Some Polynomial Theorems

by

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This paper contains a collection of 31 theorems, lemmas, and corollaries that help explain some fundamental properties of polynomials. The statements of all these theorems can be understood by students at the precalculus level, even though a few of these theorems do not appear in any precalculus text. However, to understand the proofs requires a much more substantial and more mature mathematical background, including proof by mathematical induction and some simple calculus. Of significance are the Division Algorithm and theorems about the sum and product of the roots, two theorems about the bounds of roots, a theorem about conjugates of irrational roots, a theorem about integer roots, a theorem about the equality of two polynomials, theorems related to the Euclidean Algorithm for finding the GCD of two polynomials, and theorems about the Partial Fraction Decomposition of a rational function and Descartes's Rule of Signs. It is rare to find proofs of either of these last two major theorems in any precalculus text.

1. The Division Algorithm

If $p(x)$ and $d(x) \not\equiv 0$ are any two polynomials then there exist unique polynomials $q(x)$ and $r(x)$ such that $p(x) = d(x) \cdot q(x) + r(x)$ where the degree of $r(x)$ is strictly less than the degree of $d(x)$ when the degree of $d(x) \geq 1$ or else $r(x) \equiv 0$.

Division Algorithm Proof:

We apply induction on the degree n of $p(x)$. We let m denote the degree of the divisor $d(x)$.
We will establish uniqueness after we establish the existence of $q(x)$ and $r(x)$.

If $n = 0$ then $p(x) = c$ where c is a constant.

Case 1: $m = 0$.

$d(x) = k$ where k is a constant and since $d(x) \not\equiv 0$ we know $k \neq 0$.
In this case choose $q(x) = \dfrac{c}{k}$ and choose $r(x) \equiv 0$.
Then $d(x) \cdot q(x) + r(x) = k \cdot \dfrac{c}{k} + 0 = c = p(x)$. In this case $r(x) \equiv 0$.

Case 2: $m > 0$.

In this case let $q(x) \equiv 0$ and let $r(x) = c$. Then clearly $d(x) \cdot q(x) + r(x) = d(x) \cdot 0 + c = 0 + c = c = p(x)$. In this case the degree of $r(x)$ is strictly less than the degree of $d(x)$.

Now assume there exist polynomials $q_1(x)$ and $r_1(x)$ such that $p_1(x) = d(x) \cdot q_1(x) + r_1(x)$ whenever $p_1(x)$ is any polynomial that has a degree less than or equal to k.
Let $p(x)$ be a polynomial of degree $k+1$. We assume $p(x) = a_{k+1}x^{k+1} + a_k x^k + \cdots + a_1 x + a_0$ where $a_{k+1} \neq 0$. We must show the theorem statement holds for $p(x)$.

Case 1: $m = 0$.

$d(x) = k$ where k is a constant and since $d(x) \not\equiv 0$ we know $k \neq 0$.
Let $q(x) = \dfrac{1}{k} p(x)$ and let $r(x) \equiv 0$.
Then $d(x) \cdot q(x) + r(x) = k \cdot \dfrac{1}{k} p(x) + 0 = p(x) + 0 = p(x)$. In this case $r(x) \equiv 0$.

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Case 2: $m > 0$.

Let $d(x) = d_m x^m + \cdots + d_1 x + d_0$ where $d_m \neq 0$. Note that $\frac{a_{k+1}}{d_m} \neq 0$ since both constants are nonzero. Let $p_1(x) = p(x) - \frac{a_{k+1}}{d_m} x^{k+1-m} \cdot d(x)$. Then the subtraction on the right cancels the leading term of $p(x)$ so $p_1(x)$ is a polynomial of degree k or less and we can apply the induction assumption to $p_1(x)$ to conclude there exist polynomials $q_1(x)$ and $r_1(x)$ such that $p_1(x) = d(x) \cdot q_1(x) + r_1(x)$ where the degree of $r_1(x)$ is strictly less than that of $d(x)$.

$$p_1(x) = d(x) \cdot q_1(x) + r_1(x) = p(x) - \frac{a_{k+1}}{d_m} x^{k+1-m} \cdot d(x)$$

Now we solve the 2nd equation for $p(x)$.

$$p(x) = \frac{a_{k+1}}{d_m} x^{k+1-m} \cdot d(x) \; + \; d(x) \cdot q_1(x) + r_1(x)$$

$$p(x) = d(x) \cdot \left[\frac{a_{k+1}}{d_m} x^{k+1-m} + q_{1(x)} \right] + r_1(x).$$

So we may let $q(x) = \left[\frac{a_{k+1}}{d_m} x^{k+1-m} + q_{1(x)} \right]$ and let $r(x) = r_1(x)$ and we have established the theorem holds for $p(x)$ of degree $k+1$.

The induction proof that establishes the existence part of the theorem is now complete.

To establish uniqueness, suppose $p(x) = d(x) \cdot q_1(x) + r_1(x) = d(x) \cdot q_2(x) + r_2(x)$. Then we have $d(x) \cdot [q_1(x) - q_2(x)] = r_2(x) - r_1(x)$. Call this equation (*).

Case 1: $m = 0$.

In this case both remainders must be identically zero and this means $r_1(x) \equiv r_2(x)$. In turn, this means $d(x) \cdot [\, q_1(x) - q_2(x) \,] \equiv 0$, and since $d(x) \not\equiv 0$ we must have $q_1(x) - q_2(x) \equiv 0$ which of course implies $q_1(x) \equiv q_2(x)$.

Case 2: $m > 0$.

If $[\, q_1(x) - q_2(x)] \not\equiv 0$ then we can compute the degrees of the polynomials on both sides of the (*) equation. The degree on the left side is greater than or equal to the degree of $d(x)$. But on the right side, both remainders have degrees less than $d(x)$ so their difference has a degree that is less than or equal to that of either which is less than the degree of $d(x)$. This is a contradiction. So we must have $[\, q_1(x) - q_2(x)] \equiv 0$ and when this is the case the entire left side of the (*) equation is identically 0 and we may add back $r_1(x)$ from the right side to conclude that the two remainders are also identically equal.

Q.E.D.

2. The Division Check for a Linear Divisor

Consider dividing the polynomial $p(x)$ by the linear term $(x-a)$. Then, the ***Division Check*** states that: $p(x) = (x-a)\cdot q(x) + r$

Division Check Proof:

This is just a special case of the Division Algorithm where the divisor is linear.

Q.E.D.

3. Remainder Theorem

When any polynomial $p(x)$ is divided by $(x-a)$ the remainder is $p(a)$.

Remainder Theorem Proof:

By the Division Check we have $p(x) = (x-a)\cdot q(x) + r$.
Now let $x = a$. This last equation says $p(a) = (a-a)\cdot q(a) + r$
$p(a) = 0\cdot q(a) + r = 0 + r = r$.

Q.E.D.

4. Factor Theorem

$(x-a)$ is a factor of the polynomial $p(x)$ if and only if $p(a) = 0$.

Factor Theorem Proof:

Assume $(x-a)$ is a factor of $p(x)$. Then we know $(x-a)$ divides evenly into $p(x)$. The remainder when $p(x)$ is divided by $(x-a)$ must be 0. By the Remainder Theorem this says $0 = r = p(a)$. Next, assume $p(a) = 0$. Divide $p(x)$ by $(x-a)$. By the Remainder Theorem, the remainder is $p(a) = 0$. Since the remainder is 0, the division comes out even so that $(x-a)$ is a factor of $p(x)$.

Q.E.D.

5. Maximum Number of Zeros Theorem

A polynomial cannot have more real zeros than its degree.

Maximum Number of Zeros Theorem Proof:

By contradiction. Suppose $p(x)$ has degree $n \geq 1$, and suppose $a_1, a_2, \ldots, a_n, a_{n+1}$ are n+1 roots of $p(x)$. By the Factor Theorem, since $p(a_1) = 0$ then there exists a polynomial $q_1(x)$ of degree one less than $p(x)$ such that $p(x) = (x-a_1)\cdot q_1(x)$. Now since $p(a_2) = 0$ and since $a_2 \neq a_1,$ we must have $q_1(a_2) = 0$ and again by the Factor Theorem we can write $p(x) = (x-a_1)\cdot(x-a_2)\cdot q_2(x)$ where $q_2(x)$ is of degree 2 less than $p(x)$. Now since a_3 is distinct from a_1 and a_2 we must have $q_2(a_3) = 0$ and we can continue to factor $p(x) = (x-a_1)\cdot(x-a_2)\cdot(x-a_3)\cdot q_3(x)$ where the degree of $q_3(x)$ is of degree 3 less than $p(x)$. Clearly this argument can be repeated until we reach the stage where $p(x) = (x-a_1)\cdots(x-a_n)\cdot q_n(x)$ and $q_n(x)$ is of degree n less than $p(x)$. Since $p(x)$ only had degree n in the first place, $q_n(x)$ must be of degree 0 making $q_n(x)$ some constant, say $q_n(x) = c$. Now a_{n+1} is still a zero of $p(x),$ and since a_{n+1} is distinct from all the other a_i, we must have $q_n(a_{n+1}) = 0$. The only way this can happen is if $c = 0$ and this would imply $p(x) \equiv 0$, a contradiction since we are assuming $n \geq 1$.

Q.E.D.

6. Fundamental Theorem of Algebra

a) Every polynomial of degree $n \geq 1$ has at least one zero among the complex numbers.
b) If $p(x)$ denotes a polynomial of degree $n,$ then $p(x)$ has exactly n roots, some of which may be either irrational numbers or complex numbers.

Fundamental Theorem of Algebra Proof:
This is not proved here. Gauss proved this in 1799 as his Ph.D. doctoral dissertation topic.

7. Product and Sum of the Roots Theorem

Let $p(x) = 1x^n + a_{n-1}x^{n-1} + \cdots + a_3x^3 + a_2x^2 + a_1x + a_0$ be any polynomial with real coefficients with a ***leading coefficient of 1*** where $n \geq 1$. Then a_0 is $(-1)^n$ times the product of all the roots of $p(x) = 0$ and a_{n-1} is the opposite of the sum of all the roots of $p(x) = 0$.

Product and Sum of the Roots Theorem Proof:
By the Fundamental Theorem of Algebra we know $p(x)$ has n roots which may be denoted by $r_1,\ r_2, r_3, \ldots, r_n$. Now form the product of the n factors associated with these roots. Let $q(x) = (x - r_1)(x - r_2)(x - r_3)\cdots(x - r_n)$ and multiply out all these terms. Then inspect the coefficient on x^{n-1} and inspect the constant term.

This can also be formally proved by using induction on n. When $n = 1$ we have $p(x) = x + a_0$ and in this case the only root of $p(x)$ is $r_1 = -a_0$. Since r_1 is the only root, r_1 is itself the product of all the roots. But then $(-1)^n a_0 = (-1)^1 a_0 = -a_0 = r_1$. So this establishes the part about the constant term. Note again that since r_1 is the only root, r_1 is itself the sum of all the roots and the 2nd leading coefficient is the opposite of the sum of all the roots since $a_0 = -(\ r_1)$.

It is probably more instructive to manually look at the case when $n = 2$ before setting up the induction step. Note that $(x - r_1)(x - r_2) = x^2 + (-r_1 + -r_2)x + r_1 r_2$. In this case it is immediately apparent that the 2nd leading coefficient is the opposite of the sum of all the roots and the constant term is product of all the roots. Because $p(x)$ is quadratic, in this case $n = 2$ so $(-1)^n = (-1)^2 = 1$.

Now lets assume the result is true whenever we have k roots and let $p(x)$ be a polynomial with $k + 1$ roots, say $p(x) = (x - r_1)(x - r_2)\cdots(x - r_{k+1})$. Now consider that we may write $p(x) = \{(x - r_1)(x - r_2)\cdots(x - r_k)\}(x - r_{k+1})$. Let $q(x) = \{(x - r_1)(x - r_2)\cdots(x - r_k)\}$. Then $q(x)$ has degree k and we may apply the induction hypothesis to $q(x)$. If we write $q(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_1x + a_0$ then we know $a_{k-1} = -\left(\sum_{i=1}^{k} r_i\right)$ and we know $a_0 = (-1)^k \cdot \left(\prod_{i=1}^{k} r_i\right)$.

Now $p(x) = q(x) \cdot (x - r_{k+1}) = (x - r_{k+1}) \cdot \left\{x^k + a_{k-1}x^{k-1} + \cdots + a_1x + a_0\right\}$

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$$= \left\{x^{k+1} + a_{k-1}x^k + \cdots + a_1x^2 + a_0x\right\} + \left\{(-r_{k+1})x^k + (-r_{k+1})a_{k-1}x^{k-1} + \cdots + (-r_{k+1})a_1x + (-r_{k+1})a_0\right\}$$

$$= x^{k+1} + \{a_{k-1} + (-r_{k+1})\}x^k + \cdots + (a_0 - r_{k+1} \cdot a_1)x + (-r_{k+1}) \cdot a_0.$$

Clearly $a_{k-1} + (-r_{k+1}) = -\left(\sum_{i=1}^{k} r_k\right) + (-r_{k+1}) = -\left(\sum_{i=1}^{k+1} r_i\right)$ and $(-r_{k+1}) \cdot a_0 =$

$(-r_{k+1}) \cdot (-1)^k \cdot \left(\prod_{i=1}^{k} r_i\right) = (-1)^{k+1} \cdot \left(\prod_{i=1}^{k+1} r_i\right)$ which are what we needed to establish.

Q.E.D.

8. Rational Roots Theorem

Let $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_3x^3 + a_2x^2 + a_1x + a_0$ be any polynomial with integer coefficients. If the rational number $\frac{c}{d}$ is a root of $p(x) = 0$ then c must be a factor of a_0 and d must be a factor of a_n.

Rational Roots Theorem Proof.

Let $q(x) = x_n + \frac{a_{n-1}}{a_n}x^{n-1} + \frac{a_{n-2}}{a_n}x^{n-2} + \cdots + \frac{a_3}{a_n}x^3 + \frac{a_2}{a_n}x^2 + \frac{a_1}{a_n}x + \frac{a_0}{a_n}$.

By the Product of the Roots Theorem, we know the product of the roots of this polynomial is the fraction $(-1)^n \cdot \frac{a_0}{a_n}$. Thus if $\frac{c}{d}$ is a root, c must be a factor of a_0 and d must be a factor of a_n.

Q.E.D.

9. Integer Roots Theorem

Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_3x^3 + a_2x^2 + a_1x + a_0$ be any polynomial with integer coefficients and ***with a leading coefficient of 1***. If $p(x)$ has any rational zeros, then those zeros must all be integers.

Integer Roots Theorem Proof:

By the Rational Roots Theorem we know the denominator of any rational zero must divide into the leading coefficient which in this case is 1. Thus any denominator must be ± 1 making the rational zero into a pure integer.

Q.E.D.

10. Upper and Lower Bounds Theorem

Let $p(x)$ be any polynomial with ***real coefficients and a positive leading coefficient.***

(***Upper Bound***) If $a > 0$ and $p(a) > 0$ and if in applying synthetic substitution to compute $p(a)$ all numbers in the 3rd row are positive, then a is an upper bound for all the roots of $p(x) = 0$.

(***Lower Bound***) If $a < 0$ and $p(a) \neq 0$ and if in applying synthetic substitution to compute $p(a)$ all the numbers in the 3rd row alternate in sign then a is a lower bound for all the roots of $p(x) = 0$.

[In either bound case, we can allow any number of zeros in any positions in the 3rd row except in the first and last positions. The first number is assumed to be positive and the last number is $p(a) \neq 0$. For upper bounds, we can state alternatively and more precisely that no negatives are allowed in the 3rd row. In the lower bound case the alternating sign requirement is not strict either, as any 0 value can assume either sign as required. In practice you may rarely see any zeros in the 3rd row. However, a slightly stronger and more precise statement is that the bounds still hold even when zeros are present anywhere as interior entries in the 3rd row.]

Upper and Lower Bounds Theorem Proof:

(Upper Bound). Let b be any root of the equation $p(x) = 0$. Must show $b < a$.
If $b = 0$, then clearly $b < a$ since a is positive in this case. So we assume $b \neq 0$.
If the constant term of $p(x)$ is 0, then we could factor x or a pure power of x from $p(x)$ and just operate on the resulting polynomial that is then guaranteed to have a nonzero constant term. So we can implicitly assume $p(0) \neq 0$. The last number in the third row of the synthetic substitution process is positive and it is $p(a)$. Since b is a root, we know by the Factor Theorem that $p(x) = (x - b) \cdot q(x)$ where $q(x)$ is the quotient polynomial. The leading coefficient of $p(x)$ is also the leading coefficient of $q(x)$ and since all of $q(x)$'s remaining coefficients are positive, and since $a > 0$, we must have $q(a) > 0$. Finally, $p(a) = (a - b) \cdot q(a)$. Since $q(a) > 0$, we may divide by $q(a)$ and get $(a - b) = \dfrac{p(a)}{q(a)}$. Now since $p(a)$ and $q(a)$ are both positive, $(a - b) > 0$ which implies $b < a$. Note that since the leading coefficient of $q(x)$ is positive and since $a > 0$, we don't really need all positive numbers in the last row. As long as $q(x)$'s remaining coefficients are nonnegative we can guarantee that $q(a) > 0$.

(Lower Bound). Let b be any root of the equation $p(x) = 0$. Must show $a < b$.
As in the above Upper Bound proof, we can easily dispense with the case when $b = 0$. Clearly $a < b$ when $b = 0$ because a is negative. We can further implicitly assume no pure power of x is a factor of $p(x)$ and this also allows us to assume $p(0) \neq 0$. Since $p(b) = 0$ by the Factor Theorem we may write $p(x) = (x - b) \cdot q(x)$. Substituting $x = a$ we have $p(a) = (a - b) \cdot q(a)$. Since $p(a) \neq 0$ we know $q(a) \neq 0$. So we can divide by $q(a)$ to get $(a - b) = \dfrac{p(a)}{q(a)}$.

Now $q(a)$ is either positive or negative. Because $a < 0$ and the leading term in $q(x)$ has a positive coefficient, the constant term in $q(x)$ has the same sign as $q(a)$. This fact can be established by considering the two cases of the even or odd degrees that $q(x)$ must have.

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For examples:
$q(x) = 1x^5 - 2x^4 + 3x^3 - 4x^2 + 5x - 6.$
With $a < 0$, $q(a) < 0$ and $q(a)$ and $q(x)'$s constant term agree in sign.
or
$q(x) = 1x^4 - 2x^3 + 3x^2 - 4x + 5.$
With $a < 0$, $q(a) > 0$ and again $q(a)$ and $q(x)'$s constant term agree in sign.

We might note that in these examples, it would make no difference if any of the interior coefficients were 0. This is because the first term has a positive coefficient, and all the remaining terms just add fuel to the fire with the same sign as the first term. The presence of an interior zero just means you might not get as big a fire, but the first term guarantees there is a flame!

Another note is that $p(0) = (-b) \cdot q(0),$ and since we are assuming $b \neq 0,$ we can divide this equation by $-b$ to conclude that $q(0) \neq 0$ when $p(0) \neq 0.$ So assuming neither b nor the constant term in $p(x)$ are zero guarantees that the constant term in $q(x)$ must be strictly positive or strictly negative. Since the numbers in the third row alternate in sign, $p(a)$ differs in sign from the constant term in $q(x)$. But since the constant term in $q(x)$ has the same sign as $q(a)$ we know $p(a)$ and $q(a)$ differ in sign. So $(a - b) = \dfrac{p(a)}{q(a)} < 0.$ $a < b.$

Q.E.D.

11. Intermediate Value Theorem

If $p(x)$ is any polynomial with ***real coefficients***, and if $p(a) > 0$ and $p(b) < 0$ then there is at least one real number c between a and b such that $p(c) = 0.$

Intermediate Value Theorem Proof:

This result depends on the continuity of all polynomials and is a special case of the **Intermediate Value Theorem** that normally appears in a calculus class.

12. Single Bound Theorem

Let $p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_3x^3 + a_2x^2 + a_1x + a_0$ be any polynomial with real coefficients and a leading coefficient of 1. Let $M_1 = 1 + max\{|a_0|, |a_1|, |a_2|, \ldots, |a_{n-1}|\}$ and let $M_2 = max\{1, |a_0| + |a_1| + |a_2| + \ldots + |a_{n-1}|\}$. Finally let $M = min\{M_1, M_2\}$. Then every zero of $p(x)$ lies between $-M$ and M.

Single Bound Theorem Proof:

We need to show M is an Upper Bound and we need to show $-M$ is a Lower Bound.

Case 1: $M = M_1$.

Then we know for $0 \leq i \leq n - 1$ that $M \geq 1 + |a_i|$. This implies two things. First, $M \geq 1$ and second, $M - |a_i| \geq 1$. These two inequalities are crucial and further imply that $-M \leq -1$ and $-M + |a_i| \leq -1$.

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To show M is an Upper Bound, consider the synthetic substitution calculation of $p(M)$.
We will label the second and remaining coefficients in the second row as b_i values.
We will label the second and remaining coefficients in the third row as c_i values.

$$\begin{array}{ccccccc|l} & & & & & & & \underline{M} \\ 1 & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 & \\ & M & b_{n-2} & b_{n-3} & \cdots & b_1 & b_0 & \\ \hline 1 & c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_1 & c_0 & \end{array}$$

We claim that each c_i value is not only positive, we claim each $c_i \geq 1$. Similarly we claim each $b_i \geq M$. We will establish these two claims by working from left to right across the columns in the synthetic substitution table, one column at a time.

First note that $c_{n-1} = a_{n-1} + M \geq -|a_{n-1}| + M = M - |a_{n-1}| \geq 1$.
We are done with the 2nd column.

Now we will argue about the 3rd column in the above table.
Having established in the 2nd column that $c_{n-1} \geq 1$, multiply both sides of this inequality by M to obtain: $b_{n-2} = c_{n-1} \cdot M \geq M$.

Now we basically repeat the above argument to establish the size of $c_{n-2} = a_{n-2} + b_{n-2}$.
Here we use the fact that $b_{n-2} \geq M \geq 1 + |a_{n-2}|$. So $b_{n-2} - |a_{n-2}| \geq 1$.
So $c_{n-2} = a_{n-2} + b_{n-2} \geq -|a_{n-2}| + b_{n-2} = b_{n-2} - |a_{n-2}| \geq 1$.
We are now done with the 3rd column in the table. Each next column is handled like the 3rd column.

Just to make sure you get the idea we will establish our claims for the 4th column.
Since $c_{n-2} \geq 1$, we can multiply across this inequality by M to get $c_{n-2} \cdot M \geq M$.
$b_{n-3} = c_{n-2} \cdot M \geq M$.
$c_{n-3} = a_{n-3} + b_{n-3} \geq -|a_{n-3}| + b_{n-3} = b_{n-3} - |a_{n-3}| \geq M - |a_{n-3}| \geq 1$.

Clearly we can continue working across the columns of the above table, one column at a time.
Since all the c_i coefficients are positive we know M is an Upper Bound for the zeros of $p(x)$.

Next, to show $-M$ is a Lower Bound, consider the synthetic substitution calculation of $p(-M)$.

$$\begin{array}{ccccccc|l} & & & & & & & \underline{-M} \\ 1 & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 & \\ & -M & b_{n-2} & b_{n-3} & \cdots & b_1 & b_0 & \\ \hline 1 & c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_1 & c_0 & \end{array}$$

proof continued on the next page

We claim that the coefficients in the 3rd row alternate in sign. Obviously the first coefficient is $1 > 0$.
We claim not only that the c_i alternate in sign, we claim that when $c_i < 0$ then $c_i \leq -1$. We also claim that when $c_i > 0$ then $c_i \geq 1$.

In the 2nd column of the table we have $c_{n-1} = a_{n-1} - M \leq |a_{n-1}| - M = -M + |a_{n-1}| \leq -1$. So we have established our claim within the 2nd column.

Moving over to the 3rd column we note that $b_{n-2} = c_{n-1} \cdot (-M)$ and since both of the numbers in this product are negative, we have $b_{n-2} > 0$. In fact when we start with the inequality that $c_{n-1} \leq -1$ and multiply across by the negative number $-M$ we get $c_{n-2} \cdot (-M) \geq M$. But $b_{n-2} = c_{n-2} \cdot (-M)$ so we know that $b_{n-2} \geq M$. Now lets compute c_{n-2}.
$c_{n-2} = a_{n-2} + b_{n-2} \geq a_{n-2} + M \geq -|a_{n-2}| + M = M - |a_{n-2}| \geq 1$.

Next, consider what happens in the 4th column of the above table. We just established that $c_{n-2} \geq 1$.
Multiplying across this inequality by $-M$ we get $c_{n-2} \cdot (-M) \leq -M$. But $b_{n-3} = c_{n-2} \cdot (-M)$ so we know $b_{n-3} \leq -M$. Now lets compute c_{n-3}.
$c_{n-3} = a_{n-3} + b_{n-3} \leq a_{n-3} - M \leq |a_{n-3}| - M = -M + |a_{n-3}| \leq -1$.

Clearly the above arguments may be repeated as we move across the columns of the above table. Each time we multiply by $-M$ to compute the next b_i value we have a sign change. This is primarily why the c_i values alternate in sign.

In any case, the values in the 3rd row alternate in sign and since $-M < 0$ we know $-M$ is a lower bound for any zero of the equation $p(x) = 0$.

Case 2: $M = M_2$.

Subcase 1: $M = 1$ and $|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}| < 1$.

In particular, for each i where $0 \leq i \leq n-1$ we know $0 \leq |a_i| < 1 = M$.
Then $-|a_i| + M = 1 - |a_i| > 0$.
Since $M = 1$, the Synthetic Substitution table takes on a particularly simple form. Note how the 2nd row elements are the same as the 3rd row elements shifted over one column.

								$\lfloor 1$
1	a_{n-1}	a_{n-2}	a_{n-3}	a_{n-4}	$\cdots$	a_1	a_0	
	1	c_{n-1}	c_{n-2}	c_{n-3}	$\cdots$	c_2	c_1	
1	c_{n-1}	c_{n-2}	c_{n-3}	c_{n-4}	$\cdots$	c_1	c_0	

We claim that all the c_i values are positive.
Starting in the 2nd column, $c_{n-1} = a_{n-1} + 1 \geq -|a_{n-1}| + 1 = 1 - |a_{n-1}| > 0$.

proof continued on the next page

Now consider the 3rd column. $c_{n-2} = a_{n-2} + c_{n-1} = a_{n-2} + a_{n-1} + 1 \geq -|a_{n-2}| + -|a_{n-1}| + 1 = -(|a_{n-2}| + |a_{n-1}|) + 1 > 0.$

Next consider the 4th column. $c_{n-3} = a_{n-3} + c_{n-2} = a_{n-3} + a_{n-2} + a_{n-1} + 1 \geq -|a_{n-3}| + -|a_{n-2}| + -|a_{n-1}| + 1 = -(|a_{n-3}| + |a_{n-2}| + |a_{n-1}|) + 1 > 0.$

Clearly we can continue to accumulate the sums of more and more terms and still apply the main inequality that appears in the Subcase 1: statement. So all the elements in the last row are positive and $1 = M$ is an upper bound for all the roots of $p(x) = 0$ by applying the Upper/Lower Bounds Theorem.

To establish that -1 is a lower bound we compute synthetic substitution with $-M = -1$.

$$\begin{array}{c|cccccccc} & & & & & & & & \lfloor -1 \\ 1 & a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_1 & a_0 & \\ & -1 & -c_{n-1} & -c_{n-2} & -c_{n-3} & \cdots & -c_2 & -c_1 & \\ \hline 1 & c_{n-1} & c_{n-2} & c_{n-3} & c_{n-4} & \cdots & c_1 & c_0 & \end{array}$$

Now we must establish that the c_i values in the last row alternate in sign.
Starting in the 2nd column, $c_{n-1} = a_{n-1} + (-1) \leq |a_{n-1}| + (-1) < 0.$

In the 3rd column, $c_{n-2} = a_{n-2} + (-c_{n-1}) = a_{n-2} - a_{n-1} + 1 \geq -|a_{n-2}| + -|a_{n-1}| + 1 = -(|a_{n-2}| + |a_{n-1}|) + 1 > 0.$

In the 4th column, $c_{n-3} = a_{n-3} + (-c_{n-2}) = a_{n-3} - a_{n-2} + a_{n-1} - 1 \leq |a_{n-3}| + |a_{n-2}| + |a_{n-1}| - 1 < 0.$

Clearly this argument may be repeated to establish that the coefficients in the 3rd row really do alternate in sign. So $M = -1$ is a lower bound by the Upper/Lower Bounds Theorem.

Subcase 2: $M = |a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|$ and $M \geq 1$.

To establish that M is an upper bound, we consider the synthetic substitution table for computing $p(M)$ and we will show that all the values in the last row are nonnegative.

$$\begin{array}{cccccccc} & & & & & & & \lfloor M \\ 1 & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 & \\ & M & b_{n-2} & b_{n-3} & \cdots & b_1 & b_0 & \\ \hline 1 & c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_1 & c_0 & \end{array}$$

proof continued on the next page

Now consider the 2nd column in the above table.
$c_{n-1} = a_{n-1} + M \geq -|a_{n-1}| + M = |a_0| + |a_1| + |a_2| + \cdots + |a_{n-2}| \geq 0.$

Next consider the 3rd column in the above table.
Since $M \geq 1,\ c_{n-1} \cdot M \geq c_{n-1}.$ But $b_{n-2} = c_{n-1} \cdot M$ so $b_{n-2} \geq c_{n-1}.$
Next, $c_{n-2} = a_{n-2} + b_{n-2} \geq a_{n-2} + c_{n-1} = a_{n-2} + a_{n-1} + M \geq$
$-|a_{n-2}| + -|a_{n-1}| + M = |a_0| + |a_1| + |a_2| + \cdots + |a_{n-3}| \geq 0.$

We continue to argue in the same manner for the 4th column.
Since $M \geq 1,\ c_{n-2} \cdot M \geq c_{n-2}.$ But $b_{n-3} = c_{n-2} \cdot M$ so $b_{n-3} \geq c_{n-2}.$
Next, $c_{n-3} = a_{n-3} + b_{n-3} \geq a_{n-3} + c_{n-2} \geq a_{n-3} + a_{n-2} + a_{n-1} + M \geq$
$-|a_{n-3}| + -|a_{n-2}| + -|a_{n-1}| + M = |a_0| + |a_1| + |a_2| + \cdots + |a_{n-4}| \geq 0.$

This argument may be repeated across the columns in the above table to establish that all the numbers in the last row are nonnegative. So by the Upper/Lower Bounds Theorem, M is an upper bound for all the roots of $p(x) = 0$.

Finally, consider the synthetic substitution table for computing $p(-M)$.

								$\lfloor -M$
1	a_{n-1}	a_{n-2}	a_{n-3}	a_{n-4}	$\cdots$	a_1	a_0	
	$-M$	b_{n-2}	b_{n-3}	b_{n-4}	$\cdots$	b_1	b_0	
1	c_{n-1}	c_{n-2}	c_{n-3}	c_{n-4}	$\cdots$	c_1	c_0	

We claim the nonnegative numbers in the 3rd row of this table alternate in sign. In the first column the number is 1 so we know we are starting with a positive value.

Now look at c_{n-1} in the 2nd column.
$c_{n-1} = a_{n-1} + (-M) = a_{n-1} + \{-|a_0| + -|a_1| + -|a_2| + \cdots + -|a_{n-1}|\} =$
$a_{n-1} + -|a_{n-1}| + \{-|a_0| + -|a_1| + -|a_2| + \cdots + -|a_{n-2}|\}.$
Now the last term in the above expression is obviously less than or equal to zero, and the first two terms either make 0 or make $-2 \cdot |a_{n-1}|$ so the whole expression is less than or equal to 0.

Now consider the 3rd column. We must show $c_{n-2} \geq 0$.
We take the worst case from c_{n-1} assuming this has the smallest absolute value where
$c_{n-1} = \{-|a_0| + -|a_1| + -|a_2| + \cdots + -|a_{n-2}|\}.$
Then $c_{n-2} = a_{n-2} + \{-|a_0| + -|a_1| + -|a_2| + \cdots + -|a_{n-2}|\} \cdot (-M) =$
$a_{n-2} + M \cdot |a_{n-2}| + \{-|a_0| + -|a_1| + -|a_2| + \cdots + -|a_{n-3}|\} \cdot (-M) \geq$
$a_{n-2} + |a_{n-2}| + \{-|a_0| + -|a_1| + -|a_2| + \cdots + -|a_{n-3}|\} \cdot (-M) \geq 0$
since the third term is nonnegative and the first two terms make either 0 or $2 \cdot |a_{n-2}|$.

proof continued on the next page

Now consider the 4th column. We must show $c_{n-3} \le 0$.
We take the worst case from c_{n-2} assuming c_{n-2} has the smallest absolute value where $c_{n-2} = \{-|a_0| + -|a_1| + -|a_2| + \cdots + -|a_{n-3}|\ \} \cdot (-M)$.
Then $c_{n-3} = a_{n-3} + \{-|a_0| + -|a_1| + -|a_2| + \cdots + -|a_{n-3}|\ \} \cdot (M^2) =$
$a_{n-3} - M^2 \cdot |a_{n-3}| + \{-|a_0| + -|a_1| + -|a_2| + \cdots + -|a_{n-4}|\ \} \cdot (M^2) \le$
$a_{n-3} - |a_{n-3}| + \{-|a_0| + -|a_1| + -|a_2| + \cdots + -|a_{n-4}|\ \} \cdot (M^2) \le 0$
since the third term is negative and the first two terms make either 0 or $-2 \cdot |a_{n-3}|$.

Just to make sure you get the idea we will continue with the 5th column. We must show $c_{n-4} \ge 0$. We take the worst case from c_{n-3} assuming c_{n-3} has the smallest absolute value where $c_{n-3} = \{-|a_0| + -|a_1| + -|a_2| + \cdots + -|a_{n-4}|\ \} \cdot (M^2)$.
Then $c_{n-4} = a_{n-4} + \{-|a_0| + -|a_1| + -|a_2| + \cdots + -|a_{n-4}|\ \} \cdot (-M^3) =$
$a_{n-4} + M^3 \cdot |a_{n-4}| + \{-|a_0| + -|a_1| + -|a_2| + \cdots + -|a_{n-5}|\ \} \cdot (-M^3) \ge$
$a_{n-4} + |a_{n-4}| + \{-|a_0| + -|a_1| + -|a_2| + \cdots + -|a_{n-5}|\ \} \cdot (-M^3) \ge 0$
since the third term is nonnegative and the first two terms make either 0 or $2 \cdot |a_{n-4}|$.

This argument may be repeated across the columns of the above table to conclude that the nonnegative terms in the last row alternate in sign. By the Upper/Lower Bounds Theorem we know $-M$ is a lower bound for all the zeros of $p(x) = 0$.

Q.E.D.

13. Odd Degree Real Root Theorem

If $p(x)$ has real coefficients and has a degree that is odd then it has at least one real root.

Odd Degree Real Root Theorem Proof:

Without loss of generality we assume the leading coefficient of $p(x)$ is positive. Otherwise we can factor -1 from $p(x)$ apply the theorem to the polynomial that is the other factor.
By choosing $M > 0$ sufficiently large we can establish that $p(M) > 0$ and $p(-M) < 0$.
For example, see the above Single Bound Theorem. Now apply the Intermediate Value Theorem.
There exists a number a such that $-M < a < M$ and $p(a) = 0$. a is real and is a root of $p(x)$.

Q.E.D.

14. Complex Conjugate Roots Theorem

If $p(x)$ is any polynomial with ***real coefficients***, and if $a+bi$ is a complex root of the equation $p(x)=0$, then another complex root is its conjugate $a-bi$.
(Complex number roots appear in conjugate pairs)

Complex Conjugate Roots Theorem.

This proof just depends on properties of the conjugate operator denoted by bars below.
If $p(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_3x^3+a_2x^2+a_1x+a_0=0$ then

$$\overline{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_3x^3+a_2x^2+a_1x+a_0}=\overline{0}.$$

$$\overline{a_nx^n}+\overline{a_{n-1}x^{n-1}}+\cdots+\overline{a_3x^3}+\overline{a_2\,x^2}+\overline{a_1x}+\overline{a_0}=0.$$

$$\overline{a_n}\,\overline{x}^n+\overline{a_{n-1}}\,\overline{x}^{n-1}+\cdots+\overline{a_3}\,\overline{x}^3+\overline{a_2}\,\overline{x}+\overline{a_1}\,\overline{x}+\overline{a_0}=0.$$

$a_n\,\overline{x}^n+a_{n-1}\,\overline{x}^{n-1}+\cdots+a_3\,\overline{x}^3+a_2\,\overline{x}+a_1\overline{x}+a_0=0.$ This shows $p(\overline{x})=0.$

Q.E.D.

15. Linear and Irreducible Quadratic Factors Theorem

Any polynomial $p(x)$ with real coefficients may be written as a product of linear factors and irreducible quadratic factors. The sum of all the degrees of these component factors is the degree of $p(x)$.

Linear and Irreducible Quadratic Factors Theorem Proof:

By the Fundamental Theorem of Algebra, $p(x)=c(x-r_1)(x-r_2)(x-r_3)\cdots(x-r_n)$ where the r_k denote the n roots of $p(x)$. The constant c is simply $p(x)$'s leading coefficient.
If all the r_k roots are real then $p(x)$ is a product of linear real factors only. However, if any r_k value is a complex number then it can be paired with some other r_j value which is its complex conjugate.
If we assume $r_k=a+bi$ then $r_j=a-bi$. Note that since r_k is complex, we must have $b\neq 0$.
Next we note that $r_k+r_j=2a$ and $r_k\cdot r_j=a^2+b^2$ and we compute
$(x-r_k)(x-r_j)=x^2-(r_k+r_j)x+r_kr_j=x^2-(2a)x+(a^2+b^2)$. That this last expression is an irreducible quadratic factor follows by computing its discriminant that is
$(-2a)^2-4\cdot 1\cdot(a^2+b^2)=4a^2-4a^2-4b^2=-4b^2<0$ since $b\neq 0$.
If we re-order or rename the indices of the roots so that r_k and r_j were the last two roots then we can assume that $p(x)$ now takes the form in which we put the irreducible quadratic just found at the end.
$p(x)=c(x-r_1)(x-r_2)\cdots(x-r_{n-3})\{(x^2-2ax+a^2+b^2)\}$.
Now if any two of the preceding linear factors form complex r-value conjugate roots, we treat them just like we did with the pair r_k and r_j. This produces another irreducible quadratic factor which we also place as the last rightmost factor. Clearly this process can be continued until only real linear factors remain at the beginning and only irreducible quadratic factors are at the end. There may be no linear factors at the beginning and only irreducible quadratic factors, or there may be no irreducible quadratic factors at the end and only linear factors at the beginning. It all depends on the nature of the roots r_i and how many of these roots are real and how many are complex.

Q.E.D.

16. Irrational Conjugate Roots Theorem

Let $p(x)$ be any polynomial with ***rational*** real coefficients. If $a + b\sqrt{c}$ is a root of the equation $p(x) = 0$ where $\sqrt{c}$ is irrational and a and b are rational, then another root is $a - b\sqrt{c}$. (Like complex roots, irrational real roots appear in conjugate pairs, but only when the polynomial has rational coefficients.)

Irrational Conjugate Roots Theorem Proof:

Assume $a + b\sqrt{c}$ is one root. Must show $a - b\sqrt{c}$ is also a root. If $b = 0$ we are done, so assume $b \neq 0$. Let $t(x) = \left(x - \left(a + b\sqrt{c}\right)\right) \cdot \left(x - \left(a - b\sqrt{c}\right)\right) = (x-a)^2 - b^2 c$. Then $t(x)$ is a quadratic polynomial with rational coefficients. Next, consider dividing $p(x)$ by $t(x)$. By the Division Algorithm, there is a quotient polynomial $q(x)$ and there exists a remainder polynomial $r(x)$ such that $p(x) = t(x) \cdot q(x) + r(x)$ where the degree of $r(x)$ is 1 or 0.

If we assume $r(x) = Cx + D$ then C and D must be rational. In fact, since $p(x)$ and $t(x)$ have only rational coefficients, both $q(x)$ and $r(x)$ must have only rational coefficients.

So we may write $p(x) = t(x) \cdot q(x) + Cx + D$ and when we substitute $x = a + b\sqrt{c}$ we conclude that $0 = C\left(a + b\sqrt{c}\right) + D$ from which we can further conclude that $C = D = 0$.

So we really have $p(x) = t(x) \cdot q(x)$. Finally we substitute $x = a - b\sqrt{c}$ in this last equation to conclude that $p\left(a - b\sqrt{c}\right) = 0$ which is what we needed to show.

Q.E.D.

17. Descartes's Rule of Signs Lemma 1.

If $p(x)$ has real coefficients, and if $p(a) = 0$ where $a > 0$, then $p(x)$ has at least one more sign variation than the quotient polynomial $q(x)$ has sign variations where $p(x) = (x-a)q(x)$. [When the difference in the number of sign variations is greater than 1, the difference is always an odd number.]

Descartes's Rule of Signs Lemma 1 Proof:

The following particular example shows that $q(x)$ may indeed have fewer sign variations than $p(x)$ has. In this example, $p(x)$ has three variations in sign while $q(x)$ has only two variations in sign. Had the number $+152$ in the top row been $+102$ instead, then $q(x)$ would have had even fewer sign variations as is shown in the next example.

$$\begin{array}{rrrrr|l} & & & & & \underline{2} \\ 12 & -77 & 152 & -77 & -30 & \\ & 24 & -106 & 92 & 30 & \\ \hline 12 & -53 & 46 & 15 & 0 & \end{array}$$

In the next example, $p(x)$ has four sign variations while $q(x)$ has only one sign variation. So the difference of the sign variation counts in the example below is the odd number 3.

$$\begin{array}{rrrrr|l} & & & & & \underline{2} \\ 12 & -77 & 102 & -77 & 170 & \\ & 24 & -106 & -8 & -170 & \\ \hline 12 & -53 & -4 & -85 & 0 & \end{array}$$

proof continued on the next page

Finally we show one more example before starting the formal proof. In this example, $p(x)$ has four sign variations and $q(x)$ has only three sign variations. Moreover, the coefficients in $p(x)$ and $q(x)$ match signs column by column from left to right through the constant column in $q(x)$.

					3
1	−6	11	−11	15	
	3	−9	6	−15	
1	−3	2	−5	0	

Assume the leading coefficient of $p(x)$ is positive and consider the synthetic substitution form used to compute $p(a)$. Consider the constant term in $p(x)$. If this constant term is negative as in the first example above, then in the previous column the constant term in $q(x)$ must have been positive in order for the final column numbers to add to make 0. If the constant term in $p(x)$ were positive as in the second example above, then in the previous column the constant term in $q(x)$ must have been negative in order for the final column numbers to add to make 0. So the constant terms in $p(x)$ and $q(x)$ must have opposite signs. This argument has depended on the facts that $a > 0$ and that $p(a) = 0$.

But $q(x)$ and $p(x)$ both start with the same positive coefficient. Next, reading from left to right, we claim $q(x)$ cannot change signs until $p(x)$ changes signs. Whenever $q(x)$ changes signs from one column to the next, $p(x)$ must also change signs between those same two columns. But as in the second example above (columns 2 & 3 and columns 3 & 4), $q(x)$ can keep the same sign even when $p(x)$ does change sign. But $q(x)$ can never change signs unless $p(x)$ changes signs first.

Now suppose in counting sign changes that at some point $p(x)$ changes signs when $q(x)$ does not as in the first two examples above. Then $p(x)$ has one more sign variation, and from that point forward, $p(x)$ will continue to lead in the sign variation count because each further time $q(x)$ changes signs, so does $p(x)$. We can rest our case in this case.

The other case that needs to be considered is when counting sign changes, if we reach the end of $q(x)$, and if at that point $p(x)$ and $q(x)$ have the same sign variation count, as in the third example above. Then $p(x)$ and $q(x)$ will have the same sign in the next to the last column, but then $p(x)$ will change signs one more time in its last column.

No matter how you look at it, $p(x)$ has at least one more sign variation count than $q(x)$.
If the leading coefficient of $p(x)$ is not positive, then factor out -1 from $p(x)$ and then apply the above argument to the resulting polynomial.

We have already proved $p(x)$ has at least one more sign variation than the quotient polynomial $q(x)$. To prove that the difference is always an odd number, we reiterate that the constant terms in both polynomials always differ in sign while the first terms always agree in sign. So when more than one sign variation occurs, it occurs at some interior coefficient. But changing the sign of any interior coefficient either raises or lowers the sign variation count by 2 because such a change applies to the term before it and to the term after it. So when the difference in the number of sign variations is more than 1, it must be an odd difference.

Q.E.D.

18. Descartes's Rule of Signs Lemma 2.

If $p(x)$ has real coefficients, the number of positive zeros of $p(x)$ is not greater than the number of variations in sign of the coefficients of $p(x)$.

Descartes's Rule of Signs Lemma 2 Proof:

Let $r_1, r_2, r_3, \ldots, r_k$ denote all the positive roots of the equation $p(x) = 0$.
Then we may write $p(x) = (x - r_1)(x - r_2)(x - r_3)\cdots(x - r_k) \cdot Q(x)$.
Now consider the following regrouping of these factors:

$p(x) = (x - r_1)\{(x - r_2)(x - r_3)\cdots(x - r_k) \cdot Q(x)\}$.

By Lemma 1 we know $p(x)$ has at least one more sign variation than the rightmost factor.
Let $q_1(x) = \{(x - r_2)(x - r_3)\cdots(x - r_k) \cdot Q(x)\}$. Then $p(x)$ has at least one more sign variation than $q_1(x)$ has. Moreover, since we may write
$q_1(x) = (x - r_2)\{(x - r_3)\cdots(x - r_k) \cdot Q(x)\}$
we can again apply Lemma 1 to conclude that $q_1(x)$ has one at least more sign variation than the polynomial $\{(x - r_3)\cdots(x - r_k) \cdot Q(x)\}$. Let $q_2(x) = \{(x - r_3)\cdots(x - r_k) \cdot Q(x)\}$.

Now $p(x)$ has at least one more sign variation than $q_1(x)$ and $q_1(x)$ has at least one more sign variation than $q_2(x)$, so $p(x)$ has at least two more sign variations than $q_2(x)$.

Clearly we may continue to regroup the rightmost factors and reduce the number of factors.
$q_2(x) = \{(x - r_3)\cdots(x - r_k) \cdot Q(x)\} = (x - r_3) \cdot \{(x - r_4)\cdots(x - r_k) \cdot Q(x)\}$.
So $q_2(x)$ has one or more sign variations than $q_3(x) = \{(x - r_4)\cdots(x - r_k) \cdot Q(x)\}$.

We argue that for each factor we drop, $p(x)$ has yet at least another sign variation more than the resulting rightmost factor. After dropping all k factors we conclude that $p(x)$ has k or more sign variations than does $Q(x)$, since after dropping k factors, $Q(x)$ is all that remains.

Now assume $Q(x)$ has m sign variations in its coefficients and assume $p(x)$ has n sign variations in its coefficients. Then because $m \geq 0$, $m + k \leq n$ implies $k \leq n$. The number of positive zeros of $p(x)$ is less than or equal to the number of sign variations in the coefficients of $p(x)$.

Q.E.D.

19. Descartes's Rule of Signs Lemma 3.

Let $r_1, r_2, r_3, \ldots, r_k$ denote k positive numbers and let $p(x) = \prod_{i=1}^{k}(x - r_i)$.

Then the coefficients of $p(x)$ are all alternating in sign and this polynomial has exactly k sign variations in its coefficients.

Descartes's Rule of Signs Lemma 3 Proof:

Either use induction on k or else apply Lemma 2 to conclude that $Q(x) = 1$ has k fewer variations in sign than $p(x)$. But since 1 has no variations in sign we know $p(x)$ must have k variations in sign which means all the coefficients alternate in sign.

As a simple example: $(x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + (-1)^2(r_1 \cdot r_2)$.

Using induction, if $k = 1$, then we note $(x - r_1)$ has exactly 1 sign variation. Next, assume the theorem is true for any polynomial with k factors or less, and let

$$p(x) = \prod_{i=1}^{k+1}(x - r_i) = (x - r_{k+1}) \cdot \left[\prod_{i=1}^{k}(x - r_i)\right]$$

If we consider the second factor to be the quotient polynomial then we can apply the induction assumption to conclude this quotient has exactly k sign variations. Next we apply Lemma 1 to $p(x)$ to conclude that $p(x)$ has at least one more sign variation than the quotient. This means $p(x)$ has exactly $k+1$ sign variations. Being a $(k+1)^{st}$ degree polynomial, $p(x)$ cannot have more than $k+1$ sign variations because it has only $k+2$ coefficients.

Q.E.D.

20. Descartes's Rule of Signs Lemma 4.

The number of variations in sign of a polynomial with real coefficients is even if the first and last coefficients have the same sign, and is odd if the first and last coefficients have opposite signs.

Descartes's Rule of Signs Lemma 4 Proof:

Before giving the proof we look at one example.

$p(x) = x^4 - 6x^3 + 11x^2 - 12x + 15$.

In this case, the first and last coefficients have the same sign and we can see that $p(x)$ has an even number of sign changes in its coefficients; it has 4 sign changes.

The degree of $p(x)$ is 4, it has 5 coefficients, and thus it has an *a priori* possibility of having at most 4 sign variations. If we were to change the sign of either the first or the last coefficient, we would have one less sign change, or an odd number of sign changes. If we were to increase the degree of $p(x)$ by adding just one term, then we would not add a sign change unless the sign of that new term differed from the existing leading term's sign.

We prove this theorem by strong induction on the degree n of the polynomial $p(x)$. When $n = 1$, we assume $p(x) = ax + b$. If a and b have the same sign then we have 0 or an even number of sign changes. If a and b have opposite signs then we have 1 or an odd number of sign changes. So the theorem is true when $n = 1$.

proof continued on the next page

Next, assume the theorem is true whenever $n \leq k$, and let $p(x)$ be a polynomial of degree $k+1$. Must show the theorem is true for $p(x)$.

Consider the polynomial of degree k, obtained by dropping the leading term from $p(x)$. Call this polynomial $q(x)$. The theorem is assumed true for $q(x)$ since its degree can be assumed to be either k, or even better, less than k.

There are two cases.

Case 1: $q(x)$'s leading and trailing terms have the same sign.

Then we know by the induction assumption that $q(x)$ has an even number of sign changes.
There are only two possibilities for the sign of the leading term that was dropped.
If the dropped leading term has the same sign as the leading term in $q(x)$, then there is no sign change when this term is added back. So $p(x)$ would still have an even number of sign changes and the leading and trailing terms of $p(x)$ would still agree in sign.
If the dropped leading term has a different sign from the leading term in $q(x)$, then there is one additional sign change that gets added when this term is put back. So in this case $p(x)$ would have an odd number of sign changes. But also in this case, the leading and trailing terms of $p(x)$ would have opposite signs.

Case 2: $q(x)$'s leading and trailing terms have opposite signs.

Then we know by the induction assumption that $q(x)$ has an odd number of sign changes.
There are only two possibilities for the sign of the leading term that was dropped.
If the dropped leading term has the same sign as the leading term in $q(x)$, then there is no sign change when this term is added back. So $p(x)$ would still have an odd number of sign changes and the leading and trailing terms of $p(x)$ would still have opposite signs.
If the dropped leading term has a different sign from the leading term in $q(x)$, then there is one additional sign change that gets added when this term is put back. So in this case $p(x)$ would have an even number of sign changes. But also in this case, the leading and trailing terms of $p(x)$ would have the same signs.

In either case, the theorem is true for $p(x)$ with degree $k+1$. This completes the proof by induction on n.

Q.E.D.

21. Descartes's Rule of Signs Lemma 5.

If the number of positive zeros of $p(x)$ with real coefficients is less than the number of sign variations in $p(x)$, it is less by an even number.

Descartes's Rule of Signs Lemma 5 Proof:

If the leading coefficient of $p(x)$ isn't 1, we can factor it out and just assume $p(x) = \{(x-r_1)\cdots(x-r_k)\}\cdot\{(x-n_1)\cdots(x-n_j)\}\cdot\{(x^2+b_1x+c_1)\cdots(x^2+b_lx+c_l)\}$ where the r_i denote all the positive zeros of $p(x)$, the n_i denote the all the negative zeros of $p(x)$, and the remaining factors are quadratics corresponding to all the complex-conjugate paired complex zeros of $p(x)$. Let s be the number of sign changes in the coefficients of $p(x)$. We assume $k < s$ and we must show there exists an even integer e such that $e > 0$ and $k + e = s$.
Let $f(x) = \{\{(x-n_1)\cdots(x-n_j)\}\cdot\{(x^2+b_1x+c_1)\cdots(x^2+b_lx+c_l)\}\}$.
By Lemma 2, we know the $f(x)$ polynomial has at least k fewer sign variations than $p(x)$.
We let t be the number of sign changes in the $f(x)$ polynomial. Then we know $t \geq s - k > 0$.
Next we note a special property of each of the irreducible quadratic factors. The discriminant of each quadratic must be negative, so we know $b_i^2 - 4\cdot 1\cdot c_i < 0$. So $0 \leq b_i^2 < 4c_i$ and we conclude that all the c_i coefficients are strictly positive.
Also, each $(x-n_i)$ factor of $f(x)$ may be written as $(x+p_i)$ where $p_i = -n_i$ is positive. So we may write $f(x) = \{\{(x+p_1)\cdots(x+p_j)\}\cdot\{(x^2+b_1x+c_1)\cdots(x^2+b_lx+c_l)\}\}$.
Now it is clear that the leading coefficient of $f(x)$ is $+1$, and the constant term of $f(x)$ is also positive since it is the product of all positive numbers.
The constant term of $f(x) = \left[\prod_{i=1}^{j} p_i\right]\left[\prod_{i=1}^{l} c_i\right]$. By Lemma 4, the number of sign variations in the coefficients of $f(x)$ is even. t is even. From above we have $t \geq s - k > 0$.
Therefore $t + k \geq s$. Now if it happens that $t + k = s$ then we let $e = t$ and we are done.

Otherwise, if $t + k > s$ then we have to argue about the first k factors in $p(x)$. Having just established that the constant term in $f(x)$ is positive, the sign of the constant term in $p(x)$ is the sign of $(-1)^k\cdot\left[\prod_{i=1}^{k} r_k\right]$ = the sign of $(-1)^k$ since all the r_i values are positive. If k is even then by Lemma 4, $p(x)$ has an even number of sign variations in its coefficients which means s is even. If k is odd then again by Lemma 4 we conclude s is odd.
So k and s are even together or are odd together. Now consider that $t + k - s > 0$. What kind of a positive number is this? Well t is even, and if k and s are both even then $t + k - s$ must be even. By the same token, if k and s are both odd, then $k - s$ is still even, and since t is always even, $t + (k - s)$ must be even. Therefore $t + k - s = 2v$ for some positive integer v.
Then $(t - 2v) + k = s$. Let $e = (t - 2v)$. All that remains is to show that $(t - 2v)$ is a positive even integer. First, $(t - 2v) = s - k$ and since $s - k > 0$, we know $(t - 2v)$ is a positive integer. Second, both t and $2v$ are even, so their difference $(t - 2v)$ is even. In any case, there exists a positive even integer e such that $e + k = s$. So when s is larger than k, it is larger by a positive even integer.

Q.E.D.

22. Descartes's Rule of Signs Lemma 6.

Each negative root of $p(x)$ corresponds to a positive root of $p(-x)$. That is, if $a < 0$ and a is a zero of $p(x)$, then $-a$ is a positive zero of $p(-x)$.

Descartes's Rule of Signs Lemma 6 Proof:

The graph of the function $y = p(-x)$ is just the graph of $y = p(x)$ reflected over the y-axis. So, if $a < 0$ and $p(a) = 0$, then $-a > 0,$ and when $x = -a,$ then $p(-x) = p(-(-a)) = p(a) = 0.$ So $-a$ is a positive zero of $p(-x).$

Q.E.D.

23. Descartes's Rule of Signs Theorem

Let $p(x)$ be any polynomial with ***real coefficients***.

(***Positive Roots***) The number of positive roots of $p(x) = 0$ is either equal to the number of sign variations in the coefficients of $p(x)$ or else is less than this number by an even integer.

(***Negative Roots***) The number of negative roots of $p(x) = 0$ is either equal to the number of sign variations in the coefficients of $p(-x)$ or else is less than this number by an even integer.

Note that when determining sign variations we can ignore terms with zero coefficients.

Proof of Descartes's Rule of Signs Theorem:

The statement about the number of positive roots of $p(x) = 0$ is exactly the statement of Lemma 5 that has already been proved.

To prove the statement about the number of negative roots of $p(x)$ we need only apply Lemma 6. Each negative root of $p(x)$ corresponds to a positive root of $p(-x)$ and by Lemma 5, the number of positive roots of any polynomial {like $p(-x)$ } is either equal to the number of sign variations in that polynomial { $p(-x)$ }, or is less than the number of sign variations in that polynomial { $p(-x)$ } by a positive even integer.

Q.E.D.

24. Lemma On Continuous Functions.

Let $f(x)$ and $g(x)$ be two continuous real-valued functions with a common domain that is an open interval $(a,\ b)$. Furthermore let $c \in (a,\ b)$ and assume that except when $x = c$ we have $f(x) = g(x)$ for all $x \in (a,\ b)$. Then we must also have $f(c) = g(c)$.

Proof of Lemma On Continuous Functions:

By contradiction. Assume $f(c) \neq g(c)$. Without loss of generality we may assume $f(c) < g(c)$ and choose $\epsilon = \dfrac{g(c) - f(c)}{2}$. Note that $\epsilon > 0$. By the continuity of both $f(x)$ and $g(x)$ at $x = c$, there exists a $\delta_f > 0$ and there exists a $\delta_g > 0$ such that for all $x \in (a,\ b)$

1) if $0 < |x - c| < \delta_f$ then $|f(x) - f(c)| < \epsilon$

and 2) if $0 < |x - c| < \delta_g$ then $|g(x) - g(c)| < \epsilon$

Let $\delta = min\{\delta_f,\ \delta_g\}$ and choose $x_1 \in (a,\ b)$ such that $0 < |x_1 - c| < \delta$.

Note that since x_1 is chosen so that $x_1 \neq c$, we must have $g(x_1) = f(x_1)$.
Also, by our choice of δ, parts 1) and 2) above apply so we can conclude that $f(x_1) - f(c) < \epsilon$ and with a little bit of thought, we can see that we must also have $g(c) - g(x_1) < \epsilon$.

So adding both inequalities we must have $f(x_1) - f(c) + g(c) - g(x_1) < 2\epsilon = g(c) - f(c)$.

Now since $g(x_1) = f(x_1)$ the left expression simplifies and may be rearranged so that we have $g(c) - f(c) < g(c) - f(c)$, a contradiction.

Q.E.D.

25. Theorem On the Equality of Polynomials

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_3 x^3 + a_2 x^2 + a_1 x + a_0$ and let $q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_3 x^3 + b_2 x^2 + b_1 x + b_0$ be any two real polynomials of degrees n and m respectively. If for all real numbers x, $p(x) = q(x)$ then

1) $m = n$

and 2) for all i, if $0 \leq i \leq n$ then $a_i = b_i$.

Proof of Theorem On the Equality of Polynomials:

The following informal argument can be formalized using Mathematical Induction. However, we prefer a more relaxed discussion that emphasizes technique over formality.
First note that if $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_3 x^3 + b_2 x^2 + b_1 x + b_0$ for all x, we may let $x = 0$ to conclude that $a_0 = b_0$.

proof continued on the next page

Next, we subtract the common constant term from both sides of the equation to conclude that for all x

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_3 x^3 + a_2 x^2 + a_1 x =$$
$$b_m x^m + b_{m-1} x^{m-1} + \cdots + b_3 x^3 + b_2 x^2 + b_1 x.$$

Now divide both sides of this last equation by x, assuming $x \neq 0$. Then we have:

$$a_n x^{n-1} + a_{n-1} x^{n-2} + \cdots + a_3 x^2 + a_2 x + a_1 =$$
$$b_m x^{m-1} + b_{m-1} x^{m-2} + \cdots + b_3 x^2 + b_2 x + b_1$$

for all nonzero x. However, both of these last polynomials are defined and are continuous in a neighborhood about $x = 0$, so we may apply the above Lemma with $c = 0$ to conclude this new equation is true for all x, including when $x = 0$.

Now we can repeat the above argument and let $x = 0$ to conclude that $a_1 = b_1$, and we again subtract this common constant term from both sides of the last equation to obtain the statement that for all x

$$a_n x^{n-1} + a_{n-1} x^{n-2} + \cdots + a_3 x^2 + a_2 x = b_m x^{m-1} + b_{m-1} x^{m-2} + \cdots + b_3 x^2 + b_2 x.$$

Again we divide both sides by x to obtain the simpler equation that

$$a_n x^{n-2} + a_{n-1} x^{n-3} + \cdots + a_3 x + a_2 = b_m x^{m-2} + b_{m-1} x^{m-3} + \cdots + b_3 x + b_2.$$

Even though this equation is only true for nonzero x because we just divided by x, we can apply the above Lemma to conclude this equation must also be true when $x = 0$. So again we may let $x = 0$ to conclude that $a_2 = b_2$.

Clearly this argument may be continued to repeatedly pick off each of the coefficients one by one in order until we run out of both coefficients. So every coefficient of $p(x)$ matches the same degree term coefficient of $q(x)$. That we must run out of both coefficients at the same time is because otherwise, if $p(x)$ and $q(x)$ had different degrees, we could find a 0 coefficient in one of these polynomials that would match a nonzero coefficient in the other and that would be a contradiction.

A final note about this theorem and its lemma is that the lemma is very easy for a non-calculus student to understand when continuity is presented in an intuitive way (no epsilons or deltas!). This theorem can also be proved assuming the Fundamental Theorem of Algebra, but the advantage of this alternative approach is that we don't have to assume the Fundamental Theorem of Algebra and we can introduce the fundamental property of continuity of polynomials.

Q.E.D.

26. Theorem Euclidean Algorithm for Polynomials

Let $p(x)$ and $q(x)$ be any two polynomials with degrees ≥ 1. Then there exists a polynomial $d(x)$ such that $d(x)$ divides evenly into both $p(x)$ and $q(x)$. Moreover, $d(x)$ is such that if $a(x)$ is any other common divisor of $p(x)$ and $q(x)$, then $a(x)$ divides evenly into $d(x)$. The polynomial $d(x)$ is called the Greatest Common Divisor of $p(x)$ and $q(x)$ is sometimes denoted by $GCD(p(x), q(x))$. Except for constant multiples, $d(x)$ is unique.

Proof of the Euclidean Algorithm for Polynomials

Without loss of generality we assume the degree of $p(x)$ is larger than or equal to the degree of $q(x)$. By the Division Algorithm we may write

$$p(x) = q(x) \cdot q_1(x) + r_1(x) \tag{1}$$

where $q_1(x)$ is the quotient polynomial and $r_1(x)$ is the remainder. If $r_1(x) \equiv 0$ then we stop.

Otherwise, if $r_1(x) \not\equiv 0$ we note from the above equation that any common divisor of both $q(x)$ and $r_1(x)$ must be a divisor of the right side of the above equation and therefore a divisor of the left side. Any common divisor of $q(x)$ and $r_1(x)$ must be a divisor of $p(x)$. Next, by writing

$$p(x) - q(x) \cdot q_1(x) = r_1(x)$$

we can see that every common divisor of $p(x)$ and $q(x)$ must be a divisor of $r_1(x)$ and thus a common divisor of $q(x)$ and $r_1(x)$. So $GCD(p(x), q(x)) = GCD(q(x), r_1(x))$.

We continue by applying the Division Algorithm again to write

$$q(x) = r_1(x) \cdot q_2(x) + r_2(x) \tag{2}$$

If $r_2(x) \equiv 0$ we stop. Otherwise, repeating the above reasoning, $GCD(q(x), r_1(x)) = GCD(r_1(x), r_2(x))$. Now apply the Division Algorithm again.

$$r_1(x) = r_2(x) \cdot q_3(x) + r_3(x) \tag{3}$$

If $r_3(x) \equiv 0$ we stop. Otherwise we note $GCD(r_1(x), r_2(x)) = GCD(r_2(x), r_3(x))$ and we continue to apply the Division Algorithm to get

$$r_2(x) = r_3(x) \cdot q_4(x) + r_4(x) \tag{4}$$

proof continued on the nex page

If $r_4(x) \equiv 0$ we stop. Otherwise we continue this process. However, we cannot this process forever because the degrees of the remainders $r_i(x)$ keep decreasing by 1.

$$\text{degree}(r_4(x)) < \text{degree}(r_3(x)) < \text{degree}(r_2(x)) < \text{degree}(r_1(x)) \leq \text{degree}(q(x))$$

So after applying the Division Algorithm at most the number of times that is the degree of $q(x)$ we must have some remainder become the identically zero polynomial.

We claim $GCD(p(x), q(x))$ is the last nonzero remainder. For example, suppose

$$r_{n-2}(x) = r_{n-1}(x) \cdot q_n(x) + r_n(x) \qquad (n)$$

and

$$r_{n-1}(x) = r_n(x) \cdot q_{n+1}(x) \qquad (n+1)$$

where $r_{n+1}(x) \equiv 0$ and is not written. The last equation shows $r_n(x)$ is a divisor of $r_{n-1}(x)$ so $GCD(r_{n-1}(x), r_n(x)) = r_n(x)$.

Now $GCD(p(x), q(x)) = GCD(q(x), r_1(x)) = GCD(r_1(x), r_2(x)) = GCD(r_2(x), r_3(x))$
$= \cdots = GCD(r_{n-1}(x), r_n(x)) = r_n(x)$, the last nonzero remainder.

Q.E.D.

27. Corollary to the Euclidean Algorithm for Polynomials

The GCD of any two polynomials $p(x)$ and $q(x)$ may be expressed as a linear combination of $p(x)$ and $q(x)$.

Proof of the Corollary to the Euclidean Algorithm for Polynomials.

The following were the series of equations that led up to the creation of the GCD polynomial.

$$p(x) = q(x) \cdot q_1(x) + r_1(x) \tag{1}$$
$$q(x) = r_1(x) \cdot q_2(x) + r_2(x) \tag{2}$$
$$r_1(x) = r_2(x) \cdot q_3(x) + r_3(x) \tag{3}$$
$$r_2(x) = r_3(x) \cdot q_4(x) + r_4(x) \tag{4}$$
$$\vdots$$
$$r_{n-3}(x) = r_{n-2}(x) \cdot q_{n-1}(x) + r_{n-1}(x) \tag{n-1}$$
$$r_{n-2}(x) = r_{n-1}(x) \cdot q_n(x) + r_n(x) \tag{n}$$

Now starting with the last equation, we solve for the GCD.

$$r_n(x) = r_{n-2}(x) - r_{n-1}(x) \cdot q_n(x) \tag{$*$}$$

Note this shows how to write the GCD as a linear combination of $r_{n-2}(x)$ and $r_{n-1}(x)$.

But in the next to the last equation we can solve for $r_{n-1}(x)$ and substitute into $(*)$.

$$r_n(x) = r_{n-2}(x) - [\ r_{n-3}(x) - r_{n-2}(x) \cdot q_{n-1}(x)\ ] \cdot q_n(x)$$
$$= r_{n-2}(x) - r_{n-3}(x) \cdot q_n(x) + r_{n-2}(x) \cdot q_{n-1}(x) \cdot q_n(x)$$
$$= [1 + q_{n-1}(x) \cdot q_n(x)] \cdot r_{n-2}(x) + [\,-q_n(x)] \cdot r_{n-3}(x)$$

We have now shown how to write $r_n(x)$ as a linear combination of $r_{n-2}(x)$ and $r_{n-3}(x)$.

Clearly we can continue to work backwards, and solve each next equation for the previous remainder, and then substitute that remainder (which is a linear combination of its two previous remainders) into our equation to continually write $r_n(x)$ as a linear combination of the two most recent remainders.

proof continued on the next page

As we work our way up the list, we will eventually have

$$r_n(x) = f(x) \cdot r_1(x) + g(x) \cdot r_2(x)$$

and when we solve the second equation for $r_2(x)$ and substitute we get

$$r_n(x) = f(x) \cdot r_1(x) + g(x)[\, q(x) - r_1(x) \cdot q_2(x)\,]$$
$$= f(x) \cdot r_1(x) + g(x)q(x) - g(x) \cdot r_1(x) \cdot q_2(x)$$
$$= [\, f(x) - g(x) \cdot q_2(x)\,] \cdot r_1(x) + [\, g(x)\,] \cdot q(x)$$

Lastly we solve the first equation for $r_1(x)$ and substitute and we get

$$r_n(x) = [\, f(x) - g(x) \cdot q_2(x)\,] \cdot [\, p(x) - q(x) \cdot q_1(x)\,] + [\, g(x)\,] \cdot q(x)$$
$$= [\, f(x) - g(x) \cdot q_2(x)\,] \cdot\ p(x)\ - [\, f(x) - g(x) \cdot q_2(x)\,] \cdot q(x) \cdot q_1(x) + g(x) \cdot q(x)$$
$$= [\, f(x) - g(x) \cdot q_2(x)\,] \cdot\ p(x)\ + \{[\, g(x) \cdot q_2(x) - f(x)\,] \cdot q_1(x) + g(x)\} \cdot q(x)$$

This shows that the GCD can be written as a linear combination of $p(x)$ and $q(x)$.

Q.E.D.

28. Lemma 1 for Partial Fractions

If $f(x) = \dfrac{a(x)}{b(x)c(x)}$ where $GCD(b(x), c(x)) = 1$ then there exist polynomials $d(x)$ and $e(x)$ such that

$$f(x) = \frac{d(x)}{b(x)} + \frac{e(x)}{c(x)}$$

Proof of Lemma 1 for Partial Fractions

The two polynomials $b(x)$ and $c(x)$ are called relatively prime when their GCD is 1. Of course this means that $b(x)$ and $c(x)$ have no common factor. Apply the Corollary to the Euclidean Algorithm for polynomials to construct polynomials $s(x)$ and $t(x)$ such that

$$1 = s(x) \cdot b(x) + t(x) \cdot c(x)$$

Then multiply both sides of this equation by $a(x)$ to get

$$a(x) = a(x) \cdot s(x) \cdot b(x) + a(x) \cdot t(x) \cdot c(x)$$

and finally divide both sides of this last equation by the product $b(x) \cdot c(x)$

$$\frac{a(x)}{b(x) \cdot c(x)} = \frac{a(x) \cdot s(x) \cdot b(x)}{b(x) \cdot c(x)} + \frac{a(x) \cdot t(x) \cdot c(x)}{b(x) \cdot c(x)}$$

$$f(x) = \frac{a(x)}{b(x) \cdot c(x)} = \frac{a(x) \cdot s(x)}{c(x)} + \frac{a(x) \cdot t(x)}{b(x)}$$

Now let $d(x) = a(x) \cdot s(x)$ and let $e(x) = a(x) \cdot t(x)$.

Q.E.D.

29. Lemma 2 for Partial Fractions

If $f(x) = \dfrac{p(x)}{[\,q(x)\,]^m}$ then there exists a polynomial $g(x)$ and for $1 \leq i \leq m$ there exist polynomials $s_i(x)$ each with degree less than $q(x)$ such that

$$f(x) = \frac{p(x)}{[\,q(x)\,]^m} = g(x) + \frac{s_1(x)}{q(x)} + \frac{s_2(x)}{[\,q(x)\,]^2} + \frac{s_3(x)}{[\,q(x)\,]^3} + \cdots + \frac{s_m(x)}{[\,q(x)\,]^m}$$

Proof of Lemma 2 for Partial Fractions

Apply the Division Algorithm for the first time to write :

$p(x) = q(x) \cdot [\,q_1(x)\,] + r_1(x)$. Note the degree of $r_1(x)$ is less than the degree of $q(x)$.

Now divide $q_1(x)$ by $q(x)$ to get a second quotient and a second remainder so we may write

$p(x) = q(x) \cdot [\,q(x) \cdot q_2(x) + r_2(x)\,] + r_1(x)$

$p(x) = [\,q(x)\,]^2 \cdot q_2(x) + q(x) \cdot r_2(x) + r_1(x)$

Note that the degree of $r_2(x)$ is less than the degree of $q(x)$.

Now divide $q_2(x)$ by $q(x)$ to get a third quotient and a third remainder and write

$p(x) = [\,q(x)\,]^2 \cdot [\,q(x) \cdot\ q_3(x) + r_3(x)\,]\ + q(x) \cdot r_2(x) + r_1(x)$

$p(x) = [\,q(x)\,]^3 \cdot q_3(x) + [\,q(x)\,]^2 \cdot r_3(x) + q(x) \cdot r_2(x) + r_1(x)$

We continue to divide each newest quotient $q_i(x)$ by $q(x)$ to get a newer quotient and a newer remainder and substitute for the $q_i(x)$ quotient. Each remainder has a degree smaller than the degree of $q(x)$.

$p(x) = [\,q(x)\,]^3 \cdot [\,q(x) \cdot q_4(x) + r_4(x)\,] + [\,q(x)\,]^2 \cdot r_3(x) + q(x) \cdot r_2(x) + r_1(x)$

$p(x) = [\,q(x)\,]^4 \cdot q_4(x) + [\,q(x)\,]^3 \cdot r_4(x) + [\,q(x)\,]^2 \cdot r_3(x) + q(x) \cdot r_2(x) + r_1(x)$

We may continue breaking down and substituting for each $q_i(x)$ quotient until we have

$p(x) = [\,q(x)\,]^m \cdot q_m(x) + [\,q(x)\,]^{m-1} \cdot r_m(x) + \cdots + [\,q(x)\,]^2 \cdot r_3(x) + q(x) \cdot r_2(x) + r_1(x)$

Finally we divide both sides of this last equation by $[\,q(x)\,]^m$ to get

$$f(x) = \frac{p(x)}{[\,q(x)\,]^m} = q_m(x) + \frac{r_m(x)}{q(x)} + \cdots + \frac{r_3(x)}{[\,q(x)\,]^{m-2}} + \frac{r_2(x)}{[\,q(x)\,]^{m-1}} + \cdots + \frac{r_1(x)}{[\,q(x)\,]^m}$$

Now we may let $g(x) = q_m(x)$ and let $s_i(x) = r_{m-i+1}(x)$.

Q.E.D.

30. Partial Fraction Decomposition Theorem

Let $\dfrac{p(x)}{q(x)}$ be a rational function where $p(x)$ and $q(x)$ are polynomials such that the degree of $p(x)$ is less than the degree of $q(x)$. Then there exist algebraic fractions $F_1, F_2, \ldots, F_r$ such that $\dfrac{p(x)}{q(x)} = F_1 + F_2 + \cdots + F_r$ and where each F_i fraction is one of two forms:

$\dfrac{A_i}{(a_i x + b_i)^{n_i}}$ or $\dfrac{A_k x + B_k}{(a_k x^2 + b_k x + c_k)^{m_k}}$ where A_i, a_i, b_i, A_k, B_k, a_k, b_k, c_k are all real numbers and the n_i and the m_k are positive integers and each quadratic expression $a_k x^2 + b_k x + c_k$ has a negative discriminant.

Proof of the Partial Fraction Decomposition Theorem

Since $q(x)$ is a polynomial, by the Linear and Irreducible Quadratic Factors Theorem we may write

$$q(x) = \left[\prod_{i=1}^{j} (a_i x + b_i)^{p_i}\right] \cdot \left[\prod_{k=1}^{l} (a_k x^2 + b_k x + c_k)^{q_k}\right]$$

where for each i, $(a_i x + b_i)$ is a real linear factor of $q(x)$ of multiplicity p_i and for each k, $(a_k x^2 + b_k x + c_k)$ is an irreducible quadratic factor of $q(x)$ of multiplicity q_k. The a_i, b_i are different from the a_k, b_k. Since the real linear and irreducible quadratic factors have no factors in common their GCD is 1 and we may apply Lemma 1 for Partial Fractions to write:

$$\frac{p(x)}{q(x)} = \frac{a(x)}{\left[\prod_{i=1}^{j} (a_i x + b_i)^{p_i}\right]} + \frac{b(x)}{\left[\prod_{k=1}^{l} (a_k x^2 + b_k x + c_k)^{q_k}\right]}$$

Now for each different i, each factor of the form $(a_i x + b_i)^{p_i}$ is different from the next so we may again apply Lemma 1 for Partial Fractions $j - 1$ times to split the first fraction above into a sum of j other fractions. For each of those fractions that have an exponent of 2 or higher in the denominator we apply Lemma 2 for Partial Fractions $p_i - 1$ more times to split each denominator with $(a_i x + b_i)^{p_i}$ into a sum of p_i more fractions.

For each different k, each factor of the form $(a_k x^2 + b_k x + c_k)^{q_k}$ is different from the next so we may again apply Lemma 1 for Partial Fractions $k - 1$ times to split the second fraction above into a sum of k other fractions. For each of those fractions that have an exponent of 2 or higher in the denominator we apply Lemma 2 for Partial Fractions $q_k - 1$ more times to split each denominator with $(a_k x^2 + b_k x + c_k)^{q_k}$ into a sum of q_k more fractions.

As a final note, the $g(x)$ term that appears in Lemma 2 for Partial Fractions will be the 0 polynomial because we are assuming the degree of $p(x)$ is strictly less than that of $q(x)$. So our partial fraction decomposition really does break down into a sum of pure algebraic fractions.

Q.E.D.

31. Partial Fraction Decomposition Coefficient Theorem

Let $\dfrac{p(x)}{q(x)}$ be a rational function where $p(x)$ and $q(x)$ are polynomials such that the degree of $p(x)$ is less than the degree of $q(x)$. If $x = a$ is a root of $q(x) = 0$ of multiplicity 1, then in the partial fraction decomposition of $\dfrac{p(x)}{q(x)}$ which contains a term of the form $\dfrac{A}{(x-a)}$, the constant $A = \dfrac{p(a)}{q'(a)}$.

Proof of the Partial Fraction Decomposition Coefficient Theorem:

Assume $\dfrac{p(x)}{q(x)} = \dfrac{A}{(x-a)} + f(x)$ is the partial fraction decomposition of $\dfrac{p(x)}{q(x)}$ where $f(x)$ is itself a rational function, but is such that $f(a)$ is well-defined. In fact, $f(x)$ will be continuous at $x = a$.

Then $\dfrac{(x-a)\cdot p(x)}{q(x)} = A + f(x)(x-a)$

Since $x = a$ is a simple zero of $q(x)$, $\displaystyle\lim_{x\to a}\frac{(x-a)\cdot p(x)}{q(x)}$ is an indeterminate form $\dfrac{0}{0}$ and we may thus apply L'Hopital's Rule when evaluating the limit. Taking the limit on both sides of the above equation we have

$$\lim_{x\to a}\frac{(x-a)p(x)}{q(x)} = \lim_{x\to a}[A + f(x)(x-a)]$$

$$\lim_{x\to a}\frac{1\cdot p(x) + (x-a)\cdot p'(x)}{q'(x)} = \lim_{x\to a}A \;+\; \lim_{x\to a}f(x)\cdot\lim_{x\to a}(x-a)$$

$$\lim_{x\to a}\frac{p(x)}{q'(x)} = A + \lim_{x\to a}f(x)\cdot 0$$

$$\frac{p(a)}{q'(a)} = A + f(a)\cdot 0 = A$$

Q.E.D.