# On Conjugacies of the $3x + 1$ Map Induced by Continuous Endomorphisms of the Shift Dynamical System

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**Abstract**

Lagarias showed that the shift dynamical system $S$ on the set $\mathbb{Z}_2$ of 2-adic integers is conjugate to the famous $3x+1$ map $T$ by a conjugacy $\Phi$. Thus for each continuous endomorphism $f_\infty$ of $S$ there is a corresponding endomorphism $H_f = \Phi \circ f_\infty \circ \Phi^{-1}$ of $T$ and a map $\Psi_f$ from the coimage of $H_f$ to itself defined by $\Psi_f([x]) = [T(x)]$. In this paper, we completely classify all continuous endomorphisms $f_\infty$ of $S$ for which $\Psi_f$ is conjugate to $T$. We then define an infinite family of such maps, $\Psi_{M_k}$, that are "neutral" modulo $2^{k-1}$ in the sense that each element of the domain is a complete residue system modulo $2^{k-1}$. By investigating the relationships between $T$-cycles and the $\Psi_{M_k}$-cycles that contain them, we obtain an alternate method for studying the dynamics of $T$. This method is used to prove several new results pertaining to $T$-cycles, which are then applied to yield several possible approaches to the $3x + 1$ conjecture.

*Key words:* 3x+1 conjecture, symbolic dynamics, shift map, conjugacy

## 1. Introduction

A discrete dynamical system on a set $X$ is obtained by iterating a map $f : X \to X$. For any such function $f$ and any $x \in X$, the *$f$-orbit* of $x$, denoted $\mathcal{O}_f(x)$, is the set $\{x, f(x), f^2(x), f^3(x), \ldots\}$. If $f^m(c) = c$ for some $m \in \mathbb{Z}^+$, we say $c$ is *$f$-cyclic* with *period* $m$. The *minimum period* of $c$ is the smallest positive $m$ for which this is true. In this case, we call $\mathcal{O}_f(c)$ an *$f$-cycle*. Given two maps $X \xrightarrow{f} X$ and $X \xrightarrow{h} X$, we say $h$ is an *endomorphism* of $f$ if and only if $f \circ h = h \circ f$. We say $X \xrightarrow{f} X$ and $Y \xrightarrow{g} Y$ are *conjugate* with *conjugacy* $X \xrightarrow{h} Y$ if and only if $h$ is bijective and $g \circ h = h \circ f$. Conjugate maps have the same dynamical structure. For example, two conjugate maps have the same number of cycles with any given minimum period.

Let $\mathbb{Z}_2$ be the ring of 2-adic integers. Each $x \in \mathbb{Z}_2$ can be represented as $x_0x_1\cdots$ where every $x_i \in \{0, 1\}$ and interpreted algebraically as the formal power series $\sum_{i=0}^{\infty} x_i 2^i$. Arithmetic for 2-adic integers is defined by addition and multiplication of their formal power series. Define $\mathbb{Q}_{odd}$ to be the set of rational numbers with odd denominators. It is known that $\mathbb{Q}_{odd}$ is a subring of $\mathbb{Z}_2$ (cf. [1]).

The $3x + 1$ map $T : \mathbb{Z}_2 \to \mathbb{Z}_2$ is defined by $T(x) = \frac{x}{2}$ if $x$ is even and $T(x) = \frac{3x+1}{2}$ if $x$ is odd. The famous $3x + 1$ conjecture claims that the $T$-orbit of every positive integer contains 1. Also known as the Collatz conjecture, this problem has remained unsolved since the 1930's. Erdős proclaimed mathematics "is not yet ready for such problems" and offered a monetary award for its solution [4, 7].

The shift map $S : \mathbb{Z}_2 \to \mathbb{Z}_2$ is defined by $S(x) = \frac{x}{2}$ if $x$ is even and $S(x) = \frac{x-1}{2}$ if $x$ is odd (i.e. $S(x_0x_1x_2\cdots) = x_1x_2\cdots$ for all $x_0x_1x_2\cdots \in \mathbb{Z}_2$). Lagarias [4] constructed a conjugacy from $T$ to $S$ (called $\Phi^{-1}$ in [2]) defined by $\Phi^{-1}(x) = \sum_{k=0}^{\infty}(T^k(x) \bmod 2)\cdot 2^k$. (In this paper we will use $a \bmod b$ to represent the remainder when $a$ is divided by $b$.) This raises possibilities for progress on the $3x+1$ problem by studying conjugacies and properties of $S$.

Hedlund [3] classified all continuous endomorphisms of the shift dynamical system on bisequences, and M. Monks [6] showed that analogous results apply to the shift map $S$. In particular, let $\mathcal{B}_m$ be the set of all binary sequences of length $m$, and define, for each $k \in \mathbb{Z}^+$, $F(k) = \{f : \mathcal{B}_k \xrightarrow{f} \mathcal{B}_1\}$. An element of $F(k)$ is called a *block map.* The continuous endomorphisms of $S$ are the functions $f_\infty : \mathbb{Z}_2 \to \mathbb{Z}_2$ induced by $f \in F(k)$ by letting $f_\infty(x_0x_1x_2\cdots) = y_0y_1y_2\cdots$ where $y_i = f(x_ix_{i+1}\cdots x_{i+k-1})$ for all $i \geq 0$ [6].

For $k = 1$, the endomorphism $V : \mathbb{Z}_2 \to \mathbb{Z}_2$ where $V(x) = -1 - x$ is conjugate to the only autoconjugacy of $T$ other than the identity map, namely $\Omega = \Phi \circ V \circ \Phi^{-1}$. This can be used to construct the function $\Psi$ sending the unordered pair $\{x, \Omega(x)\}$ to $\{T(x), T(\Omega(x))\}$ for all $x \in \mathbb{Z}_2$ [5]. K. Monks and Yazinski [5] used $\Psi$ to state a "parity-neutral" form of the $3x+1$ conjecture. They also defined a $T$-cycle to be "self-conjugate" if it maps to itself by $\Omega$, and showed that the only such cycle of positive integers is $\{1, 2\}$.

For $k = 2$, M. Monks [6] examined the "discrete derivative" map $D$ (induced by the block map sending any two-digit binary string to the absolute value of the difference of the two digits), and used it to prove that $\Psi$ is conjugate to $T$. For $k > 2$, M. Monks left open the question for further study, stating "it would be of interest to study the dynamics of other continuous endomorphisms of $S$ and their applications" [6].

In this paper, we answer this question in several ways. First, we completely classify all of the continuous endomorphisms of $S$ that induce a map conjugate to $T$ that is analogous to $\Psi$. We then generalize $D$ to an infinite family of these maps, and use them to extend both the "parity-neutral" and "self-conjugacy" results of [5].

## 2. Main Results

In this section we state our main results. Proofs are postponed until Section 3.

Let $k$ be a positive integer and $f \in F(k)$. Define $H_f : \mathbb{Z}_2 \to \mathbb{Z}_2$ by $H_f = \Phi \circ f_\infty \circ \Phi^{-1}$. Let $\overset{f}{\equiv}$ be the equivalence relation defined on $\mathbb{Z}_2$ induced by $H_f$, i.e. $a \overset{f}{\equiv} b$ if and only if $H_f(a) = H_f(b)$ for any $a, b \in \mathbb{Z}_2$. The equivalence class of $a \in \mathbb{Z}_2$ of this relation will be denoted as $[a] = \{x : x \overset{f}{\equiv} a\}$. Let $\mathbb{Z}_2/f$ be the coimage of $H_f$, i.e. the set of all equivalence classes of $\overset{f}{\equiv}$. Define $\Psi_f : \mathbb{Z}_2/f \to \mathbb{Z}_2/f$ by $\Psi_f([a]) = [T(a)]$. We will prove that $\Psi_f$ is a well-defined function from $\mathbb{Z}_2/f$ to itself in Section 3 (Lemma 17).

We can now state our first main theorem.

**Theorem 1.** *For all $k \in \mathbb{Z}^+$ and $f \in F(k)$, $\Psi_f$ is conjugate to $T$ if and only if $f_\infty$ is onto.*

This completely classifies the continuous endomorphisms of $S$ that induce a map $\Psi_f$ conjugate to $T$. In this notation, the map $\Psi$ of [5] is $\Psi_d$ where $d$ is the block map for which $d_\infty$ is the discrete derivative $D$. Since $D$ is onto, we obtain the result of [6] that $\Psi$ is conjugate to $T$ as a special case of Theorem 1. Note that an equivalent definition of $d : \mathcal{B}_2 \to \mathcal{B}_1$ is $d(x_0x_1) = (x_0 + x_1) \bmod 2$. Thus a natural generalization of $d$ is the block map $M_k \in F(k)$ defined by

$$M_k(x_0x_1 \cdots x_{k-1}) = \left( \sum_{i=0}^{k-1} x_i \right) \bmod 2$$

(so that $d = M_2$). The fact that $\Psi_d$ is conjugate to $T$ also extends to all $\Psi_{M_k}$.

**Theorem 2.** *For all $k \in \mathbb{Z}^+$, $(M_k)_\infty$ is onto.*

So from Theorem 1 we obtain the desired result.

**Corollary 3.** *For all $k \in \mathbb{Z}^+$, $\Psi_{M_k}$ is conjugate to $T$.*

The map $\Psi$ of [5] is "parity-neutral" in the sense that every equivalence class in $\mathbb{Z}_2/d$ contains exactly one even and one odd element. Our next result generalizes this to all $\Psi_{M_k}$.

**Corollary 4.** *Every equivalence class in $\mathbb{Z}_2/M_k$ has exactly $2^{k-1}$ elements, no two of which are congruent modulo $2^{k-1}$.*

It follows that there is exactly one element in each equivalence class in a given residue class modulo $2^{k-1}$.

Having constructed an infinite family of maps $\Psi_{M_k}$ conjugate to $T$, we can use these maps to gain insights into the $3x+1$ problem. A standard way to obtain information about the dynamical structure of $T$ is by studying the dynamical structure of a conjugate map and using the conjugacy to interpret these results as facts about $T$. However, in the case of $\Psi_{M_k}$ we have another means of obtaining information about $T$. We say that a $T$-orbit *folds into* a $\Psi_{M_k}$-orbit if each element of the $T$-orbit is an element of some equivalence class in the $\Psi_{M_k}$-orbit. In this case we also say that the $\Psi_{M_k}$-orbit *contains* the $T$-orbit. Our next result describes how $T$-cycles fold.

**Theorem 5.** *A $T$-cycle with minimum period $m$ folds into a $\Psi_{M_k}$-cycle whose minimum period $n$ divides $m$. Furthermore, $m \mid nk$ if $\gcd(n, k) = 1$, and $m \mid 2\operatorname{lcm}(n, k)$ otherwise.*

**Example 6.** *The $T$-cycle $\left\{\frac{5}{7}, \frac{11}{7}, \frac{20}{7}, \frac{10}{7}\right\}$ with minimum period 4 folds into the $\Psi_{M_2}$-cycle $\left\{\left\{\frac{5}{7}, \frac{20}{7}\right\}, \left\{\frac{11}{7}, \frac{10}{7}\right\}\right\}$ with minimum period 2.*

A *fixed point* of a map $f : X \to X$ is any $x \in X$ for which $f(x) = x$. We will also refer to an $f$-cycle with period 1 as a *fixed point* of $f$. Using Theorem 5, we obtain the following fascinating result about the fixed points of $\Psi_{M_k}$.

**Theorem 7.** *A $T$-cycle folds into one of the fixed points of $\Psi_{M_k}$ if and only if its minimum period is a divisor of $k$. Furthermore, the fixed points contain only these $T$-cycles.*

**Example 8.** *The two fixed points for* $\Psi_{M_3}$ *are* $\left\{\frac{4}{5},\frac{2}{5},\frac{1}{5},-1\right\}$ *and* $\{-5,-7,-10,0\}$*. These consist of all the* $T$*-cycles with minimum period dividing* 3*, i.e.* $\{0\},\{-1\},\left\{\frac{4}{5},\frac{2}{5},\frac{1}{5}\right\}$ *and* $\{-5,-7,-10\}$*. Note that this example also illustrates Corollary 4.*

We shall prove (Lemma 24) that only $T$-cycles fold into $\Psi_{M_k}$-cycles. A natural question, then, is to ask how many $T$-cycles fold into a given $\Psi_{M_k}$-cycle.

**Theorem 9.** *If exactly* $r$ *distinct* $T$*-cycles fold into a given* $\Psi_{M_k}$*-cycle, then*

$$\left\lceil \frac{2^{k-1}}{k} \right\rceil \leq r \leq 2^{k-1}.$$

*Furthermore, the upper bound is attained for all* $k$*, and the lower bound is attained whenever* $k$ *is prime.*

**Example 10.** *By Theorem 9, every* $\Psi_{M_2}$*-cycle contains either one or two* $T$*-cycles. For example, the* $\Psi_{M_2}$*-cycle*

$$\left\{\left\{\tfrac{1}{13},-\tfrac{38}{11}\right\},\left\{\tfrac{8}{13},-\tfrac{19}{11}\right\},\left\{\tfrac{4}{13},-\tfrac{23}{11}\right\},\left\{\tfrac{2}{13},-\tfrac{29}{11}\right\}\right\}$$

*contains the two* $T$*-cycles* $\left\{\frac{1}{13},\frac{8}{13},\frac{4}{13},\frac{2}{13}\right\}$ *and* $\left\{-\frac{38}{11},-\frac{19}{11},-\frac{23}{11},-\frac{29}{11}\right\}$.

*Similarly, every* $\Psi_{M_3}$*-cycle contains between two and four* $T$*-cycles. For example, the* $\Psi_{M_3}$*-cycle*

$$\left\{\left\{\tfrac{28}{55},\tfrac{38}{55},-\tfrac{65}{17},-\tfrac{179}{17}\right\},\left\{\tfrac{14}{55},\tfrac{19}{55},-\tfrac{89}{17},-\tfrac{260}{17}\right\},\left\{\tfrac{7}{55},\tfrac{56}{55},-\tfrac{125}{17},-\tfrac{130}{17}\right\}\right\}$$

*contains the two* $T$*-cycles* $\left\{\frac{28}{55},\frac{14}{55},\frac{7}{55},\frac{38}{55},\frac{19}{55},\frac{56}{55}\right\}$ *and* $\left\{-\frac{65}{17},-\frac{89}{17},-\frac{125}{17},-\frac{179}{17},-\frac{260}{17},-\frac{130}{17}\right\}$.

The result of K. Monks and Yazinski [5] (that the only self-conjugate $T$-cycle of positive integers is $\{1,2\}$) can be restated as follows: $\{1,2\}$ is the only $T$-cycle of positive integers such that some $\Psi_{M_2}$-cycle contains only that $T$-cycle. This holds for all $\Psi_{M_k}$ as follows (since if $r=1$ then $k<3$ by Theorem 9).

**Corollary 11.** *If* $k\geq 3$*, then every* $\Psi_{M_k}$*-cycle contains more than one* $T$*-cycle.*

Thus, this generalizes the self-conjugacy result of [5].

## 3. Proofs

### *3.1. Maps Conjugate to T*

In this section we will prove Theorem 1. We will begin with some technical lemmas. Several of the proofs in this section are generalizations of proofs by Hedlund [3] and M. Monks [6].

The metric $\delta$ on $\mathbb{Z}_2$ is defined for $x,y\in\mathbb{Z}_2$ as $\delta(x,y)=\frac{1}{2^i}$, where $i$ is the subscript of the first binary digit in which $x$ and $y$ differ. Thus, if a sequence $x_1$, $x_2$, … of 2-adic integers has the property that each term $x_i$ shares at least $i$ initial digits with a given 2-adic integer $x$, then the sequence converges to $x$ since in this case $\delta(x_i,x)<\frac{1}{2^i}$ for all $i\in\mathbb{Z}^+$.

It is well-known that any infinite sequence of 2-adic integers must contain a convergent subsequence. To see this, consider a sequence of 2-adic integers $x_1, x_2, \ldots$. Infinitely many of these must begin with either a 0 or a 1. Assume without loss of generality that infinitely many begin with 0. Let $y_1$ be any one of these 2-adic integers. Either infinitely many of these begin with 00 or infinitely many begin with 01. Whichever the case, let $y_2$ be a 2-adic integer beginning with those two digits. We can continue this process indefinitely, so the subsequence $y_1, y_2, \ldots$ has the property that $y_i$ matches at least the first $i$ digits of $y_{i+1}$. Let $y$ be the 2-adic integer with the property that the $i^{th}$ digit of $y$ is the same as the $i^{th}$ digit of $y_i$. The sequence $y_1, y_2, \ldots$ then converges to $y$.

For the next few lemmas, let $k \in \mathbb{Z}^+$ and $f \in F(k)$. We define, for all $m \in \mathbb{Z}^+$, the map $f_m : \mathcal{B}_{m+k-1} \to \mathcal{B}_m$ by $f_m(x_0x_1 \cdots x_{m+k-2}) = y_0y_1 \cdots y_{m-1}$ where $y_i = f(x_ix_{i+1} \cdots x_{i+k-1})$ for all $i \geq 0$. The following lemma provides a condition for $f_\infty$ to be onto.

**Lemma 12.** *If $f_m$ is onto for all $m \in \mathbb{Z}^+$, then $f_\infty$ is onto.*

**Proof.** Let $f \in F(k)$ and assume that for all $m \in \mathbb{Z}^+$, $f_m$ is onto. Let $x = x_0x_1 \cdots \in \mathbb{Z}_2$ and $a_m = x_0x_1 \cdots x_{m-1}$ for all $m \geq 1$. By our assumption, for each $m$ there exists a string $b_m \in \mathcal{B}_{m+k-1}$ with $f_m(b_m) = a_m$. For all $i \in \mathbb{Z}^+$, let $y_i$ be the 2-adic integer whose binary representation is formed by concatenating infinitely many 0's to the end of the binary string $b_i$. Also let $c_i = f_\infty(y_i)$. Then the sequence $c_1, c_2, \ldots$ converges to $x$, as $c_i$ matches at least the first $i$ digits of $x$ for all $i \in \mathbb{Z}^+$ by definition of $f_\infty$. Since the sequence $y_1, y_2, \ldots$ must contain a convergent subsequence $y_{s_1}, y_{s_2}, \ldots$, we can let $y = \lim_{j\to\infty} y_{s_j}$. Then because $f_\infty$ is continuous,

$$f_\infty(y) = f_\infty(\lim_{j\to\infty} y_{s_j}) = \lim_{j\to\infty} f_\infty(y_{s_j}) = \lim_{j\to\infty} c_{s_j} = x.$$

Thus $f_\infty$ is onto. ■

The next lemma proves the converse of Lemma 12 by constructing a 2-adic integer that is not in the range of $f_\infty$ given a binary sequence of length $m$ that is not in the range of $f_m$. We write the 2-adic integer $x_0x_1 \cdots x_{m-1}x_0x_1 \cdots x_{m-1}x_0 \cdots$ as $\overline{x_0x_1 \cdots x_{m-1}}$.

**Lemma 13.** *Let $m \in \mathbb{Z}^+$ and $x = x_0x_1 \cdots x_{m-1} \in \mathcal{B}_m$. If $x$ is not in the range of $f_m$, then $\overline{x_0x_1 \cdots x_{m-1}}$ is not in the range of $f_\infty$.*

**Proof.** We will prove the contrapositive. Let $x' = \overline{x_0x_1 \cdots x_{m-1}}$. Assume there exists $y' = y_0y_1 \cdots \in \mathbb{Z}_2$ with $f_\infty(y') = x$. By the definition of $f_\infty$, $f(y_iy_{i+1} \cdots y_{i+k-1}) = x_i$ for all $i \geq 0$. Let $y = y_0y_1 \cdots y_{m+k-2}$. Then $f_m(y) = x$, so we are done. ■

Define $G_f : \mathbb{Z}_2/f \to \mathbb{Z}_2$ by $G_f([a]) = H_f(a)$ for all $a \in \mathbb{Z}_2$. We can see that $G_f$ is a well-defined function as it is defined for every element of $\mathbb{Z}_2/f$, and $H_f$ maps any two elements in a given class to the same value by definition.

**Lemma 14.** *$G_f$ is one-to-one.*

**Proof.** Let $A, B \in \mathbb{Z}_2/f$, and assume $G_f(A) = G_f(B)$. We can write $A = [a]$ and $B = [b]$ for some $a, b \in \mathbb{Z}_2$. Then by definition of $G_f$, $H_f(a) = H_f(b)$. This implies $a \overset{f}{\equiv} b$, so $[a] = [b]$. Thus $A = B$, and $G_f$ is one-to-one. ■

We will use $G_f$ as the conjugacy between $\Psi_f$ and $T$, so we need to prove it is bijective.

**Lemma 15.** *If $f_\infty$ is onto, then $G_f$ is bijective.*

**Proof.** Assume $f_\infty$ is onto. Since $\Phi$ is a conjugacy, both $\Phi$ and $\Phi^{-1}$ are onto. Then $H_f$ is onto, since it is defined as $\Phi \circ f_\infty \circ \Phi^{-1}$. Let $x \in \mathbb{Z}_2$. There exists $a \in \mathbb{Z}_2$ such that $H_f(a) = x$. Let $A = [a] \in \mathbb{Z}_2/f$. Then by definition of $G_f$, $G_f(A) = G_f([a]) = H_f(a) = x$, and so $G_f$ is onto. By Lemma 14, $G_f$ is one-to-one. Therefore $G_f$ is bijective. ∎

In order to prove that $\Psi_f$ is well-defined, we must prove that $H_f$ commutes with $T$.

**Lemma 16.** *$H_f \circ T = T \circ H_f$*

**Proof.** $H_f = \Phi \circ f_\infty \circ \Phi^{-1}$ by definition of $H_f$. Since $\Phi$ is a conjugacy from $T$ to $S$, we have $\Phi^{-1} \circ T = S \circ \Phi^{-1}$ and $\Phi \circ S = T \circ \Phi$. M. Monks [6] showed that $f_\infty$ is an endomorphism of $S$, so $f_\infty \circ S = S \circ f_\infty$. Thus,

$$\begin{aligned} H_f \circ T &= \Phi \circ f_\infty \circ \Phi^{-1} \circ T \\ &= \Phi \circ f_\infty \circ S \circ \Phi^{-1} \\ &= \Phi \circ S \circ f_\infty \circ \Phi^{-1} \\ &= T \circ \Phi \circ f_\infty \circ \Phi^{-1} \\ &= T \circ H_f. \end{aligned}$$

∎

Using this we can now prove the lemma mentioned in Section 2.

**Lemma 17.** *$\Psi_f$ is a well-defined function from $\mathbb{Z}_2/f$ to itself.*

**Proof.** Define the relation $\Psi_f \subseteq \mathbb{Z}_2/f \times \mathbb{Z}_2/f$ to be the set $\{([a], [T(a)]) : a \in \mathbb{Z}_2\}$. For every $A \in \mathbb{Z}_2/f$, $A = [a]$ for some $a \in \mathbb{Z}_2$, and $\{A, [T(a)]\} \in \Psi_f$. Hence $\mathbb{Z}_2/f$ is the domain of $\Psi_f$. Assume $([a], [T(a)]), ([b], [T(b)]) \in \Psi_f$ with $[a] = [b]$. By the definition of equivalence classes, $H_f(a) = H_f(b)$. This implies $T(H_f(a)) = T(H_f(b))$, so by Lemma 16, $H_f(T(a)) = H_f(T(b))$. Then $T(a) \overset{f}{\equiv} T(b)$ so $[T(a)] = [T(b)]$. Thus, $\Psi_f$ is a well-defined function. ∎

We now show that if $\Psi_f$ is conjugate to $T$ by any conjugacy, then $G_f$ bijectively maps $T$-cycles to $\Psi_f$-cycles with the same minimum period.

**Lemma 18.** *Assume $\Psi_f$ is conjugate to $T$ and $X \in \mathbb{Z}_2/f$ such that $X$ is $\Psi_f$-cyclic with minimum period $m \in \mathbb{Z}^+$. Then $G_f(X)$ is $T$-cyclic with minimum period $m$.*

**Proof.** We can write $X = [x]$ for some $x \in \mathbb{Z}_2$. We are given that $\Psi_f^m([x]) = [x]$, which implies $[T^m(x)] = [x]$. Thus we have $T^m(x) \overset{f}{\equiv} x$, and so $H_f(T^m(x)) = H_f(x)$. Then by Lemma 16, $T^m(H_f(x)) = H_f(x)$. Therefore $T^m(G_f(X)) = G_f(X)$, so $G_f(X)$ is $T$-cyclic with period dividing $m$.

Now we must show that $m$ is the minimum period of $G_f(X)$ for $T$. We will proceed by induction on $m$ to prove that $G_f$ maps the $\Psi_f$-cycles with minimum period $m$ bijectively to $T$-cycles with minimum period $m$.

First, assume $m = 1$. Since we assumed $\Psi_f$ is conjugate to $T$, there are the same number of fixed points for each map. Since $m \mid 1$ implies $m = 1$, the fixed points of $\Psi_f$ map to the fixed points of $T$ by $G_f$. By Lemma 14, $G_f$ is one-to-one, and so $G_f$ must map the fixed points of $\Psi_f$ each to a unique fixed point of $T$.

Now let $m \geq 2$ and assume that for all integers $1 \leq i \leq m-1$, $G_f$ maps $\Psi_f$-cycles with minimum period $i$ bijectively to $T$-cycles with the same minimum period. We have shown that $G_f$ maps the elements of a $\Psi_f$-cycle with minimum period length $m$ to elements of a $T$-cycle with minimum period dividing $m$. Assume that some $\Psi_f$-cycle with minimum period $m$ maps to a $T$-cycle with minimum period $d$ where $d \mid m$ and $d \neq m$. By our induction hypothesis, some $\Psi_f$-cycle with minimum period $d$ maps to this $T$-cycle. However, this is a contradiction since $G_f$ is one-to-one. Thus $d = m$. This completes the proof. ■

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** ($\Leftarrow$) Let $f \in F(k)$, and assume $f_\infty$ is onto. Let $A \in \mathbb{Z}_2/f$. Then $A = [a]$ for some $a \in \mathbb{Z}_2$. By Lemma 16, we have $H_f(T(a)) = T(H_f(a)) = T(G_f([a]))$. Thus, by the definitions of $\Psi_f$ and $G_f$, we obtain

$$\begin{aligned} G_f(\Psi_f(A)) &= G_f(\Psi_f([a])) \\ &= G_f([T(a)]) \\ &= H_f(T(a)) \\ &= T(G_f([a])) \\ &= T(G_f(A)). \end{aligned}$$

Thus $G_f \circ \Psi_f = T \circ G_f$. By Lemma 15, $G_f$ is bijective, so $\Psi_f$ is conjugate to $T$ with conjugacy $G_f$.

($\Rightarrow$) We prove this by contradiction. Let $f \in F(k)$ such that $f_\infty$ is not onto, and suppose $\Psi_f$ is conjugate to $T$ with some conjugacy $h$. By Lemma 12, there exists a positive integer $m$ for which $f_m$ is not onto, i.e. there exists an $x = x_0x_1\cdots x_{m-1} \in \mathcal{B}_m$ that is not in the range of $f_m$. Thus, by Lemma 13, $x' = \overline{x_0x_1\cdots x_{m-1}}$ is not in the range of $f_\infty$. Since $\Phi^{-1}$ is bijective, $\Phi(x')$ is not in the range of $H_f$. Consequently, $\Phi(x')$ is not in the range of $G_f$. So $\Phi(x')$ is $T$-cyclic with period $m$ (and some minimum period $d \mid m$) and is not in the range of $G_f$.

If $\Psi_f$ and $T$ are conjugate, they have the same finite number of cycles with minimum period $d$. Each $\Psi_f$-cycle with minimum period $d$ is sent by $G_f$ to a distinct $T$-cycle with the same period by Lemma 18. Also, since $\Phi(x')$ is not in the range of $G_f$, some $T$-cycle with minimum period $d$ is not mapped to via $G_f$ from any $\Psi_f$-cycle. This is a contradiction, so $\Psi_f$ is not conjugate to $T$ by any conjugacy.

Therefore $\Psi_f$ and $T$ are conjugate if and only if $f_\infty$ is onto. ■

*3.2. Generalizing D*

Having classified the endomorphisms that induce maps conjugate to $T$, we now prove the results pertaining to $M_k$, our generalization of $D$. Let $k \in \mathbb{Z}^+$.

**Proof of Theorem 2.** Let $x = x_0x_1\cdots \in \mathbb{Z}_2$, and let $y_0y_1\cdots y_{k-2} \in \mathcal{B}_{k-1}$. For $i \geq 0$, define $y_{i+k-1} = \left(x_i - \sum_{j=i}^{i+k-2} y_j\right) \bmod 2$. Notice that this is the unique solution modulo 2 to the equation $x_i = \left(\sum_{j=i}^{i+k-1} y_j\right) \bmod 2$. Therefore $M_k(y_i y_{i+1} \cdots y_{i+k-1}) = x_i$ for all $i \geq 0$. Letting $y = y_0 y_1 \cdots$, we have $(M_k)_\infty(y) = x$, and so $(M_k)_\infty$ is onto. This completes the proof. ■

Notice that the construction of $y$ in this proof was completely determined by the choice of $y_0 y_1 \cdots y_{k-2} \in \mathcal{B}_{k-1}$, and that any such choice allows for the construction of such a $y$. Thus every binary string of length $k-1$ is represented in each class exactly once as the starting block of one of its 2-adic elements. From the formal power series representation of any 2-adic integer $x = x_0 x_1 \cdots = \sum_{i=0}^{\infty} x_i 2^i$, we obtain $x \bmod 2^a = \sum_{i=0}^{a-1} x_i 2^i = x_0 x_1 \cdots x_{a-1}$. Thus every remainder modulo $2^{k-1}$ is represented exactly once in every class in $\mathbb{Z}_2/M_k$, proving Corollary 4.

*3.3. Folding*

In this section we prove the remaining theorems about the folding of $T$-cycles into $\Psi_{M_k}$-cycles. For clarity, we separate the technical parts of Theorem 5 into the next two lemmas. As before, let $k \in \mathbb{Z}^+$.

**Lemma 19.** *$H_{M_k}$ sends any $T$-cycle with minimum period $m$ to a $T$-cycle with minimum period $n$ dividing $m$.*

**Proof.** Let $x$ be $T$-cyclic with minimum period $m$, and let $y = H_{M_k}(x)$. By Lemma 16,

$$\begin{aligned} T^m(y) &= T^m(H_{M_k}(x)) \\ &= H_{M_k}(T^m(x)) \\ &= H_{M_k}(x) \\ &= y. \end{aligned}$$

Thus $y$ is $T$-cyclic with minimum period dividing $m$, and we are done. ■

Now that we know that $H_{M_k}$ sends $T$-cycles to $T$-cycles, we can prove a further restriction on the periods of these cycles.

**Lemma 20.** *Suppose a $T$-cycle with minimum period $m$ maps to a $T$-cycle with minimum period $n$ by $H_{M_k}$. Then $m \mid nk$ if $\gcd(n,k) = 1$, and $m \mid 2\operatorname{lcm}(n,k)$ otherwise.*

**Proof.** Let $x$ be $T$-cyclic with minimum period $m$ and $y = H_{M_k}(x)$. Then $y$ is $T$-cyclic with some minimum period $n$ by Lemma 19. Let $g = \gcd(n,k)$, $\Phi^{-1}(x) = \overline{x_0 x_1 \cdots x_{m-1}}$, $\Phi^{-1}(y) = \overline{y_0 y_1 \cdots y_{n-1}}$, and $(M_2)_\infty(\Phi^{-1}(y)) = z = z_0 z_1 \cdots$. By the definition of $H_{M_k}$, $y = \Phi((M_k)_\infty(\Phi^{-1}(x)))$, and so $\Phi^{-1}(y) = (M_k)_\infty(\Phi^{-1}(x))$. Therefore,

$$(M_k)_\infty(\overline{x_0 x_1 \cdots x_{m-1}}) = \overline{y_0 y_1 \cdots y_{n-1}}.$$

So by definition of $(M_k)_\infty$, for all $i \geq 0$,

$$(x_i + x_{i+1} \cdots + x_{i+k-1}) \bmod 2 = y_i$$

and

$$(x_{i+1} + \cdots + x_{i+k-1} + x_{i+k}) \bmod 2 = y_{i+1}.$$

Adding these two equations gives

$$(x_{i+k} + x_i) \bmod 2 = (y_i + y_{i+1}) \bmod 2 = z_i$$

by definition of $(M_2)_\infty$, and so $x_{i+k} = (z_i + x_i) \bmod 2$. Applying this repeatedly to $x_{i+nk/g}$, we obtain

$$\begin{aligned} x_{i+nk/g} &= \left(z_{i+(n/g-1)k} + x_{i+(n/g-1)k}\right) \bmod 2 \\ &= \left(z_{i+(n/g-1)k} + z_{i+(n/g-2)k} + x_{i+(n/g-2)k}\right) \bmod 2 \\ &\vdots \\ &= \left(z_{i+(n/g-1)k} + z_{i+(n/g-2)k} + \cdots + z_i + x_i\right) \bmod 2 \\ &= \left(x_i + \sum_{j=0}^{n/g-1} z_{i+jk}\right) \bmod 2. \end{aligned}$$

Let $\hat{z}_i = \left(\sum_{j=0}^{n/g-1} z_{i+jk}\right) \bmod 2$ for all $i \geq 0$. Then

$$x_{i+nk/g} = (\hat{z}_i + x_i) \bmod 2. \tag{1}$$

Now consider $\hat{z}_{i+nk/g}$. Since $\Phi^{-1}(y)$ is $S$-cyclic with minimum period $n$, $z$ has period $n$ (not necessarily minimum) by Lemma 19 and the fact that $\Phi$ is the conjugacy between $S$ and $T$. Using this in conjunction with the fact that $\frac{k}{g}$ is an integer gives

$$\hat{z}_{i+nk/g} = \left(\sum_{j=0}^{n/g-1} z_{i+jk+n(k/g)}\right) \bmod 2 = \left(\sum_{j=0}^{n/g-1} z_{i+jk}\right) \bmod 2 = \hat{z}_i. \tag{2}$$

Then, by (1) and (2)

$$\begin{aligned} x_{i+2nk/g} &= \left(\hat{z}_{i+nk/g} + x_{i+nk/g}\right) \bmod 2 \\ &= \left(\hat{z}_{i+nk/g} + \hat{z}_i + x_i\right) \bmod 2 \\ &= (\hat{z}_i + \hat{z}_i + x_i) \bmod 2 \\ &= x_i \end{aligned}$$

for all $i$. So $m \mid \frac{2nk}{g}$ and thus $m \mid 2\,\mathrm{lcm}(n,k)$.

If $g = 1$, then for any $i \geq 0$ the subscripts of the terms in the sum $\left(\sum_{j=0}^{n-1} z_{i+jk}\right) \bmod 2$ include all residue classes modulo $n$. Thus $\left(\sum_{j=0}^{n-1} z_{i+jk}\right) \bmod 2 = \left(\sum_{j=0}^{n-1} z_j\right) \bmod 2$ since $z$ is

$S$-cyclic with period $n$. Using this, we have

$$\begin{aligned}
\hat{z}_i &= \left(\sum_{j=0}^{n-1} z_{i+jk}\right) \bmod 2 \\
&= \left(\sum_{j=0}^{n-1} z_j\right) \bmod 2 \\
&= \left(\sum_{j=0}^{n-1} (y_j + y_{j+1})\right) \bmod 2 \\
&= ((y_0 + y_1) + (y_1 + y_2) + \cdots + (y_{n-1} + y_n)) \bmod 2 \\
&= (y_0 + y_n) \bmod 2 \\
&= (y_0 + y_0) \bmod 2 \\
&= 0.
\end{aligned}$$

So by (1), $x_i = \hat{z}_i + x_i = x_{i+nk/g} = x_{i+nk}$. Thus $m \mid nk$. ■

The following lemma provides a useful relationship between how a $T$-cycle folds into a $\Psi_{M_k}$-cycle and what it maps to by $H_{M_k}$.

**Lemma 21.** *A $T$-cycle maps to a $T$-cycle with minimum period $n$ by $H_{M_k}$ if and only if it folds into a $\Psi_{M_k}$-cycle with minimum period $n$.*

**Proof.** Let $m \in \mathbb{Z}^+$, let $C$ be a $T$-cycle with period $m$, and let $c \in C$. Then, using Lemma 16,

$$\begin{aligned}
H_{M_k}(c) = T^m(H_{M_k}(c)) &\iff H_{M_k}(c) = H_{M_k}(T^m(c)) \\
&\iff c \overset{M_k}{\equiv} T^m(c) \\
&\iff [c] = [T^m(c)] \\
&\iff [c] = \Psi^m_{M_k}([c]).
\end{aligned}$$

Thus $H_{M_k}(c)$ is $T$-cyclic with period $m$ if and only if $[c]$ is $\Psi_{M_k}$-cyclic with period $m$. Note that by the definition of $\Psi_{M_k}$, $[T^i(c)] = \Psi^i_{M_k}([c])$ for all $i \in \mathbb{Z}^+$, so all elements of $C$ are elements of an equivalence class in the same $\Psi_{M_k}$-cycle as $[c]$. Therefore $H_{M_k}(C)$ is a $T$-cycle with minimum period $n$ if and only if $C$ folds into a $\Psi_{M_k}$-cycle having minimum period $n$. ■

We can now prove Theorem 5.

**Proof of Theorem 5.** We are given a $T$-cycle with period $m$ that folds into a $\Psi_{M_k}$-cycle with minimum period $n$. By Lemma 21, the $T$-cycle maps to a $T$-cycle with minimum period $n$ by $H_{M_k}$. Thus, Lemmas 19 and 20 give us the desired result. ■

This theorem also gives us a useful inequality about the periods $m$ and $n$.

**Corollary 22.** *If a $T$-cycle with minimum period $m$ folds into a $\Psi_{M_k}$-cycle with minimum period $n$, then $n \leq m \leq nk$.*

**Proof.** By Lemma 19, $n \mid m$ and so $n \le m$. By Theorem 5, if $\gcd(n,k) = 1$, then $m \mid nk$ so $m \le nk$. If $\gcd(n,k) \ge 2$, $m \mid 2\operatorname{lcm}(n,k)$ so $m \le \frac{2nk}{\gcd(n,k)} \le nk$. This completes the proof. ■

Before we prove Theorem 7, we must show that $\Psi_{M_k}$-cycles contain only $T$-cycles. The following lemma is the first step toward that result.

**Lemma 23.** *If $A = \{a_1, a_2, \ldots, a_{2^{k-1}}\} \in \mathbb{Z}_2/M_k$, then*

$$\Psi_{M_k}(A) = \{T(a_1), T(a_2), \ldots, T(a_{2^{k-1}})\}.$$

**Proof.** By the definition of $\Psi_{M_k}$, $T(a_i) \in \Psi_{M_k}(A)$ for all $1 \le i \le 2^{k-1}$. Assume $x, y \in A$ with $T(x) = T(y)$, and let $\Phi^{-1}(x) = x_0x_1\cdots$ and $\Phi^{-1}(y) = y_0y_1\cdots$. Then $\Phi^{-1}(T(x)) = \Phi^{-1}(T(y))$, and since $\Phi^{-1}$ is a conjugacy from $T$ to $S$, $S(\Phi^{-1}(x)) = S(\Phi^{-1}(y))$. Thus $x_1x_2\cdots = y_1y_2\cdots$, and in particular

$$(x_1 + x_2 + \cdots + x_{k-1}) \bmod 2 = (y_1 + y_2 + \cdots + y_{k-1}) \bmod 2. \tag{3}$$

Since $x$ and $y$ are in the same equivalence class, $H_{M_k}(x) = H_{M_k}(y)$. By definition of $H_{M_k}$, $\Phi((M_k)_\infty(\Phi^{-1}(x))) = \Phi((M_k)_\infty(\Phi^{-1}(y)))$. So $(M_k)_\infty(x_0x_1\cdots) = (M_k)_\infty(y_0y_1\cdots)$ since $\Phi$ is bijective. By the definition of $(M_k)_\infty$, we have

$$(x_i + x_{i+1} + \cdots + x_{i+k-1}) \bmod 2 = (y_i + y_i + \cdots + y_{i+k-1}) \bmod 2$$

for any $i \ge 0$. In particular,

$$(x_0 + x_1 + \cdots + x_{k-1}) \bmod 2 = (y_0 + y_1 + \cdots + y_{k-1}) \bmod 2.$$

However, adding this to (3) proves $x_0 = y_0$. Therefore $x = y$.

Thus all of $T(a_1), T(a_2), \ldots, T(a_{2^{k-1}})$ are distinct, and since $\Psi_{M_k}(A)$ must have $2^{k-1}$ elements by Corollary 4, we have $\Psi_{M_k}(A) = \{T(a_1), T(a_2), \ldots, T(a_{2^{k-1}})\}$. ■

We apply this to cycles to prove the desired result.

**Lemma 24.** *$\Psi_{M_k}$-cycles contain only $T$-cycles (i.e. any element of a $\Psi_{M_k}$-cyclic class is $T$-cyclic).*

**Proof.** Assume $A \in \mathbb{Z}_2/M_k$ is $\Psi_{M_k}$-cyclic with minimum period $n$, and let $a \in A$. By Lemma 23, there exists a unique $b_1 \in \mathbb{Z}_2$ such that $a = T(b_1)$ and $\Psi_{M_k}([b_1]) = A$ where $[b_1]$ is in the same $\Psi_{M_k}$-cycle as $A$. Similarly, for any $i \in \mathbb{Z}^+$ there exists a unique $b_i$ such that $a = T^i(b_i)$ and $\Psi^i_{M_k}([b_i]) = A$ where $[b_i]$ is in the same $\Psi_{M_k}$-cycle as $A$. Consider the sequence $b_1, b_2, \ldots, b_{2^{k-1}n+1}$. By Corollary 4, there are $2^{k-1}n$ total elements in the classes in a $\Psi_{M_k}$-cycle with minimum period $n$. Thus, by the pigeonhole principle there exist $p, q \in \{1, 2, \ldots, 2^{k-1}n + 1\}$ with $p < q$ such that $b_p = b_q$. By the definition of $b_i$, we have $T^{q-p}(b_p) = b_q = b_p$, so $b_p$ is $T$-cyclic. Since $a = T^p(b_p)$, this implies $a$ is $T$-cyclic. ■

With this understanding of the elements of $\Psi_{M_k}$-cyclic classes, we can prove Theorem 7.

**Proof of Theorem 7.** Assume a $T$-cycle with minimum period $m$ folds into a fixed point of $\Psi_{M_k}$. Applying Theorem 5 with $n = 1$, we have $m \mid k$. Therefore any $T$-cycle that folds into a fixed point must have (not necessarily minimum) period $k$. By Corollary 4, the two fixed points of $\Psi_{M_k}$ have a total of $2^k$ elements. There are $2^k$ 2-adic integers that are $S$-cyclic with period $k$ (one for each binary string of length $k$). However, $\Phi$ is a bijection between the set of $T$-cyclic 2-adic integers with period $k$ and $S$-cyclic 2-adic integers with period $k$, so there are $2^k$ $T$-cyclic 2-adic integers with period $k$ as well. Thus every $T$-cycle with period $k$ must fold into a fixed point (or else there would not be a total of $2^k$ elements by Lemma 24). ■

We end with the proof of Theorem 9.

**Proof of Theorem 9.** Assume $A$ is $\Psi_{M_k}$-cyclic with minimum period $n$. Then the elements of $A$ are each contained in one of $r$ distinct $T$-cycles for some positive integer $r$ by Lemma 24. Each of these $T$-cycles has minimum period between $n$ and $nk$ by Corollary 22, so the total number of elements in all of the $r$ $T$-cycles is between $rn$ and $rnk$. The total number of elements in every class in the $\Psi_{M_k}$-cycle containing $A$ is $2^{k-1}n$ by Corollary 4. Thus we have $rn \leq 2^{k-1}n \leq rnk$. Solving this for $r$ gives $\left\lceil \frac{2^{k-1}}{k} \right\rceil \leq r \leq 2^{k-1}$.

We now show that the upper bound is attained for all $k$. Let $k$ be arbitrary and let $z = \overline{\underbrace{000\cdots0}_{k}\underbrace{111\cdots1}_{k}}$. Assume $x \in {G_{M_k}}^{-1}(\Phi(z))$ with $\Phi^{-1}(x) = x_0x_1x_2\cdots$. By the definition of $G_{M_k}$, we have $H_{M_k}(x) = \Phi(z)$, and so $(M_k)_\infty(x_0x_1x_2\cdots) = (M_k)_\infty(\Phi^{-1}(x)) = z$. Then for all $i \geq 0$,

$$\sum_{j=i}^{i+k-1} x_j = \begin{cases} 0 & \text{if } i \bmod 2k = 0, 1, 2, \ldots, k-1 \\ 1 & \text{if } i \bmod 2k = k, k+1, k+2, \ldots, 2k-1 \end{cases}$$

so

$$\sum_{j=i}^{i+2k-1} x_j = \sum_{j=i}^{i+k-1} x_j + \sum_{j=i+k}^{i+2k-1} x_j = 1.$$

Similarly,

$$\sum_{j=i+1}^{i+2k} x_j = 1$$

so

$$x_i - x_{i+2k} = \sum_{j=i}^{i+2k-1} x_j - \sum_{j=i+1}^{i+2k} x_j = 0.$$

Hence $x$ is $T$-cyclic with minimum period dividing $2k$. By Theorem 5, $2k$ divides the period of $x$ since $\mathcal{O}_T(x)$ folds into a $\Psi_{M_k}$-cycle with minimum period $2k$. Thus $x$ has minimum period $2k$, and hence every cycle contained in $\mathcal{O}_{\Psi_{M_k}}({G_{M_k}}^{-1}(z))$ has period $2k$. Since there are exactly $2^k k$ total elements in all of the equivalence classes in $\mathcal{O}_{\Psi_{M_k}}({G_{M_k}}^{-1}(z))$ by Corollary 4, it contains exactly $2^{k-1}$ $T$-cycles. Thus the upper bound is attained.

We now show that the lower bound is attained for prime $k$. When $k = 2$, the lower bound is attained, as the fixed point $\{1, 2\}$ of $\Psi_{M_2}$ contains only one $T$-cycle. Let $k$ be an odd prime. Since $k$ is odd,

$$\begin{aligned} H_{M_k}(-1) &= \Phi((M_k)_\infty(\Phi^{-1}(-1))) \\ &= \Phi((M_k)_\infty(\overline{11\cdots 1})) \\ &= \Phi(\overline{11\cdots 1}) \\ &= -1. \end{aligned}$$

Similarly, $H_{M_k}(0) = 0$, so the two fixed points of $T$ each fold into a different fixed point of $\Psi_{M_k}$. By Theorem 7, the two fixed points of $\Psi_{M_k}$ contain all $T$-cycles with period $k$, i.e. all $T$-cycles with minimum period 1 or $k$. In either of these fixed points, the remaining $2^{k-1} - 1$ elements must be accounted for by $T$-cycles of period $k$. Therefore each fixed point contains $\frac{2^{k-1}-1}{k} + 1$ total $T$-cycles. Since $\frac{2^{k-1}-1}{k}$ is an integer, we can write $\frac{2^{k-1}-1}{k} + 1 = \left\lceil \frac{2^{k-1}}{k} \right\rceil$, so the lower bound is attained. ■

## 4. Conclusion

As promised in the introduction, we have completely classified all continuous endomorphisms $f_\infty$ of $S$ that induce $\Psi_f$ conjugate to $T$ and generalized the map $D$ of [6] to an infinite class of such maps, $(M_k)_\infty$. We have shown that each class in $\mathbb{Z}_2/M_k$ contains exactly one element in each residue class modulo $2^{k-1}$, generalizing the concept of a "parity-neutral" map as defined in [5]. We have also generalized the concept of a "self-conjugate" $T$-cycle by showing no $\Psi_{M_k}$-cycle can contain only one $T$-cycle for $k > 2$.

These results provide several new insights into the $3x + 1$ problem. In order to study the dynamics of $T$, one can study the dynamics of any conjugate map and the conjugacy between them. We have constructed infinitely many new maps $\Psi_f$ conjugate to $T$. Future investigation of the dynamics of the maps $\Psi_f$ for surjective $f_\infty$ and their respective conjugacies $G_f$ may provide further insight into the $3x + 1$ conjecture. For example, to prove the $3x+1$ conjecture it would suffice to show that for some surjective $f_\infty$, $\mathcal{O}_{\Psi_f}(G_f^{-1}(x))$ contains $G_f^{-1}(1)$ for any positive integer $x$.

Additionally, we have demonstrated a separate method of obtaining information about the dynamics of $T$ by studying the ways that $T$-orbits fold into $\Psi_{M_k}$-orbits. One such property is that all and only those $T$-cycles with period $k$ fold into the fixed points of $\Psi_{M_k}$. For example, any $T$-cycle with minimum period $m$ folds into the same fixed point of $\Psi_{M_{4m}}$ as $\{1, 2\}$, since $(M_{4m})_\infty$ sends both $\overline{x_0 x_1 \cdots x_{m-1}}$ and $\overline{10}$ to $\overline{0}$. Thus, if it can be shown that no two positive integer $T$-cycles can fold into the same equivalence class, then the only $T$-cycle of positive integers is $\{1, 2\}$. It is our hope that further study of the conjugate maps $\Psi_f$ and the ways $T$-orbits fold into $\Psi_f$-orbits will lead to greater insights or even a proof of the $3x + 1$ conjecture.

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