Enumerating Anchored Permutations with Bounded Gaps

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Abstract

Say that a permutation of $1, 2, \ldots, n$ is *k-bounded* if every pair of consecutive entries in the permutation differs by no more than k. Such a permutation is *anchored* if the first entry is 1 and the last entry is n. We show that the generating function for the enumeration of k-bounded anchored permutations is always rational, mirroring the known result on (non-anchored) k-bounded permutations due to Avgustinovich and Kitaev. We then explicitly determine the recursive formulas of minimal depth for the number of anchored k-bounded permutations of n for $k = 2$ and $k = 3$, resolving a conjecture listed on the Online Encyclopedia of Integer Sequences (entry A249665). We additionally show that the number of anchored k-bounded permutations of n is asymptotically $O(k^n)$ as a function of n for a given k.

1 Introduction

Suppose one starts on the first stair of a staircase with n steps labeled $1, \ldots, n$ in order, and at each step one either steps forwards or backwards by at most k steps, such that every stair

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is used exactly once and the climb ends on the nth stair. How many distinct such ways are there to climb the stairs?

This question can be stated more precisely as follows. For a positive integer k, define a **k-bounded** permutation of $[n] = \{1, 2, \ldots, n\}$ to be a bijection $\pi : [n] \to [n]$ such that for all $i \in \{1, 2, \ldots, n-1\}$ we have

$$|\pi(i) - \pi(i+1)| \leq k.$$

We say that a permutation is **anchored** if $\pi(1) = 1$ and $\pi(n) = n$. We are interested in enumerating the k-bounded anchored permutations in terms of k and n.

Example 1. The permutation $1, 4, 2, 3, 6, 5, 7, 8, 9$ is a 3-bounded anchored permutation of $\{1, 2, \ldots, 9\}$, since the first entry is 1, the last entry is 9, and no pair of consecutive entries differs by more than 3.

Several related questions have been previously explored. Positive stair climbing problems were studied by Goins and Washington [4], extending the well-known fact that the number of ways to climb a staircase of length n using positive steps of $+1$ or $+2$ each time is the nth Fibonacci number.

Avgustinovich and Kitaev [1] studied *k-determined permutations*, which they show are equivalent to (the inverses of) $(k-1)$-bounded, *non-anchored* permutations, as well as certain Hamiltonian paths in graphs. They resolve a conjecture of Plouffe [9] by providing the generating function for 2-bounded non-anchored permutations, which were originally defined as *key permutations* [8]. Avgustinovich and Kitaev further show that the generating function of the k-determined permutations of $[n]$ is always rational for any k, using the transfer-matrix method described by Stanley [10, Ch. 4].

In this paper, we show that the generating function is still rational for *anchored* k-bounded permutations, for any k. Furthermore, we resolve the question of finding a minimal-depth linear recurrence for the enumeration of 3-bounded anchored permutations, which was originally posed as a conjecture on the Online Encyclopedia of Integer Sequences, entry A249665 [7].

1.1 Main results

Let $A_n^{(k)}$ be the number of k-bounded anchored permutations of $[n]$. For $k = 1$, there is clearly only one 1-bounded anchored permutation for each n, namely the identity permutation. In this paper, we resolve the cases $k = 2$ and $k = 3$ completely by finding recursions for $A_n^{(2)}$ and $A_n^{(3)}$. We also adapt the methods in [1] to show that the generating function

$$A^{(k)}(x) = \sum A_n^{(k)} x^n$$

of k-bounded anchored permutations is rational for any k, which in particular implies that there always exists a finite-depth homogeneous linear recurrence relation for enumerating these permutations. We then apply techniques from algebraic graph theory to obtain an asymptotic upper bound for $A_n^{(k)}$.

Our main results are summarized in the following theorems.

Theorem 2. *The generating function $A^{(k)}(x)$ is a rational function, that is, $A^{(k)}(x) \in \mathbb{Z}(x)$.*

Due to standard results on rational generating functions (see [10, Ch. 4]), we immediately obtain the following corollary.

Corollary 3. *For any fixed k, the numbers $A_n^{(k)}$ satisfy a finite-depth homogeneous linear recurrence in n with integer coefficients.*

The proof of Theorem 4 relies on the transfer-matrix method, giving an expression for the rational function that is explicit, but may not be a reduced fraction. It therefore gives an explicit recursion for $A_n^{(k)}$ for all k, but this recursion may not be the minimal depth recursion. We give the reduced generating function and minimal recursion for $k = 2$ and $k = 3$ in the next two theorems.

For simplicity, we define $R_n = A_n^{(2)}$ and $A_n = A_n^{(3)}$ in the theorems below.

Theorem 4. *Let R_n be the number of 2-bounded anchored permutations of $[n]$. Then the sequence $(R_n)_{n\geq 1}$ is given by the recurrence $R_1 = 1$, $R_2 = 1$, $R_3 = 1$, and*

$$R_n = R_{n-1} + R_{n-3} \tag{1}$$

for all $n \geq 4$. The generating function of the sequence is

$$R(x) = \sum_{n=1}^{\infty} R_n x^n = \frac{x}{1 - x - x^3}.$$

This sequence R_n is also known as *Narayana's cows sequence* [6], and the particular interpretation as 2-bounded anchored permutations is stated without proof (in a slightly different but equivalent form) in Flajolet and Sedgewick [3, p. 373]. Note the similarity to the Fibonacci recurrence. It is interesting that for steps of $+1$ and $+2$ only, the recurrence is precisely the Fibonacci sequence, and here, with the added steps of -1 and -2 where every step is reached, only one index differs by one from the Fibonacci recurrence.

We provide two separate proofs of Theorem 4. The first uses the transfer-matrix method used to prove Theorem 2, which we include in order to illustrate the method. The second is a direct, elegant combinatorial proof.

Theorem 5. *Let A_n be the number of 3-bounded anchored permutations of $[n]$. Then the sequence $(A_n)_{n\geq 1}$ is given by the recurrence $A_1 = 1$, $A_2 = 1$, $A_3 = 1$, $A_4 = 2$, $A_5 = 6$, $A_6 = 14$, $A_7 = 28$, $A_8 = 56$, and*

$$A_n = 2A_{n-1} - A_{n-2} + 2A_{n-3} + A_{n-4} + A_{n-5} - A_{n-7} - A_{n-8} \tag{2}$$

for all $n \geq 9$. The generating function of the sequence is

$$A(x) = \frac{x - x^2 - x^4}{1 - 2x + x^2 - 2x^3 - x^4 - x^5 + x^7 + x^8}.$$

Lastly, we determine an upper bound for the Perron-Frobenius eigenvalue of the transfer matrix mentioned above. We use this to obtain an asymptotic bound for $A_n^{(k)}$.

Theorem 6. *For any k, we have that $A_n^{(k)}$ is $O(k^n)$ as $n \to \infty$.*

We prove Theorems 2, 4, 5, and 6 in Sections 2, 3.2, 3.3, and 4, respectively.

2 Rationality of the generating function for all k

We now prove Theorem 2. To do so, we first recall the basic well-known notions of permutation patterns, as well as important definitions and facts from [1]. Throughout this section we write S_m for the set of all permutations of $\{1, 2, \dots, m\}$. We also fix positive integers k and n with $k \le n$ throughout.

Definition 7. A **consecutive pattern** of length k of a permutation $\pi \in S_n$ is a permutation $\tau \in S_k$ for which the relative order of the entries $\tau(1), \dots, \tau(k)$ exactly matches that of k consecutive entries $\pi(i+1), \pi(i+2), \dots, \pi(i+k)$ of π for some $i \in \{0, 1, \dots, n-k\}$.

For instance, the consecutive patterns of length 3 of the permutation $5, 1, 4, 3, 2$ are $3, 1, 2$, $1, 3, 2$, and $3, 2, 1$ (from left to right). We can think of these consecutive patterns as defining a path in the following graph defined in [1], which is similar to the well-known *de Bruijn graphs*.

Definition 8 ([1])**.** The **graph of pattern overlaps of length** k is the directed graph $\mathcal{P}_k$ whose vertex set is S_k and for which there is a directed edge from τ to σ if and only if $\tau(2), \dots, \tau(k)$ has the same relative order as $\sigma(1), \dots, \sigma(k-1)$.

If $k \le n$, then any permutation π of $\{1, 2, \dots, n\}$ corresponds to a unique path in $\mathcal{P}_k$, given by considering the sequence of consecutive patterns of length k in π from left to right. For example, the path corresponding to $5, 1, 4, 3, 2$ in $\mathcal{P}_3$ is

$$3, 1, 2 \longrightarrow 1, 3, 2 \longrightarrow 3, 2, 1.$$

However, not all paths determine a unique permutation, as there may be more than one permutation corresponding to the same path. For instance, $5, 2, 4, 3, 1$ has the same path as the example $5, 1, 4, 3, 2$ above. This naturally leads one to consider the following definition.

Definition 9 ([1])**.** A permutation π is k**-determined** if its corresponding path in the graph $\mathcal{P}_k$ uniquely determines π.

Note that the permutations $5, 1, 4, 3, 2$ and $5, 2, 4, 3, 1$ from our example above, which have the same path in $\mathcal{P}_3$, are not 2-bounded. It turns out that the condition of being k-bounded is equivalent to being $(k+1)$-determined, as follows.

Lemma 10. *Let $\pi \in S_n$. The following are equivalent:*

- *π is $(k+1)$-determined*
- *For all $x \in \{1, 2, \dots, n-1\}$, we have $|\pi^{-1}(x) - \pi^{-1}(x+1)| \le k$*
- *π^{-1} is k-bounded.*

Moreover, π^{-1} is anchored if and only if π is anchored, and so the anchored k-bounded permutations of n are in bijection with the anchored $(k+1)$-determined permutations.

Proof. The equivalence of the first two statements was proven in Theorem 1 of [1]. The remaining statements follow immediately from the definitions of k-bounded and anchored. □

Lemma 10 allows us to focus instead on the generating function of the anchored $(k+1)$-determined permutations of n. In order to apply the transfer-matrix method, we first recall the notion of a **prohibited pattern**.

Definition 11. ([1]) A **k-prohibited pattern** is a consecutive pattern of the form $x, X, (x+1)$ or $(x+1), X, x$ where X is a permutation of $\{1, 2, \ldots, |X|+2\} - \{x, x+1\}$ with $|X| \geq k$.

Finally, for any j, define $\mathcal{P}_{j,k}$ to be the graph formed by deleting all vertices from $\mathcal{P}_j$ that contain a consecutive pattern that is k-prohibited.

Example 12. The permutation $5, 2, 4, 3, 1$ contains an occurrence of the 2-prohibited pattern $2, 4, 3, 1$.

We say a prohibited pattern P is **k-irreducible** if it has no consecutive patterns of length less than $|P|$ which are k-prohibited.

Lemma 13 ([1]). *Any k-irreducible, k-prohibited pattern has length at most $2k+1$. Moreover, the $(k+1)$-determined permutations in S_n correspond bijectively to paths of length $n-2k$ in the graph $\mathcal{P}_{2k+1,k}$.*

We now give a further criterion to determine which paths in $\mathcal{P}_{2k+1,k}$ correspond to the *anchored* $(k+1)$-determined permutations.

Theorem 14. *A $(k+1)$-determined permutation is anchored if and only if its corresponding path in $\mathcal{P}_{2k+1,k}$ starts at a vertex whose pattern starts with 1 and ends at a vertex whose pattern ends with $2k+1$.*

Proof. The forward implication is clear: if a permutation in S_n is anchored, with first entry 1 and last entry n, then its first consecutive pattern must start with 1 and its last consecutive pattern (of length $2k+1$) must end with $2k+1$.

Conversely, let $\pi \in S_n$ be $(k+1)$-determined and suppose its first pattern of length $2k+1$ starts with 1 and its last ends with $2k+1$. Assume for contradiction that $\pi(1) > 1$. Let $x = \pi(1)$. Then due to the first pattern starting with 1, the entry $x-1$ does not occur among the first $2k+1$ values of π. But then $|\pi^{-1}(x-1) - \pi^{-1}(x)| \geq 2k+1 > k$, which contradicts $(k+1)$-determinability by Lemma 10. It follows that $\pi(1) = 1$, and a similar argument shows that $\pi(n) = n$. Thus π is anchored. □

Theorem 2 now follows using the transfer-matrix method by counting paths in a directed graph. Namely, Theorem 4.7.2 in [10] shows that the generating function for the number of paths of length $n-2k$ between any two fixed vertices in a finite directed graph (in this case $\mathcal{P}_{2k+1,k}$) is rational. Summing the rational generating functions from all possibilities of valid starting and ending vertices given by Theorem 14 gives the desired result.

3 Structure and enumeration for $k = 2$ and $k = 3$

We now use Theorem 14 to prove Theorem 4, the recursion for $k = 2$.

In particular, set $k = 2$ and define M to be the adjacency matrix of the graph $\mathcal{P}_{2k+1,k} = P_{5,2}$, that is, the matrix having $M_{ij} = 1$ if there is an edge from vertex i to vertex j, and $M_{ij} = 0$ otherwise. Define $p_{ij}(n)$ to be the number of paths of length n from vertex i to vertex j in the directed graph $\mathcal{P}_{5,2}$. Then the transfer-matrix method states that

$$G_{ij}(x) := \sum_{n=0}^{\infty} p_{ij}(n)x^n = \frac{(-1)^{i+j}\det(I - xM; j, i)}{\det(I - xM)}$$

where $\det(I - xM; j, i)$ is the determinant of the minor of the matrix $I - xM$ formed by deleting the jth row and ith column.

Note that the sum of the generating functions $G_{ij}(x)$, over all starting vertices i whose permutation starts with 1 and over all ending vertices j whose permutation ends with 5, is precisely the generating function

$$\sum_{n=0}^{\infty} R_{n+5}x^n,$$

since the paths of length n in $\mathcal{P}_{5,2}$ correspond to permutations of length $n + 5$. Using a computer, one easily finds that this summation equals

$$(3 + x + 2x^2)/(1 - x - x^3).$$

Since we can explicitly calculate that $R_0 = 0$, $R_1 = R_2 = R_3 = 1$, and $R_4 = 2$, we have that

$$\begin{aligned}
\sum_{n=0}^{\infty} R_n x^n &= x + x^2 + x^3 + 2x^4 + x^5 \left(\sum_{n=0}^{\infty} R_{n+5}x^n\right) \\
&= x + x^2 + x^3 + 2x^4 + x^5(3 + x + 2x^2)/(1 - x - x^3) \\
&= x/(1 - x - x^3),
\end{aligned}$$

as desired.

Surprisingly, the same computation for $k = 3$ proved to be computationally intractable due to the significantly larger size of the adjacency matrix. We therefore provide a direct combinatorial proof for the minimal depth recursion for $k = 3$. To do so, and to establish a combinatrial proof for $k = 2$, we require some additional notation.

3.1 Notation

A **gap** of a permutation π is a difference $\pi(i + 1) - \pi(i)$ between two consecutive entries. We will always write our gaps with a $+$ or $-$ sign in front to indicate the sign, even if the sign is clear, to distinguish gaps from entries. For instance, we would say that the first gap of the permutation $1, 3, 2, 4$ is $+2$, and the second gap is -1. We sometimes refer to the gaps of a sequence that is not a permutation as well, defined in the same way as consecutive differences between entries.

A sequence whose gaps are all between $-k$ and $+k$ is said to be **blocked** or **stuck** at the end if the last entry a has the property that $a \pm 1, \ldots, a \pm k$ all either occur in the sequence or are less than or equal to 0. For instance, if $k = 3$, the sequence $1, 3, 4, 6, 5, 2$ is blocked at 2; the next possible positive integer that has not been used is 7, which is more than a gap of k away.

The **graph** of a permutation of $\{1, \ldots, n\}$ is the plot of all points $(i, \pi(i))$ in the plane. The **main diagonal** is the line with equation $y = x$. Note that a point in the graph of a permutation is on the main diagonal if and only if it is a fixed point of the permutation.

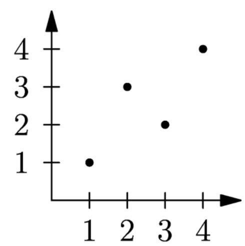

Figure 1: The graph of the 2-bounded permutation $1, 3, 2, 4$.

3.2 Combinatorial proof for $k = 2$

We now give a purely combinatorial proof of Theorem 4. To get a handle on the 2-bounded anchored permutations, we first prove the following lemma. It is worth noting that a weaker version of the lemma suffices to prove recursion (1), but the stronger statement explicitly describes the structure of a 2-bounded permutation.

Lemma 15. *Let π be an anchored 2-bounded permutation of $[n]$. Then there exists a subset $I \subseteq \{2, \ldots, n-2\}$ such that*

1. *Any pair of numbers in I differ by at least three, and*
2. *For all $i \in [n]$,*

$$\pi(i) = \begin{cases} i+1, & \text{if } i \in I; \\ i-1, & \text{if } i-1 \in I; \\ i, & \text{otherwise.} \end{cases}$$

In other words, the graph of the permutation can only deviate from the diagonal $x = y$ in consecutive pairs, with an up-step of 2 and a down-step of 1, before returning to the diagonal with an up-step of 2. (See Figure 2.)

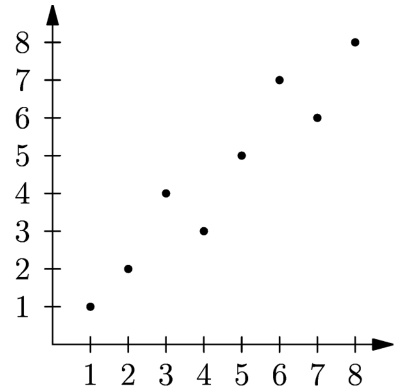

Figure 2: The 2-bounded permutation graphed above, $1, 2, 4, 3, 5, 7, 6, 8$, has subset $I = \{3, 6\}$ as the set of indices i for which $\pi(i) = i + 1$.

Proof. The lemma is clearly true when $n = 1$. We proceed by strong induction on n. Assume that the lemma holds for all positive integers $n' < n$, and let π be an anchored 2-bounded permutation of $[n]$.

If π is the identity permutation then $I = \emptyset$ and we are done, so we may assume that π is not the identity. Let i be the smallest index for which $\pi(i) \neq i$. Note that $i \in \{2, \ldots, n-2\}$ because π is anchored. Then since $\pi(j) = j$ for all $j < i$, the gap from $\pi(i-1)$ to $\pi(i)$ cannot be -1, -2, or $+1$. It therefore must be $+2$, and we have

$$\pi(i) = \pi(i-1) + 2 = i - 1 + 2 = i + 1.$$

Now, the next gap, from $\pi(i)$ to $\pi(i+1)$, can either be -1, $+1$, or $+2$. We claim that it is not $+1$ or $+2$. If the gap were $+1$, then $i+1$ and $i+2$ both occur before the value i appears in the permutation. So for some $i + 1 < j < n$, $\pi(j) = i$. But then the value of $\pi(j+1)$ must be at least $i+3$ (since all other possible values are already used), and this contradicts 2-boundedness. Otherwise, if the gap between $\pi(i)$ and $\pi(i+1)$ is $+2$, so that $\pi(i+1) = i+3$, then the only way to reach i in the permutation is via a -2 step from $i+2$. So there exists $i + 1 < j < n$ such that $\pi(j) = i$ and $\pi(j-1) = i+2$. But then the value of $\pi(j+1)$ must be at least $i+4$ (since all other possible values are already used), and this contradicts 2-boundedness.

It follows that the gap at i is -1, so $\pi(i+1) = i$. The only possible value for $\pi(i+2)$ is then $i + 2$ (with a $+2$ step from the previous), which is on the diagonal again with all smaller numbers having occurred to the left of it. The remaining entries form a 2-bounded, anchored permutation of $\{i+2, i+3, \ldots, n\}$, which has a corresponding subset $I' \subseteq \{i+3, i+4, \ldots, n-2\}$ that satisfies the conditions above by the inductive hypothesis. Since i is at least 3 less than any element of I', we see that setting $I = \{i\} \cup I'$ gives a valid subset that corresponds to π. □

We now can prove Theorem 4.

Proof. It is easily checked that $R_1 = R_2 = R_3 = 1$. Let $n \geq 4$. Then any anchored 2-bounded permutation π of $[n]$ either starts with $1, 2$ or $1, 3$. In the former case, there are R_{n-1} ways of completing the permutation, since any 2-bounded way of completing it that ends at n is an anchored permutation of $\{2, \dots, n\}$.

In the latter case, by Lemma 15, the first four entries of the permutation must be $1, 3, 2, 4$, and then the remaining entries starting from 4 form a 2-bounded anchored permutation of $\{4, 5, \dots, n\}$. It follows that there are R_{n-3} possibilities if the permutation starts with $1, 3$.

These cases combine to show that $R_n = R_{n-1} + R_{n-3}$.

The generating function now follows from a straightforward calculation. We have

$$\begin{aligned} R(x) - xR(x) - x^3R(x) &= \sum_{n=1}^{\infty} R_n x^n - \sum_{n=2}^{\infty} R_{n-1} x^n - \sum_{n=4}^{\infty} R_{n-3} x^n \\ &= x + x^2 + x^3 - (x^2 + x^3) + \sum_{n=4}^{\infty} (R_n - R_{n-1} - R_{n-3}) x^n \\ &= x + \sum_{n=4}^{\infty} 0 \cdot x^n \\ &= x, \end{aligned}$$

and it follows that $R(x) = x/(1 - x - x^3)$. □

3.3 Combinatorial proof for $k = 3$

As in Theorem 5, we define A_n to be the number of 3-bounded anchored permutations of $[n]$. In the 2-bounded case, we saw that there is one possible pattern in which the permutations can veer from the identity, and used that to generate the recursion. Similarly, in the 3-bounded case, we will need to single out a certain special sequence that interferes with an otherwise regular pattern that the permutations must follow.

Definition 16. The **Joker** is the sequence $3, 1, 4, 2, 5$. We say the Joker **appears** in a 3-bounded permutation if for some i, the ith through $(i + 4)$th entries of the permutation are $i + 2, i, i + 3, i + 1, i + 4$.

Aside from the Joker, the 3-bounded permutations turn out to follow a predictable pattern in terms of runs of $+3$ and -3 steps. We will use this structure to devise a three-term recurrence for A_n.

Definition 17. Define B_n to be the number of 3-bounded permutations π of $\{1, 2, \dots, n\}$ that start with either $\pi(1) = 1$ or $\pi(1) = 2$ (so they are not necessarily anchored) and end at $\pi(n) = n$.

Definition 18. Define C_n to be the number of 3-bounded permutations π of $\{1, 2, \dots, n\}$ that start with $\pi(1) = 3$, end with $\pi(n) = n$, and do not start with the Joker as the first five terms.

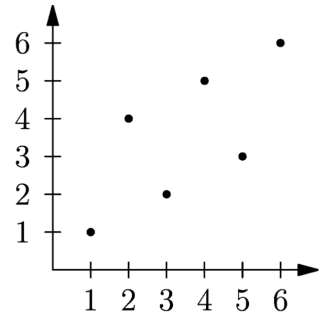

Figure 3: The Joker appears in the above permutation, in its second through sixth entries.

We claim that for all $n \geq 6$, the sequences A_n, B_n, and C_n satisfy the following recurrence relations:

$$\begin{aligned} A_n &= B_{n-1} + C_{n-1} + A_{n-5}, \\ B_n &= A_n + B_{n-2} + A_{n-3} + B_{n-4} + C_{n-2}, \\ C_n &= A_{n-3} + B_{n-3} + A_{n-4} + B_{n-5} + C_{n-3}. \end{aligned}$$

To prove these relations, we first prove the following structure lemma.

Lemma 19. *Suppose π is a 3-bounded anchored permutation of $[n]$, and that the first i entries form a 3-bounded anchored permutation of $[i]$, so that $\pi(1) = 1$, $\pi(i) = i$, and the numbers $1, \ldots, i$ comprise the first i entries of the permutation in some order. If the next step is $+3$, then one of the following two patterns occurs starting at entry i:*

1. *The Joker appears as entries i through $i + 4$.*

2. *There is a positive integer m and a gap $d \in \{\pm 1, \pm 2\}$ such that the sequence of gaps after i is*
$$+3, +3, \ldots, +3, d, -3, -3, \ldots, -3, \overline{d}, +3, +3, \ldots, +3$$
where the first run of $+3$'s has length m, the run of -3's has length m' where $m' = m-1$ if $d < 0$ and $m' = m$ if $d > 0$, the last run of $+3$'s has length m' as well, and
$$\overline{d} = \begin{cases} +1, & \text{if } d = 1 \text{ or } d = -2; \\ -1, & \text{if } d = 2 \text{ or } d = -1. \end{cases}$$

We call such a pattern a ***cascading 3-pattern****.*

Proof. First, note that since π restricts to a permutation on $\{1, \ldots, i\}$, we can assume for simplicity that $i = 1$. Now, suppose the next gap is $+3$, so $\pi(2) = 4$.

Let m be the length of the run of consecutive gaps of $+3$ starting from 1 before a gap d not equal to $+3$ occurs. Notice that d cannot be -3 or else the same entry would occur twice

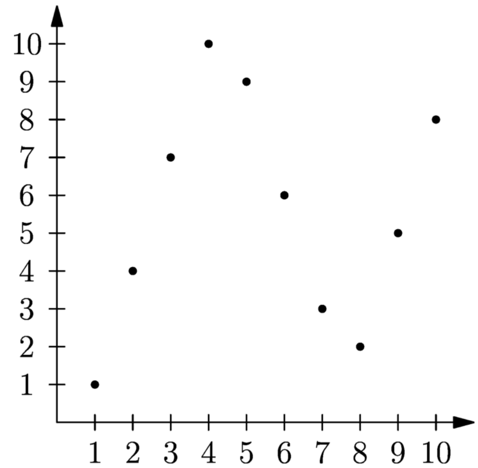

Figure 4: An example of a cascading 3-pattern, with $m = 3$ and $d = -1$.

in the permutation, and so $d \in \{\pm 1, \pm 2\}$. We will prove that one of the two possibilities above hold by induction on m.

Base Case. Suppose $m = 1$. We consider several subcases based on the value of d.

If $d = -2$, then the first three entries of the sequence are $1, 4, 2$, and the next entry may be 5 or 3. If the next entry is 5 and the fifth entry is larger than 5, then the only way to reach 3 later in the permutation is by a gap of -3 from 6, in which case we would be stuck at 3, having used 1, 2, 4, 5, and 6 already. Thus, if π starts with $1, 4, 2, 5$ then it must continue $1, 4, 2, 5, 3, 6$, which is the Joker. Otherwise, it starts $1, 4, 2, 3$, which is a cascading 3-pattern for $m = 1$ and $d = -2$.

If $d = -1$, suppose for contradiction that the next gap is positive, so that the first four entries are either $1, 4, 3, 5$ or $1, 4, 3, 6$. Then 2 must be reached from a gap of -3 from 5, at which point the permutation is stuck. Thus the next gap must be -1 as well, and the permutation must start $1, 4, 3, 2, 5$, which is a cascading 3-pattern for $m = 1$ and $d = -1$.

If $d = +1$, suppose for contradiction that the next gap is positive, so that the first four entries are either $1, 4, 5, 6$ or $1, 4, 5, 7$ or $1, 4, 5, 8$. Then to reach 2 or 3, there must be a gap of -3 from 6, at which point the permutation is blocked by 4, 5, and 6 and ends at 2 or 3, a contradiction. It follows that the next gap is -2 or -3, and in fact it must be -3 so as to reach the entry 2 without being blocked. Thus, the first five entries are $1, 4, 5, 2, 3, 6$, which is a cascading 3-pattern with $m = 1$ and $d = +1$.

Finally, if $d = +2$, suppose for contradiction that the next gap is positive or -1. Then as in the case above, the permutation becomes blocked once it reaches 2 or 3. So the next gap must be -3 and we have $1, 4, 6, 3$ as the first four entries. We must then have 2 as the fifth entry, or else the sequence would get blocked at 2 later, so the first six entries are $1, 4, 6, 3, 2, 5$, which is a cascading 3-pattern with $m = 1$ and $d = +2$.

Induction step. Suppose $m > 1$ and assume the lemma holds for $m' = m - 1$. Then π starts with $1, 4, 7$. We claim that the entries 2 and 3 must be adjacent in π. Suppose they are not adjacent. If 3 comes first, then the only way to reach 2 is by a -3 gap from 5 (since 1 and 4 are already used) at which point the permutation would be stuck at 2, a contradiction. If 2 comes first, then since 7 comes after 4 we must have reached the 2 using a -3 gap from 5. But then the only possible entry that can follow the 2 is 3, and they are in fact adjacent.

Now, consider the adjacent positions of the 2 and 3. Then the other entry adjacent to 2 must be 5, and 6 must be adjacent to 3 as well, so the 5 and 6 surround the 2 and 3. It follows that if we remove 2, 3, and 4 from the permutation and shift all entries larger than 4 down by 3, we obtain a permutation π' that starts at 1 with a $+3$ gap to 4 (which was the 7 in π). Since the 5 and 6 surrounded the 2 and 3 in π, they become 2 and 3 and are adjacent in π'. All other pairs of adjacent entries in π' still have a difference of at most 3, because they did in π and were both translated down by 3. Thus, π' is a 3-bounded anchored permutation starting with $m-1$ gaps of $+3$, and by the induction hypothesis it must either start with the Joker or a cascading 3-pattern.

Since the 2 and 3 are adjacent in π' it cannot start with the Joker and so it must be of the second form. It follows that π also starts with a cascading 3-pattern, formed by inserting one more $+3$ and -3 and $+3$ into each of the runs of 3's that comprise the gaps of π'. □

We now prove each of the recurrence relations as their own lemma.

Lemma 20. *We have* $A_n = B_{n-1} + C_{n-1} + A_{n-5}$.

Proof. Any 3-bounded anchored permutation either starts with a gap of $+1$, $+2$, or $+3$. If it starts with $+1$ or $+2$, together the number of possibilities are equal to the number of 3-bounded permutations of $\{2, \ldots, n\}$ that start with either 2 or 3, which is exactly B_{n-1}.

If it starts with $+3$, then by Lemma 19 it either starts with the Joker sequence or is a cascading 3-pattern. If it starts with the Joker, then $\pi(6) = 6$ and the first six entries are a permutation of $[6]$, so the entries after the fifth form a 3-bounded anchored permutation of $\{6, 7, \ldots, n\}$. There are therefore A_{n-5} possibilities in this case. Otherwise, the number of possibilities is equal to the number of 3-bounded permutations of $\{2, \ldots, n\}$ that start with 4 and end at n but do not start with the Joker, which is exactly C_{n-1}. The recursion follows. □

Lemma 21. *We have* $B_n = A_n + B_{n-2} + A_{n-3} + B_{n-4} + C_{n-2}$.

Proof. We now wish to enumerate the 3-bounded permutations that start at either 1 or 2 and end at n. The number starting at 1 is A_n, which is the first term in the recurrence.

For those starting at 2, if the next entry is 1 then the third entry can either be 3 or 4. We now wish to count 3-bounded permutations of $\{3, \ldots, n\}$ that start at either 3 or 4 and end at n, which is exactly B_{n-2}.

If the first two entries are $2, 3$, then if the next gap is positive it follows that the 1 can only be reached by a gap of -3 from 4, at which point the permutation is stuck. It follows that the next gap is negative, and it must be a gap of -2. So the first four entries are $2, 3, 1, 4$, and the remaining entries starting from 4 form a 3-bounded anchored permutation of $\{4, \ldots, n\}$. Thus, there are A_{n-3} possibilities in this case.

If the first two entries are $2, 4$, then 1 can either be reached from a gap of -3 from 4, or later from a gap of -2 from 3. But the latter option becomes stuck at 1, and so there must be a gap of -3 from 4 to 1. It follows that the permutation starts $2, 4, 1, 3$ and then continues with a 3-bounded permutation of $\{5, \ldots, n\}$ that starts at either 5 or 6. There are therefore B_{n-4} such possibilities.

Finally, if the first two entries are $2, 5$, then the 1 must occur at some point in π and must be surrounded by 3 and 4. If we remove the 1, then, we obtain a 3-bounded permutation of $\{2, \ldots, n\}$ starting at 2 and with a starting gap of $+3$, with the 3 and 4 adjacent. By Lemma 19, the 3 and 4 will always be adjacent in such a permutation with a starting gap of $+3$ unless it starts with the Joker pattern, and so, removing the 1 and the 2, we see that there are exactly C_{n-2} possibilities in this case. □

Notice that the final step in the above proof was analogous to the final step of the proof of Lemma 20. Deleting the 1 from the permutation resulted in the C_{n-1} term in the A_n recurrence, just as deleting the 1 and the 2 from the permutation resulted in the C_{n-2} term in the B_n recurrence. We will use this trick once more below, deleting the 1, 2, and 3, resulting in a C_{n-3} term in the C_n recurrence.

Lemma 22. *We have* $C_n = A_{n-3} + B_{n-3} + A_{n-4} + B_{n-5} + C_{n-3}$.

Proof. We wish to enumerate the 3-bounded permutations that start at 3 and end at n but do not start with the Joker sequence $3, 1, 4, 2, 5$. The second entry can either be 1, 2, 4, 5, or 6.

Notice that if we add a 0 to the front of the permutation, we will get a 3-bounded anchored permutation of $\{0, \ldots, n\}$ that starts with a gap of $+3$. By Lemma 19, since the permutation does not start with the Joker, it must start with a cascading 3-pattern.

Thus, if the first gap after the 3 is not $+3$, then d is determined and the 3-pattern is determined as well. In particular, if the first two entries are $3, 1$ then the permutation must start with $3, 1, 2$, and so the entries after the third form a 3-bounded permutation of $\{4, \ldots, n\}$ that starts at either 4 or 5 and ends at n. There are exactly B_{n-3} such entries in this case.

If the first two entries are $3, 2$ then since the start is a cascading 3-pattern, the first four entries are $3, 2, 1, 4$. The entries starting at 4 form a 3-bounded permutation of $\{4, \ldots, n\}$ starting at 4 and ending at n, giving us A_{n-3} more possibilities.

If the first two entries are $3, 4$, then by the cascading 3-pattern the first five entries are $3, 4, 1, 2, 5$. The entries starting at 5 form a 3-bounded permutation of $\{5, \ldots, n\}$ starting at 5 and ending at n, giving us A_{n-4} more possibilities.

If the first two entries are $3, 5$, the cascading 3-pattern tells us that the first five entries are $3, 5, 2, 1, 4$, with the next entry either 6 or 7. The entries starting after the fifth form a 3-bounded permutation of $\{6, \ldots, n\}$ starting at either 6 or 7 and ending at n, giving us B_{n-5} more possibilities.

Finally, if the first two entries are $3, 6$, then since it is a cascading 3-pattern the 1 and 2 must be adjacent in π. Removing the 1, 2, and 3 then gives a 3-bounded permutation of $\{4, \ldots, n\}$ that starts at 6 and ends at n but avoids the Joker. There are C_{n-3} such possibilities, and the proof is complete. □

We can now eliminate C_n from these recurrences to form a two-term recurrence. Putting $n-1$ in the recurrence for B_n, we have $B_{n-1} = A_{n-1} + B_{n-3} + A_{n-4} + B_{n-5} + C_{n-3}$, which nearly matches the recurrence for C_n. From this we conclude $C_n = A_{n-3} + B_{n-1} - A_{n-1}$. We can now substitute for the C terms in the A and B recurrences to obtain the following relationships:

$$A_n = B_{n-1} + A_{n-4} + B_{n-2} - A_{n-2} + A_{n-5}, \tag{3}$$

$$B_n = A_n + B_{n-2} + B_{n-3} + B_{n-4} + A_{n-5}. \tag{4}$$

Notice that our proofs above actually show that these recursions hold for all n, even $n \leq 5$, where we set $A_j = B_j = 0$ for any $j \leq 0$. Thus, we can unwind the recursions to find the first few values of A_n and B_n, as follows.

n	1	2	3	4	5	6	7	8
A_n	1	1	1	2	6	14	28	56
B_n	1	1	2	4	10	22	45	93

We now have the tools to prove Theorem 5.

Proof. We first find the generating function for $\{A_n\}$, and use this to find the single-term recurrence for the sequence.

Let $A(x) = \sum_{n=1}^{\infty} A_n x^n$ and $B(x) = \sum_{n=1}^{\infty} B_n x^n$. Then we have

$$\begin{aligned}
A(x) &= x + x^2 + x^3 + 2x^4 + 6x^5 + \sum_{n=6}^{\infty} A_n x^n \\
x^2 A(x) &= x^3 + x^4 + x^5 + \sum_{n=6}^{\infty} A_{n-2} x^n \\
x^4 A(x) &= x^5 + \sum_{n=6}^{\infty} A_{n-4} x^n \\
x^5 A(x) &= \sum_{n=6}^{\infty} A_{n-5} x^n,
\end{aligned}$$

and

$$
\begin{aligned}
B(x) &= x + x^2 + 2x^3 + 4x^4 + 10x^5 + \sum_{n=6}^{\infty} B_n x^n \\
xB(x) &= \quad x^2 + x^3 + 2x^4 + 4x^5 + \sum_{n=6}^{\infty} B_{n-1} x^n \\
x^2B(x) &= \quad x^3 + x^4 + 2x^5 + \sum_{n=6}^{\infty} B_{n-2} x^n \\
x^3B(x) &= \quad x^4 + x^5 + \sum_{n=6}^{\infty} B_{n-3} x^n \\
x^4B(x) &= \quad x^5 + \sum_{n=6}^{\infty} B_{n-4} x^n.
\end{aligned}
$$

We can now utilize the recursions (3) and (4) to make the infinite summations cancel and keep track of the smaller terms, obtaining the following two equations:

$$
\begin{aligned}
A(x) - xB(x) - x^2B(x) + x^2A(x) - x^4A(x) - x^5A(x) &= x, \\
B(x) - A(x) - x^2B(x) - x^3B(x) - x^4B(x) - x^5A(x) &= 0.
\end{aligned}
$$

Solving these two equations for $A(x)$ and $B(x)$ gives us that

$$
A(x) = \frac{x - x^2 - x^4}{1 - 2x + x^2 - 2x^3 - x^4 - x^5 + x^7 + x^8}.
$$

Finally, we can multiply both sides of the above relation by the denominator of the fraction, and we find that for $n \geq 8$, A_n satisfies the recursion

$$
A_n = 2A_{n-1} - A_{n-2} + 2A_{n-3} + A_{n-4} + A_{n-5} - A_{n-7} - A_{n-8},
$$

as desired. □

4 Asymptotic bounds

We now establish asymptotic bounds for $A_n^{(k)}$. To do so, recall that a directed graph is **strongly connected** if there is a directed path from any vertex to any other vertex. We will make use of known results on the spectra of the adjacency matrix of strongly connected directed graphs. As $\mathcal{P}_{2k+1,k}$ itself is not strongly connected, we begin by restricting to a certain strongly connected component of $\mathcal{P}_{2k+1,k}$ before applying the transfer-matrix method.

Definition 23. Define $U_{2k+1,k}$ (resp. $W_{2k+1,k}$) to be the set of vertices in $\mathcal{P}_{2k+1,k}$ whose pattern starts with 1 (resp. ends with $2k+1$).

By Theorem 14, the sets $U_{2k+1,k}$ and $W_{2k+1,k}$ are the sets of possible starting and ending vertices for a path in $\mathcal{P}_{2k+1,k}$ that determines an anchored $(k+1)$-determined permutation.

Definition 24. Let $V'_{2k+1,k}$ denote the set of vertices v in $\mathcal{P}_{2k+1,k}$ for which there exists a directed path containing v that starts at some vertex $u \in U_{2k+1,k}$ and ends at some vertex $w \in W_{2k+1,k}$. Then we define $\mathcal{P}'_{2k+1,k}$ to be the vertex-induced subgraph of $\mathcal{P}_{2k+1,k}$ on the vertices $V'_{2k+1,k}$.

Lemma 25. *The graph $\mathcal{P}'_{2k+1,k}$ is strongly connected, and moreover, it is the strongly connected component in $\mathcal{P}_{2k+1,k}$ of the vertex labeled by the identity permutation* $\mathrm{id}_{[2k+1]} \in S_{2k+1}$.

Proof. Let $v \in V'_{2k+1,k}$ be a vertex in $\mathcal{P}'_{2k+1,k}$. Then there exists $u \in U_{2k+1,k}$ and $w \in W_{2k+1,k}$ for which there is a directed path $u \to v \to w$ in $\mathcal{P}_{2k+1,k}$.

Let e denote the identity vertex $\mathrm{id}_{[2k+1]}$. We first show that there are directed paths $e \to v$ and $v \to e$ in $\mathcal{P}_{2k+1,k}$. To show that there is a directed path $e \to v$, it suffices to construct a path $e \to u$, since there is a path $u \to v$. By the definition of $U_{2k+1,k}$, we have that $u = u_1, \ldots, u_{2k+1}$ is a permutation pattern having $u_1 = 1$, and the inverse of u is k-bounded since it is a non-prohibited pattern. We can therefore define the permutation $p = p_1, \ldots, p_{4k+2}$ by

$$p_i = \begin{cases} i, & i \le 2k+1; \\ u_{i-2k-1} + 2k + 1, & i > 2k+1. \end{cases}$$

In other words, we increase each of the entries of u by $2k+1$ and append the result to e, to form the permutation p of length $4k+2$.

Note that $2k+1$ and $2k+2$ are adjacent in p, since $u_1 = 1$ by assumption. Furthermore, e and u both have k-bounded inverses. Thus, no pair of consecutive numbers $i, i+1$ occur more than a distance of k apart in p. Hence p^{-1} is k-bounded as well, and so p is $(k+1)$-determined by Lemma 10. It follows that there is a path in $\mathcal{P}_{2k+1,k}$ giving the consecutive patterns of p, and this path starts at e and ends at u, as desired.

To show there is a path from v to e, it similarly suffices to find a path from w to e. Since w ends with $2k+1$, we can add $2k+1$ to each entry of e and append it to the end of w to form a longer permutation, and a similar argument as above gives the desired path from w to e in $\mathcal{P}_{2k+1,k}$.

Note that the directed paths $v \to e$ and $e \to v$ lie entirely in $\mathcal{P}'_{2k+1,k}$ (since $e \in U_{2k+1,k}$ and $e \in W_{2k+1,k}$). Thus, if v and t are any two vertices in $\mathcal{P}'_{2k+1,k}$, they are connected by a directed path $v \to e \to t$. Hence $\mathcal{P}'_{2k+1,k}$ is strongly connected.

Finally, given any vertex v in the strongly connected component of e in $\mathcal{P}_{2k+1,k}$, some path $e \to v \to e$ exists by the definition of strongly connected. Hence $v \in V'_{2k+1,k}$ since $e \in U_{2k+1,k} \cap W_{2k+1,k}$. It follows that $\mathcal{P}'_{2k+1,k}$ is indeed equal to the strongly connected component of e. □

Now, let M' be the adjacency matrix of ${\mathcal{P}'}_{2k+1,k}$. By the transfer-matrix method and Theorem 14, the denominator of one rational expression for the generating function

$$A^{(k)}(x) := \sum A_n^{(k)} x^n$$

is given by $\det(I - xM')$. The reversed polynomial of $\det(I - xM')$ is simply the characteristic polynomial $\det(M' - xI)$ of M', whose roots are given by the eigenvalues of M'.

Using the well-known Perron-Frobenius theorem, we will show that M' has a unique eigenvalue r with largest absolute value, and that r is a positive real number with multiplicity 1. Then, by considering the partial fraction decomposition of the rational generating function (after factoring the denominator completely over $\mathbb{C}$), we will see that $A_n^{(k)}$ is bounded above by $c \cdot r^n$ for some constant $c > 0$ for sufficiently large n.

The "nonnegative" version of the Perron-Frobenius theorem can be stated as follows. Recall that the **spectral radius** of a matrix is the largest absolute value of any complex eigenvalue. The following theorem summarizes several of the results described in Section 8.4 of [5].

Theorem 26 (Perron-Frobenius)**.** *Let M be a matrix having nonnegative real entries, such that for any i and j there is some power M^n for which $M_{ij}^n > 0$. Then there is a unique positive real eigenvalue, of multiplicity one, equal to the spectral radius of M.*

Moreover, the number of eigenvalues of M equal to the spectral radius is equal to the ***period*** *of M, defined as greatest common divisor of all exponents n for which some diagonal entry $(M^n)_{i,i}$ is nonzero.*

Note that the condition on M in the theorem above, for an adjacency matrix of a directed graph, is precisely equivalent to the graph being strongly connected.

Lemma 25 therefore allows us to apply the Perron-Frobenius theorem to the adjacency matrix M' of the strongly connected graph $\mathcal{P}'_{2k+1,k}$. In particular, note that if i is the index of the identity vertex e, we see that $M'_{i,i} = 1$ since there is a directed edge from e to itself. Hence the period of M' is 1, and so there is a unique maximal eigenvalue r, and this eigenvalue is positive and real of multiplicity one.

We now wish to find an upper bound for r. Frobenius obtained classical bounds for the spectral radius of an adjacency matrix of a strongly connected digraph. In particular, it must be less than or equal to the maximum outdegree of the vertices. (See Theorem 2.1 in the survey paper [2].)

Therefore, to prove Theorem 6, it only remains to show that ${\mathcal{P}'}_{2k+1,k}$ has a maximum outdegree of k.

Lemma 27. *The largest outdegree of any vertex in ${\mathcal{P}'}_{2k+1,k}$ is k.*

Proof. Let $v = v_1 \cdots v_{2k+1}$ be a vertex in ${\mathcal{P}'}_{2k+1,k}$. Let

$$a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k$$

be the permutation pattern of $v_2, v_3, \ldots, v_{2k+1}$.

In the larger graph $\mathcal{P}_{2k+1}$, the vertex v has outdegree of exactly $2k+1$ since any number $m \in \{1, 2, \ldots, 2k+1\}$ can be appended to the pattern $a_1, \ldots, a_k, b_1, \ldots, b_k$, incrementing all entries $a_i \geq m$ or $b_j \geq m$ by 1, in order to form a vertex that v points to. However, at least $k+1$ of the values of m lead to k-prohibited permutation patterns. Specifically, if we insert any of $a_1, a_2, \ldots, a_k$, or $1 + \max\{a_1, a_2, \ldots, a_k\}$ as the last entry, we will create a k-prohibited pattern, and the above listed numbers are distinct. Thus, the maximum outdegree in the smaller graph ${\mathcal{P}'}_{2k+1,k}$ is no greater than $(2k+1)-(k+1)=k$.

Finally, we show that the identity pattern $e = 1, 2, 3, \dots, 2k+1$ has outdegree exactly k. Indeed, consider the permutations of length $2k+2$ of the form

$$1, 2, 3, \dots, i-1, i+1, \dots, 2k+2, i$$

. Such a permutation is $(k+1)$-determined if and only if $i \in \{k+3, \dots, 2k+2\}$, and each of these k permutations gives a directed edge $e \to v$ for some pattern v. Thus e has outdegree k in $\mathcal{P}_{2k+1,k}$. To see that each of these vertices v is also in $\mathcal{P}'_{2k+1,k}$, note that we may extend the permutation above to

$$1, 2, 3, \dots, i-1, i+1, \dots, 2k+2, i, 2k+3, 2k+4, \dots, 4k+3,$$

which is $(k+1)$-determined and therefore gives a path from v back to the identity in $\mathcal{P}_{2k+1,k}$. Hence $v \in \mathcal{P}'_{2k+1,k}$ as desired. □

Thus, we have $r \le k$, and the proof of Theorem 6 is complete.

Remark 28. In the cases $k = 2$ and $k = 3$, we can directly factor the (degree-reversed) denominators of the rational generating functions stated in Theorems 4 and 5 to find that the largest roots are approximately 1.466 and 2.114 respectively, when rounded up to three decimal places. Thus $R_n = O(1.466^n)$ and $A_n = O(2.114^n)$, which agrees with the upper bounds of $O(2^n)$ and $O(3^n)$ given by Theorem 6.

Remark 29. A lower bound for $A_n^{(k)}$, similar to that in [1] for the non-anchored case, may be obtained as follows. Consider a permutation $\pi : [n] \to [n]$ with the following properties:

1. The map π fixes the elements $1, k+2, 2k+3, \dots$, namely those of the form $(k+1)a+1$ for some nonnegative integer a.

2. It restricts to a permutation on each of the consecutive blocks of elements

$$\{2, 3, 4, \dots, k+1\}, \{k+3, k+4, \dots, 2k+2\}, \dots, \{(k+1)(b-1)+1, \dots, (k+1)b\}$$

   where $(k+1)b+1$ is the largest fixed point of the form described above.

3. The remaining elements $(k+1)b+1, (k+1)b+2, \dots, n$ are also fixed by π.

Then π^{-1} is k-bounded. Since the second property above gives $k!$ options for the permutations on each block, and there are $\lfloor \frac{n-1}{k+1} \rfloor$ such blocks, we have that there are at least

$$(k!)^{\lfloor (n-1)/(k+1) \rfloor}$$

k-bounded permutations of length n.

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