subset of A has a least element.

***Tactic**: Whenever possible put the elements of your problem in some order. Focus on the largest and smallest (i.e. extreme) elements in this order as they may be constrained in interesting ways.*

The most common orderings and partial orderings encountered in problems include:

- $<$ and $\leq$ on sets of real numbers
- $\subseteq$ and $\subsetneq$ on sets of sets
- alphabetical and lexicographic ordering on strings and tuples
- ordering polynomials by degree
- ordering complex numbers by absolute value

Note that you can often order your elements in more than one way, so that you should use the strategy of *staying loose* when considering orderings as one particular ordering might be more useful than another.

One ordering principle that can be very helpful when dealing with sets of natural numbers is actually an axiom of the natural numbers:

Well Ordering Principle: Every set of natural numbers has a least element.

Example: The handshake problem 1.1.4

Example: Given that 7 distinct positive integers add up to 100, prove that some three of them add up to at least 50.

Example: Given $2n + 2$ points in the plane, no three collinear, prove that two of them determine a line that separates n of the points from the other n.

The Pigeonhole Principle

Pigeonhole Principle: If you have n pigeons in k holes some hole contains at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons and some hole contains at most $\left\lfloor \frac{n}{k} \right\rfloor$ pigeons.

This is more powerful than it looks. Note that the pigeonhole principle does not tell you which hole contains the number of pigeons indicated, nor that any hole contains exactly that number of pigeons. It is just tells you that *some* hole satisfies the condition given.

Example: Given $n + 1$ positive integers prove that there are two of them whose difference is divisible by n.

Example: Show that if there are n people at a party, then two of them know the same number of people (among those present, assuming "knowing" is symmetric).

Example: Show that if five distinct points are placed inside an equilateral triangle of side length two, there are two distinct points that are a distance less than or equal to one apart.

Example: Prove that there exist integers a, b, c not all zero and each of absolute value less than one million such that

$$\left| a + b\sqrt{2} + c\sqrt{3} \right| < 10^{-11}$$

Invariants

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*An **invariant** is any quantity or property that remains the same under some operation or is the same for all elements of a set.*

Example: Consider a point $P = (x, y)$ on the unit circle. As P moves around the circle it's x and y coordinates change, but the value of $x^2 + y^2$ is always 1.

Example: There is one stone at each vertex of a square. We are allowed to change the number of stones according to the following rule: We may take away any number of stones from any vertex and add twice as many stones to the pile at one of the adjacent vertices. Is it possible to get 2004, 2003, 2005, and 2004 stones at consecutive vertices after a finite number of moves?

***Parity** and **congruence** are common invariants.*

Example: For example, the operations $f(n) = 3n + 1$ always maps an integer to another integer having the same parity after two iterations. The operation $g(n) = 10 - n$ always maps a number to an integer that is congruent to n mod 5 after two iterations.

Example: Let there be nine lattice points in a three-dimensional Euclidean space. Show there is a lattice point on the interior of one of the line segments joining two of these points.

Example: Is it possible to start with a knight in some corner of a chessboard and reach the opposite corner by a sequence of legal moves that pass through every square exactly once?

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*A **monovariant** is any quantity that strictly increases or strictly decreases under some operation.*

A monovariant often has positive integer values and is strictly decreasing, so that after finitely many steps it is zero.

Example: In the stone-and-square problem above, the total number of stones on all four corners is a monovariant. It goes up by one for each stone removed.

Example: At a round table are 2004 girls, playing a game with a deck of n cards. Initially one girl holds all the cards. At each turn, if at least one girl holds at least two of these cards, one of these girls must pass a card to each of her two neighbors. The game ends when, and only when each girl is holding at most one card. Prove that the game will end if and only if $n < 2004$.

Crossover Tactics

Graph Theory

*A **graph** is a is a pair of sets (N,E) where E is a set of one or two element subsets of N. A **directed graph** (or **digraph**) is a pair of sets (N,E) where $E \subseteq N \times N$. The elements of N are called **nodes** and the elements of E are called **edges**.*

Graphs and digraphs are usually represented pictorially by drawing the nodes as points and connecting nodes x and y with an arrow whenever (x,y) or a curve if $\{x,y\}$ is an edge. The positioning of the nodes and the shapes of the edges drawn is irrelevant.

Graphs vs. Relations: Since every relation on N can be represented as a subset of $N \times N$, the set E in any digraph is a relation on N. Similarly, given a relation R on N we can construct the digraph of that relation. Thus both are just different representations of the same mathematical concept. Symmetric relations can be represented by a graph instead of a digraph. This explains why graph theory is a crossover tactic!

Graph mini-lexicon:

- If N is finite we say the graph or digraph (N,E) is also **finite**. The number of nodes in a finite graph or digraph is called the **order** of the graph.
- A graph or digraph is **simple** if it has not edges from a point to itself.
- (N',E') is a **subgraph** of graph (N,E) if (N',E') is a graph and $N' \subseteq N$ and $E' \subseteq E$.
- A **path** from x_1 to x_n in a graph is a sequence of nodes $x_1, x_2, \ldots, x_n$ having an edge between any two consecutive terms in the sequence (these edges are said to be in the path) where no edge is in the path more than once. In the case where $x_1 = x_n$ we say the path is a **cycle**.
- A graph having a path between every two vertices is a **connected graph**.
- A connected graph with no cycles is called a **tree**.
- The **degree** of a node in a finite graph is the number of edges connected to that node. In a finite digraph we have the **out-degree** and **in-degree** of each node which is the number of edges leaving and entering the node, respectively.
- A **bipartite graph** is a graph whose nodes can be partitioned into two subsets U and V such that every edge connects an element of U to an element of V.

- An **Eulerian path** is a path in a graph that contains every edge.
- A **Hamiltonian path** is a path in a graph that contains every vertex.
- A graph is **complete** if there is an edge connecting every pair of nodes. A bipartite graph is **complete** if there is an edge between every pair (u, v) with $u \in U$ and $v \in V$.

Playbook Facts about Graphs

Connected components: The relation "there is a path from x to y" partitions a graph into a disjoint union of connected subgraphs. These are called the connected components of the graph.

Handshake Lemma: In any finite graph the sum of the degrees of the nodes is twice the number of edges.

Existence of Eulerian paths and cycles: A graph has an Eulerian path if and only if it is connected and the number of nodes of odd degree is either two or zero. A path has an Eulerian cycle if and only if it is connected and every vertex has even degree.

Existence of Hamiltonian cycles - Dirac's Theorem: A simple graph with n nodes has an Hamiltonian cycle if the degree of every node is at least $n/2$.

Existence of Hamiltonian cycles - Ore's Theorem: A simple graph with n nodes has an Hamiltonian cycle if whenever two nodes are not connected by an edge the sum of their degrees is at least n.

Ramsey's Theorem: Let $N(a, b)$ be the smallest number such that any group of $N(a, b)$ people must contain either a mutual friends or b mutual strangers.
a. $N(a, b) = N(b, a)$
b. $N(a, 2) = a$
c. $N(a, b) \leq N(a-1, b) + N(a, b-1)$

Examples

Example: There are 59 people at a party. Prove that someone shook hands an even number of times.

Example: Show that in any group of six people there are either three who are mutual friends or three who are mutual strangers.

Example: There are n people at a party. For any two people at the party who are not friends, the sum of the number of people at the party that each is friends with is at least n. Prove everyone at that party can be seated at a round table so that nobody sits next to anyone who is not their friend. (You may assume that nobody is their own friend.)

Note: this is actually the proof of Ore's Theorem in disguise!

Complex Numbers

Definition Let $\mathbb{C} = \mathbb{R}^2$. For each $(x,y) \in \mathbb{C}$ we formally write $(x,y) = x + yi$. This form, $x + yi$, is called the ***standard form*** of the complex number (x,y).

Definition Let $x + yi, a + bi \in \mathbb{C}$, then:
1. $\overline{x + yi} = x - yi$. (This is called the ***complex conjugate***.)
2. $|x + yi| = \sqrt{x^2 + y^2}$. (This is called the ***complex norm***.)
3. $\operatorname{Arg}(x + yi) =$ the angle in $[0 \ldots 2\pi)$ of (x,y) in polar form (not defined for $x = y = 0$). (This is called the ***Argument*** of $x + yi$.)
4. $\operatorname{Re}(x + yi) = x$. (This is called the ***real part*** of $x + yi$.)
5. $\operatorname{Im}(x + yi) = y$. (This is called the ***imaginary part*** of $x + yi$.)
6. $(x + yi) + (a + bi) = (x + a) + (y + b)i$. (This is the definition of ***addition*** in $\mathbb{C}$.)
7. $(x + yi)(a + bi) = (xa - yb) + (ya + xb)i$. (This is the definition of ***multiplication*** in $\mathbb{C}$.)

Notation We can abbreviate $0 + yi$ as yi, $x + 0i$ as x, $x + 1i$ as $x + i$, and $x - 1i$ as $x - i$ with no ambiguity in the above definitions. With this notation $i = (0,1)$ and $i^2 = -1$. It is easy to verify that the usual laws of addition and multiplication (associative, commutative, distributive, identity, etc.) hold for the complex numbers as well.

Definition Let $\theta \in \mathbb{R}$. Then $e^{i\theta} = \cos\theta + i\sin\theta$

Definition Let $x + yi \in \mathbb{C} - \{0\}$. The ***standard polar form*** of $x + yi$ is $re^{i\theta}$ where $r = |x + yi|$ and $\theta = \operatorname{Arg}(x + yi)$.

Definition The ***distance*** between two complex numbers z, w is denoted $d(z,w)$ and is defined to be $d(z,w) = |z - w|$.

Theorem Let $\theta, \gamma \in \mathbb{R}$
1. $e^{i\theta}e^{i\gamma} = e^{i(\theta+\gamma)}$.
2. $|e^{i\theta}| = 1$.
3. $\overline{e^{i\theta}} = e^{i(-\theta)}$.

Theorem Let $z, z_1, z_2 \in \mathbb{C}$. Then:
1. $|z_1 z_2| = |z_1||z_2|$
2. $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$ i.e. the conjugate of a product is the product of conjugates.
3. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ i.e. the conjugate of a sum is the sum of the conjugates.
4. $z\,\overline{z} = |z|^2$
5. $|z| = |\overline{z}|$
6. If $z = re^{i\theta}$ in polar form, then $\overline{z} = re^{i(-\theta)}$

Complex Transformations of the Plane

Definition *A **transformation** of a set S is a bijection from S to S.*

Remark *In other branches of mathematics a transformation of S is often called a **permutation** of S.*

Some Useful Geometric Transformations Let $w \in \mathbb{C}$ and $\theta, k \in \mathbb{R}$.

Translation by w: $T(z) = z + w$

Rotation by θ radians counterclockwise about the origin: $T(z) = e^{i\theta}z$

Reflection across the x-axis: $T(z) = \overline{z}$

Homothety by positive factor k with respect to the origin: $T(z) = kz$

Inversion* **with respect to the unit circle**: $T(z) = \frac{1}{\overline{z}}$

*Inversion is a transformation of the extended complex plane $\mathbb{C}^{+} = \mathbb{C} \cup \{\infty\}$ with $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

Remark *You can compose these functions to obtain many useful transformations!*

Examples

Example: (*Arithmetic*) Prove that if an integer can be written as a sum of two squares, then so can any positive integer power of that integer.

Example: (*Algebra*) Factor $z^5 + z + 1$ (as a polynomial with integer coefficients).

Example: (*Trigonometry*) Express $\cos(5\theta)$ in terms of $\cos(\theta)$.

Example: (*Geometry*) Let $ABCD$ be a convex quadrilateral and construct a square on each side lying outside of the quadrilateral. Show that the two line segments connecting the centers of the two pairs of opposite squares are the same length and perpendicular to each other.

Generating Functions

A generating function is a clothesline on which we hang up a sequence of numbers for display. - HERBERT WILF (FROM GENERATINGFUNCTIONOLOGY)

Definition *The **generating function** of a sequence of integers* $a_0, a_1, a_2, \ldots$ *is the formal power series*

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

Remark *A generating function defines a function*

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

defined for all x *for which the series converges.*

Remark *Variations such as* $\sum_{n=0}^{\infty} \frac{a_n}{n!}x^n$ *are sometimes useful.*

Main Tools of the Generatingfunctionologist

Geometric Series: $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$

Manipulating ordinary generating functions: Let $A(x) = \sum_{n=0}^{\infty} a_nx^n$ and $B(x) = \sum_{n=0}^{\infty} b_nx^n$

1. **a.** **i.** $A(x) = B(x)$ if and only if $a_n = b_n$ for all n.

ii. $\frac{A(x)}{(1-x)} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k \right) x^n$

iii. $A(x)B(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_kb_{n-k} \right) x^n$

iv. $xA'(x) = \sum_{n=0}^{\infty} na_nx^n$

v. ${}^{\backprime}A(x) = C + \sum_{n=1}^{\infty} \frac{a_{n-1}}{n}x^n$

Partial Fraction Decomposition: If $p(x) \in \mathbb{R}[x]$ has degree less than $k + 2m$ and $l_1(x) \cdots l_k(x) \in \mathbb{R}[x]$ are irreducible linear polynomials and $q_1(x) \cdots q_m(x) \in \mathbb{R}[x]$ are irreducible quadratic polynomials then there exist real numbers $A_1, \ldots, A_k, B_1, \ldots, B_k, C_1, \ldots, C_k$ such that

$$\frac{p(x)}{l_1(x) \cdots l_k(x) q_1(x) \cdots q_m(x)} = \frac{A_1}{l_1(x)} + \cdots + \frac{A_k}{l_k(x)} + \frac{B_1x + C_1}{q_1(x)} + \cdots + \frac{B_mx + C_m}{q_m(x)}$$

Taylor Series: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$

Examples

Example: Let $a_0, a_1, a_2, \ldots$ be a sequence of positive integers satisfying $a_0 = 1$ and $a_n = 2a_{n-1} + 3^n$ for $n \geq 1$. Find a closed formula for a_n.

Example: Prove the hockey stick identity using generatingfunctionological methods.

$$\binom{k}{k}+\binom{k+1}{k}+\binom{k+2}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1}$$

Example: Let C_n be the number of ways to distribute n cookies to three children so that no child has fewer than two cookies or more than four cookies, and there is no shortage of cookies or cookies left over after the distribution. Compute C_n for all n.

Tools

Thinking on your feet: Mental Arithmetic

Problem solvers become adroit at mental arithmetic. It is something that is developed over time as you practice. Along the way you will learn and develop many tricks for doing arithmetic in your head. Here is a list of a few common ones. There are plenty more! We will go over them in class.

Prime Factorization is your best friend!

The problem solver often prefers to think of a positive integer as a product of primes, not as it's base ten representation.

- **Fundamental Theorem of Arithmetic**: Every positive integer n can be written uniquely as a product of prime powers in increasing order of the primes, i.e. there is a unique sequence of nonnegative integer exponents $e_1, e_2, \ldots$ such that

$$n = 2^{e_1}3^{e_2}5^{e_3}7^{e_4}\cdots$$

- **Divisibility Tests**: are very useful for finding the prime factorization of small numbers
- **Modular Arithmetic**: divisibility tests also give the remainder for integer division. The sum of the remainders is the remainder of the sum and the product of remainders is the remainder of the product.
- **Primality Criteria**: a positive integer p is prime if it is not divisible by any prime n such that $n^2 \leq p$.
- **Applications to Arithmetic**: multiplying fractions, reducing fractions, computing gcd and lcm, modular arithmetic. It is often useful to leave numbers in their prime factorization form when doing arithmetic rather than multiplying the prime factorization out to get the base ten representation.
- **Cancel**, **cancel**, **cancel**: when multiplying fractions always cancel first!
- **Primes are Good Luck**!: always check if your phone number, home address, runner number, lotto number, etc. is prime. They are good luck!

Memorization: not fun, but useful

There is no doubt that memorization is required to be good at mental arithmetic. Everyone memorizes their times tables for example. There are some other things that come up a lot and can be quite valuable to a problem solver. Of course, the list is not limited to these items, and anything you do beyond this list is certainly going to be of value to you. But these are some of the common things that problem solvers memorize.

- Squares
- Cubes
- Factorials
- Powers of two
- Decimal representations of fractions
- Prime numbers
- Useful Prime Factorizations
- Decimal approximations of $\pi, e, \sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots, \ln(2), \ln(3), \ldots$ etc.
- Pythagorean triples

Specific tricks

There are also a host of specific tricks that only apply to a limited situation, but that situation comes up often enough to make the tricks worthwhile.

- Decimal Representations of Sevenths
- Powers of eleven
- Squaring two digit numbers that end in five
- Squaring numbers that are close to a number whose square you know
- Multiplying by a number that ends in 9 or 1
- Left to right operations, especially subtraction
- Fractions are usually easier than decimals
- Famous numbers like 105 and 1001
- converting eventually repeating decimals to fractions