INTEGRAL-FREE ANTIDERIVATIVES

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You cannot avoid paradise.
You can only avoid seeing it.
— Charlotte Joko Beck

1. Introduction

It is rare that anyone applauds mathematical notation, and rarer still for that applause to come from my freshmen calculus students. But that is indeed what occurred when I first presented some notation I had devised to assist them with antidifferentiation. With the interest in calculus reform today, this calculus *notation* reform provides a concrete, tangible way to greatly improve the teaching and learning of antidifferentiation.

It is common practice to use f' to denote the derivative of a function of a single variable, f. Thus it seems quite natural to define a similar notation for antidifferentiation.

Definition 1.1. *Let* $\backprime f$ *denote an antiderivative of* f*, i.e. define*

$$\backprime f = \int f(x)\, dx.$$

We can read $\backprime f$ as "an antiderivative of f" or simply "antiprime of f". Throughout this article we will assume f, g, u, and v are C^∞ functions of a single variable x on some fixed open interval. Also, for the time being let's not fuss over constants of integration (we will discuss them later in detail)[1].

Armed with this notation we have the Magic Formula

$$\backprime f' = f. \qquad (\textit{Magic Formula})$$

Also notice that antipriming is linear, i.e. $\backprime(f+g) = \backprime f + \backprime g$ and $\backprime(cf) = c\backprime f$ for $c \in \mathbb{R}$.

Let $\backprime$ have higher operator precedence than function evaluation, so that $\backprime f(u)$ means $(\backprime f)(u)$ and not $\backprime(f(u))$ [2]. In this notation the rules for u-substitution and integration by parts become simply

$$\backprime(f(u)\,u') = \backprime f(u) \qquad (\textit{The Substitution Rule})$$

and

$$\backprime(uv') = uv - \backprime(vu'). \qquad (\textit{Integration by Parts})$$

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[1] Specifically, let $f \equiv g$ if and only if $f = g + C$ for some constant function C. We then write $f = g$ as an abbreviation for $f \equiv g$ (where the meaning of $f = g$ is determined by context).

[2] As is common practice, we will not overemphasize the distinction between the function f and the expression $f(x)$. For example, we write $\backprime(f(u))$ instead of $\backprime(f \circ u)$.

Using this notation, we can improve on the Parts formula. If we substitute ${}^{\backprime}v$ for v in the Parts formula we obtain a product rule for antidifferentiation

$$(\textit{Product Rule}) \qquad {}^{\backprime}(uv) = {}^{\backprime}vu - {}^{\backprime}\left({}^{\backprime}vu'\right)$$

Notice that this requires only one more character to write (and memorize[3]) than the Parts formula, but as we will see, greatly simplifies calculations for the student. Considering that the Product Rule in traditional calculus notation is

$$\int u(x)\,v(x)\;dx = u(x)\int v(x)\;dx - \int\left(\int v(x)\,dx\right)u'(x)\,dx$$

it is not surprising that it has not been commonly taught in elementary calculus courses. Yet the antiprime notation makes this formula not only workable, but actually easier to use than the Parts formula, as we shall now illustrate.

2. Calculations Made Easy

2.1. **The Death of Integration by Parts.** Sometimes one example is worth a thousand words. In this section we present two thousand words worth.

Example 2.1. *Evaluate* ${}^{\backprime}(x\cos(x))$.

Solution: Let $u = x$ *and* $v = \cos(x)$. *Then* $u' = 1$ *and* ${}^{\backprime}v = \sin(x)$. *Thus*

$$\begin{aligned} {}^{\backprime}(x\cos(x)) &= {}^{\backprime}(uv) \\ &= {}^{\backprime}vu - {}^{\backprime}\left({}^{\backprime}vu'\right) \\ &= x\sin(x) - {}^{\backprime}\sin(x) \\ &= x\sin(x) + \cos(x) \end{aligned}$$

Excellent! Notice there is not a single $\int$ or du or dv or dx in the entire solution! (Insert student applause here.) The student does not have to "choose dv" or any such silliness.

Consider how simple this is from the students' perspective. They are faced with the antiderivative of a product. They name one factor u and the other v. They compute u' and ${}^{\backprime}v$. They substitute into the Product Rule formula and simplify. This is much more natural and straightforward than choosing u and dv or even u and v' for the factors.

Can it be a fluke? What about the dreaded use-Parts-twice-and-solve type of exercise?

Example 2.2. *Evaluate* ${}^{\backprime}(e^x\sin(x))$

Solution: Let $u = \sin(x)$ *and* $v = e^x$. *Then* $u' = \cos(x)$ *and* ${}^{\backprime}v = e^x$. *Thus*

$$\begin{aligned} {}^{\backprime}(e^x\sin(x)) &= {}^{\backprime}(uv) \\ &= {}^{\backprime}vu - {}^{\backprime}\left({}^{\backprime}vu'\right) \\ &= e^x\sin(x) - {}^{\backprime}(e^x\cos(x)) \end{aligned}$$

Now let $u = \cos(x)$ *and* $v = e^x$. *Then* $u' = -\sin(x)$ *and* ${}^{\backprime}v = e^x$. *Thus*

$$\begin{aligned} {}^{\backprime}(e^x\cos(x)) &= {}^{\backprime}(uv) \\ &= {}^{\backprime}vu - {}^{\backprime}\left({}^{\backprime}vu'\right) \\ &= e^x\cos(x) + {}^{\backprime}(e^x\sin(x)) \end{aligned}$$

[3] When written in the form shown the string ${}^{\backprime}vu$ appears twice on the right hand side giving the mantra-like formula "*antiprime vee you minus antiprime of antiprime vee you prime*"

Combining these results yields

$$\backprime\left(e^{x}\sin(x)\right)=e^{x}\sin(x)-e^{x}\cos(x)-\backprime\left(e^{x}\sin(x)\right)$$

which we then solve for $\backprime\left(e^{x}\sin(x)\right)$

$$\backprime\left(e^{x}\sin(x)\right)=\tfrac{1}{2}e^{x}\left(\sin(x)-\cos(x)\right).$$

Still not a single du, dv, or $\int$ anywhere to be found! This solution is much more direct and clean than the usual mess. Down with Integration by Parts! Long live the Product Rule in antiprime notation!

2.2. **The Substitute-o-matic.** The other major techniques of antidifferentiation commonly taught in college calculus courses are u-substitution and inverse/trig substitution. Armed with the antiprime notation we can devise a simple uniform procedure for computing antiderivatives by either method which is nearly algorithmic in nature.

Recall that the Substitution Rule in this notation is:

$$\backprime\left(f(u)\,u'\right)=\backprime f(u)$$

In this setting we will refer to u' as the *prime factor.* Suppose we want to use the Substitution Rule to evaluate $\backprime(g(x))$. We wish to choose $u=h(x)$ so that $g(x)=f(u)\,u'$ for some f. Then we can *always* insert a prime factor into the expression by multiplying $g(x)$ by $\frac{u'}{h'(x)}$ since this fraction is equal to one[4]. This gives us an algorithmic approach to u-substitution (resp. inverse substitution) antidifferentiation exercises.

Algorithm: To evaluate $\backprime(g(x))$ by substitution (resp. inverse substitution)

1. Choose $u=h(x)$ (resp. $x=h(u)$).
2. Differentiate to obtain $u'=h'(x)$ and rewrite this in the form $\frac{u'}{h'(x)}=1$ (resp. differentiate to obtain $1=h'(u)\,u'$).
3. Write $\backprime(g(x))=\backprime(g(x)\cdot 1)=\backprime\left(g(x)\cdot\frac{u'}{h'(x)}\right)$ (resp. $\backprime(g(x)\cdot h'(u)\,u')$) and simplify.
4. Substitute to obtain an expression of the form $\backprime(f(u)\,u')$.
5. Apply the Substitution Rule and simplify.

Thus, the goal in every substitution problem is to choose a u (or $h(u)$), insert the prime factor into the expression, simplify, substitute as usual and apply the Substitution Rule. This procedure is quite easy in practice and unifies the many different substitution antidifferentiation techniques under one roof, as illustrated in the following typical examples.

Example 2.3. *Evaluate* $\backprime\left(xe^{x^{2}}\right)$.

[4]Technically, we will multiply by $q(x)=\begin{cases}\frac{u'}{h'(x)} & \text{if } h'(x)\neq 0\\ 1 & \text{if } h'(x)=0\end{cases}$. But we never need worry about the case $h'(x)=0$ as long as $g(x)=0$, since then $g(x)=0=f(h(x))\,h'(x)=f(u)\,u'$ anyway.

Solution: Let $u = x^2$. *Then* $u' = 2x$, *so* $\frac{u'}{2x} = 1$. *Thus*

$$\begin{aligned}
\backprime\left(xe^{x^2}\right) &= \backprime\left(xe^{x^2}\cdot 1\right) \\
&= \backprime\left(xe^{x^2}\cdot\frac{u'}{2x}\right) \\
&= \backprime\left(e^{x^2}\cdot\frac{u'}{2}\right) \\
&= \tfrac{1}{2}\backprime\left(e^{u}u'\right) \\
&= \tfrac{1}{2}e^{u} \\
&= \tfrac{1}{2}e^{x^2}.
\end{aligned}$$

Example 2.4. *Evaluate* $\backprime\left(x\sqrt{x+1}\right)$.

Solution. Let $u = x+1$, *so that* $x = u-1$. *Then* $u' = 1$. *Thus*

$$\begin{aligned}
\backprime\left(x\sqrt{x+1}\right) &= \backprime\left(x\sqrt{x+1}\cdot 1\right) \\
&= \backprime\left(x\sqrt{x+1}\cdot u'\right) \\
&= \backprime\left((u-1)\sqrt{u}\cdot u'\right) \\
&= \backprime\left(\left(u^{3/2}-u^{1/2}\right)u'\right) \\
&= \tfrac{2}{5}u^{5/2}-\tfrac{2}{3}u^{3/2} \\
&= \tfrac{2}{5}(x+1)^{5/2}-\tfrac{2}{3}(x+1)^{3/2}.
\end{aligned}$$

Example 2.5. *Evaluate* $\backprime\left(\sqrt{4-x^2}\right)$.

Solution: Let $x = 2\sin(u)$. *Then* $1 = 2\cos(u)\,u'$. *Thus*

$$\begin{aligned}
\backprime\left(\sqrt{4-x^2}\right) &= \backprime\left(\sqrt{4-x^2}\cdot 1\right) \\
&= \backprime\left(\sqrt{4-x^2}\cdot 2\cos(u)\,u'\right) \\
&= 2\backprime\left(\sqrt{4-(2\sin(u))^2}\cdot\cos(u)\,u'\right) \\
&= 4\backprime\left(\cos^2(u)\,u'\right) \\
&= 2\backprime\left((1+\cos(2u))\,u'\right).
\end{aligned}$$

Now let $v = 2u$. *Then* $v' = 2u'$, *so* $\frac{v'}{2u'} = 1$. *Thus*

$$\begin{aligned}
2\backprime\left((1+\cos(2u))\,u'\right) &= 2\backprime\left((1+\cos(2u))\,u'\cdot\frac{v'}{2u'}\right) \\
&= \backprime\left((1+\cos(v))\,v'\right) \\
&= v+\sin(v) \\
&= 2u+\sin(2u) \\
&= 2\arcsin\left(\frac{x}{2}\right)+\sin\left(2\arcsin\left(\frac{x}{2}\right)\right) \\
&= 2\arcsin\left(\frac{x}{2}\right)+\tfrac{1}{2}x\sqrt{4-x^2}.
\end{aligned}$$

Now you just can't beat that! Just as with the Parts examples, there is not a differential or integral sign to be found in any of these examples.

Notice that in some cases following the algorithm is overkill, although it still does work. Compare, for example,

Example 2.6. *Evaluate* ${}^{\backprime}\left(2x\cos\left(x^2\right)\right)$.

Solution: Let $u = x^2$. *Then* $u' = 2x$, *so* $\frac{u'}{2x} = 1$. *Thus*

$$\begin{aligned} {}^{\backprime}\left(2x\cos\left(x^2\right)\right) &= {}^{\backprime}\left(2x\cos\left(x^2\right)\cdot\frac{u'}{2x}\right) \\ &= {}^{\backprime}\left(\cos\left(u\right)u'\right) \\ &= {}^{\backprime}\cos\left(u\right) \\ &= \sin\left(u\right) \\ &= \sin\left(x^2\right). \end{aligned}$$

with the slightly more efficient:

Example 2.7. *Evaluate* ${}^{\backprime}\left(2x\cos\left(x^2\right)\right)$.

Solution: Let $u = x^2$. Then $u' = 2x$. Thus

$$\begin{aligned} {}^{\backprime}\left(2x\cos\left(x^2\right)\right) &= {}^{\backprime}\left(\cos\left(u\right)u'\right) \\ &= {}^{\backprime}\cos\left(u\right) \\ &= \sin\left(u\right) \\ &= \sin\left(x^2\right). \end{aligned}$$

In this case there is no need to compute $\frac{u'}{2x}$ and substitute since u' itself appeared as a factor. Shortcuts such as this will surely be noticed by many students, whereas the algorithm provides even weak students with a consistent method for attacking *all* substitution type antiderivatives.

The advantage of this method is that it builds on every student's ability to algebraically simplify quotients. The disadvantage is the need to justify multiplication by $\frac{u'}{h'(x)}$ when $h'(x) = 0$ and $g(x) = 0$. However, all of the usual methods of substitution are still available in the antiprime notation, so nothing is lost. Yet we have gained a cleaner notation and no longer need to justify the use of differentials in indefinite integrals as algebraic objects. Having a single uniform method to attack a wide variety of problems frees us from having to catalogue a plethora of ad-hoc procedures, so that we can spend time on more important concepts.

3. Concepts Revealed

3.1. **Improved Exposition.** As if helping the student to master the computational techniques of antidifferentiation isn't enough, this notation can also assist their understanding of the material by clarifying the exposition. Consider the proof of the Parts formula.

Proof. Begin with the product rule,

$$(uv)' = uv' + vu'$$

apply ${}^{\backprime}$ to both sides,

$$ {}^{\backprime}(uv)' = {}^{\backprime}(uv' + vu')$$

apply the Magic Formula and linearity,

$$uv = {}^{\backprime}(uv') + {}^{\backprime}(vu')$$

and solve for ${}^{\backprime}(uv')$ to obtain

$$ {}^{\backprime}(uv') = uv - {}^{\backprime}(vu')\,. $$

∎

This derivation is so simple that a typical freshmen calculus student could easily understand and reproduce it. To obtain the Product Rule we need only substitute ${}^{\backprime}v$ for v in the Parts formula and simplify using the Magic Formula.

What about the Substitution Rule?

Proof. Begin with the chain rule,

$$ (F(u))' = F'(u)\,u' $$

apply ${}^{\backprime}$ to both sides,

$$ {}^{\backprime}(F(u))' = {}^{\backprime}(F'(u)\,u') $$

simplify with the Magic Formula, and rewrite to obtain

$$ {}^{\backprime}(F'(u)\,u') = F(u)\,. $$

Finally substituting ${}^{\backprime}f$ for F and simplifying with the Magic Formula gives the result

$$ {}^{\backprime}(f(u)\,u') = {}^{\backprime}f(u)\,. $$

∎

Again, something a typical student can easily follow and even reproduce.

3.2. **The Fundamental Theorem.** There is a pedagogical problem I run into whenever I teach the fundamental theorem. The traditional notation for the definite integral, $\int_a^b f(x)\ dx$, can be thought of as a mnemonic device for recalling the definition as a limit of Riemann sums ($\int \leftrightarrow \sum$, $dx \leftrightarrow \Delta x_i$, etc.). But when we define the symbol $\int f(x)\ dx$ to stand for an antiderivative of f, what happens to the mnemonic correspondence between $\int$ and $\sum$? between dx and Δx_i? Thus when we state the fundamental theorem in the traditional notation, we are effectively saying

$$ \int_a^b f(x)\ dx = \int f(x)\,dx\Big|_a^b $$

which appears to many students to be almost a content-free definition. After all, it just says integrals are related to integrals!

On the other hand, the antiprime notation preserves the mnemonic nature of both the definite integrals and antiderivatives, because the ${}^{\backprime}f$ notation is certainly reminiscent of the f' notation for derivatives only being "undone". For a similar reason I constantly try to avoid the term *indefinite integral* in lecture, and use instead the term, *antiderivative.* Thus instead of claiming that the fundamental theorem says that "indefinite integrals and definite integrals are related", it is better to say that "integrals are related to antiderivatives". The word *integral* is reserved in lecture to mean *definite integral* only.

In the antiprime notation the fundamental theorem becomes

(*Fundamental Theorem I*)
$$ \int_a^x f(t)\ dt \equiv {}^{\backprime}f(x) $$

or

$$\text{(\textit{Fundamental Theorem II})} \qquad \int_a^b f(x)\ dx = {}^\backprime f(b) - {}^\backprime f(a)\,.$$

In this form it is much more clear to the novice that there is some content to the theorem because it is clearly relating "that symbol that looks like the limit of Riemann sums" to "those symbols that look like antiderivatives".

4. The Math Behind the Magic

Everything should be made as simple as possible,
but not one bit simpler.

—Albert Einstein

4.1. **What happened to the $+C$'s?** At this point you may be reaching for your red pens to take off a few points for the missing constants of integration (shouldn't we call them constants of antidifferentiation?) in the article thus far. In particular, one might argue that the magic formula should be ${}^\backprime f' = f + C$ instead of what we have presented. However, we said at the outset that we would define $f \equiv g$ to mean

$$\text{For some constant function } C,\ \ f = g + C$$

and that we would further abbreviate $f \equiv g$ by writing $f = g$, the actual meaning of $f = g$ being determined in each case by context. Thus the Magic Formula, ${}^\backprime f' = f$, is just an abbreviation for ${}^\backprime f' \equiv f$ which is clearly true. Similarly the $=$ in the Substitution, Parts, and Product Rules should all be $\equiv$, as well as many of the equal signs in the examples and proofs discussed thus far. Thus, by this convention, ${}^\backprime(\cos(x) + 2) = \sin(x) + 2x$ is correct even without the $+C$ if we interpret the $=$ as $\equiv$.

This is not a new practice in calculus or something new that has been introduced as a result of our change of notation. To see that this is true we should consider more carefully what we mean by the symbol $\int f(x)\ dx$.

4.2. **What is $\int f(x)\ dx$ anyway?** The next time you are at lunch with a group of mathematicians, ask each of them how they define the symbol $\int f(x)\ dx$. As this is one of the most frequently taught symbols in college mathematics today, one would think that all mathematicians would agree on a standard uniform definition for this symbol. But this is hardly the case!

In particular, the definition should allow us to interpret, in a precise mathematical way, common statements like

$$\int \cos(x)\ dx = \sin(x) + C \tag{4.1}$$

and

$$\int f(x) + g(x)\ dx = \int f(x)\ dx + \int g(x)\ dx \tag{4.2}$$

and

$$\int 1 + 2\sin^4(x)\ dx = x + 2\int \sin^4(x)\ dx. \tag{4.3}$$

In my own informal surveys two answers were often given:

1. $\int f(x)\ dx$ is a function. In particular it is a symbol which stands for an antiderivative of f.

2. $\int f(x)\ dx$ is a set of functions. In particular, it is the set of all antiderivatives of f. In other words, $\int f(x)\ dx$ is an equivalence class in the quotient group $C^{\infty}(a,b)/K$ where we consider $C^{\infty}(a,b)$ as an additive group and K is the subgroup of the constant functions under addition.

Now if we use the latter definition when teaching calculus, then how do we interpret (4.1), (4.2), and (4.3) for the students? In (4.1) we have a set on the left hand side of the equals sign, but a function on the right. So we must make some notational convention which allows us to interpret the right hand side as a set in some fashion (e.g. treat it as the set $\{\sin(x)+C : C \in \mathbb{R}\}$ or treat C as the set of constant functions and $\sin(x)$ as a function and think of $\sin(x)+C$ as coset notation for the equivalence class, etc.). But then in an equation like (4.2) we are talking about addition of sets, and in (4.3) it is even worse, because we are talking about the addition of a function and a scalar multiple of a set on the right hand side. So if we adhere to the interpretation of $\int f(x)\ dx$ as a set we are forced to define addition of sets of functions, scalar multiplication of sets of functions, addition of functions to sets of functions, to show that scalar multiplication distributes over addition, etc. If we do not teach this then it must be swept under the rug for the sake of simplifying the topic.

On the other hand, if we assume the first definition of $\int f(x)\ dx$, i.e. that $\int f(x)\ dx$ is a symbol which stands for some antiderivative or f, then (4.1) is certainly true for some value of C. However, (4.2) is certainly not true in general for arbitrary antiderivatives of f, g, and $f+g$ (e.g. let $f(x)=\cos(x)$, $g(x)=1$, and suppose $\int f(x)+g(x)\ dx = \sin(x)+x+1$, $\int f(x)\ dx = \sin(x)$, and $\int g(x)\ dx = x$). Once again we have run into problems which somehow must be swept under the rug.

Even the heuristic convention used thus far in this article —using $f=g$ in two different ways— is just another mechanism for sweeping this same concern under the rug. Is there no simple solution?

4.3. **Cleaning Under the Rug.** Perhaps the beauty and elegance of the antiprime notation also provides us with a good opportunity to help our students even further by removing this sort of mathematical chicanery from our courses.

So let's take it again from the top, but this time we will never use $f=g$ as an abbreviation for $f \equiv g$.

Definition 4.1. *Let* ${}^{\backprime}f$ *denote an antiderivative of* f*.*

Thus ${}^{\backprime}f$ is a function, namely some antiderivative of f. Note that we don't know which antiderivative ${}^{\backprime}f$ is, just as we don't know which function in $C^{\infty}(a,b)$ the symbol f stands for.

Definition 4.2. *We define a relation* $\equiv$ *on* $C^{\infty}(a,b)$ *by*

$$f \equiv g \text{ if and only if for some constant function } C,\ f = g + C.$$

Thus we have the not-so-magical formulas:

Remark 4.1. *For any functions* f, g, *and* h *and constant* c

1. $({}^{\backprime}f)' = f$.
2. ${}^{\backprime}(f') \equiv f$.
3. *If* $f \equiv g$ *then* $f+h \equiv g+h$.
4. *If* $f \equiv g$ *then* $cf \equiv cg$.

5. *If* $f = g$ *then* $`f \equiv `g$.
6. $`(f+g) \equiv `f + `g$.
7. $`(cf) \equiv c`f$.
8. $f \equiv f$.
9. *If* $f \equiv g$ *and* $g \equiv h$ *then* $f \equiv h$.
10. *If* $f \equiv g$ *then* $g \equiv f$.

A typical calculus student could easily verify these properties directly from the definition of $\equiv$. By #1, #2, and #8 we see that for either interpretation of the symbol $`f'$ we have the not-so-magical formula

$$`f' \equiv f.$$

Remarks #8-10 simply state that $\equiv$ is an equivalence relation.

Let us continue in this vein. For example, reconsider the proof of the Product Rule.

Proof. Begin with the product rule,

$$(uv)' = uv' + vu'$$

apply $`$ to both sides (remark #5 above),

$$`(uv)' \equiv `(uv' + vu')$$

apply remarks #2, #6, and #9,

$$uv \equiv `(uv') + `(vu')$$

and solve for $`(uv')$ via remarks #3 and #10 to obtain

$$`(uv') \equiv uv - `(vu').$$

Substitute $`v$ for v

$$`\left(u\left(`v\right)'\right) \equiv uv - `\left(`vu'\right)$$

and apply remark #1

$$`(uv) \equiv uv - `\left(`vu'\right).$$

∎

Calculations can be placed on a similar secure foundation.

Example 4.1. *Evaluate* $`(x\cos(x))$.

Solution: Let $u = x$ *and* $v = \cos(x)$. *Then* $u' = 1$ *and* $`v \equiv \sin(x)$. *Thus*

$$\begin{aligned} `(x\cos(x)) &= `(uv) \\ &\equiv `vu - `\left(`vu'\right) \\ &\equiv x\sin(x) - `\sin(x) \\ &\equiv x\sin(x) + \cos(x) \end{aligned}$$

But wait? How can we justify the next to last $\equiv$ in the example above? After all, it is not true that if $f \equiv g$ then we can automatically substitute f for g in any expression or statement. For example, $f \equiv g$ does not imply $f^2 \equiv g^2$ or even that $`f \equiv `g$. So how can we justify substituting $\sin(x)$ for $`v$ in this example? Well, one can easily verify that this step is always justified when using the Product Rule for antiderivatives because $f \equiv g$ implies $fu - `(fu') \equiv gu - `(gu')$. This is a fine point that typically goes unnoticed in the traditional integral notation for integration by parts.

Notice that we could require that the student write

$$\backprime(x\cos(x)) = x\sin(x) + \cos(x) + C \text{ for some real number } C$$

as the last step in the evaluation if we wish to preserve constants of integration (constants of antidifferentiation!).

Also, while we are cleaning house, we can eliminate the heuristic of multiplying by $\frac{u'}{h'(x)}$ in substitution problems if we wish. For example,

Example 4.2. *Evaluate* $\backprime(\cos(2\sin(x))\cos(x))$.

Solution: Let $u = 2\sin(x)$. Then $u' = 2\cos(x)$. Thus $\cos(x) = \frac{1}{2}u'$

$$\begin{aligned}\backprime(\cos(2\sin(x))\cos(x)) &= \backprime\left(\cos(u)\tfrac{1}{2}u'\right)\\ &\equiv \tfrac{1}{2}\backprime\cos(u)\\ &\equiv \tfrac{1}{2}\sin(u)\\ &= \tfrac{1}{2}\sin(2\sin(x)).\end{aligned}$$

Thus, in both exposition and computation we can eliminate our heuristics[5] and safely walk through the paradise we have created on secure mathematical ground (although that ground might be slightly more awkward to tread upon!). In accelerated or honors calculus courses this should certainly be done. In a business/applied calculus course, this is probably overkill and using the heuristics of abbreviating $f \equiv g$ by $f = g$ and multiplying by $\frac{u'}{h'(x)}$ will most likely be in the students' best interest. In a typical freshman calculus course for scientific/math oriented majors the decision of whether to use the heuristics or use the full exposition should probably be made by the instructor in that course, since they are familiar with the nature of the students and the course at their particular school.

4.4. **Extending the Notation.** A natural question to ask is if this notation can be extended to higher order antiderivatives or functions of more than one variable.

It is a common practice to write

$$f^{(n)}$$

to denote the n^{th} derivative of f. There are two obvious ways to extend this notation to antiderivatives. We could either write

$$^{(n)}f$$

for an n^{th} order antiderivative of f, or else we could write

$$f^{(-n)}$$

for the same thing. While the former notation is more consistent with the $\backprime$ notation, it has the disadvantage that the meaning of an expression such as

$$g^{(2)}f$$

is ambiguous, whereas in the latter notation this meaning is clear.

[5]While we are at it, we should also emphasize the distinction between a function f and the expression $f(x)$. But the interested reader who attempts this will soon find that the gain in rigor is soundly offset by a painful loss of efficiency and clarity.

There is a fine point that one must beware of when dealing with higher order antiderivatives. As mentioned above, it is not true that $f \equiv g$ implies ${}^{\backprime}f \equiv {}^{\backprime}g$. For example the calculation ${}^{\backprime\backprime}x \equiv {}^{\backprime}\left(\frac{1}{2}x^2\right) \equiv \frac{1}{6}x^3$ is certainly wrong. Instead we should have

$$\begin{aligned} {}^{\backprime\backprime}x &= {}^{\backprime}\left(\frac{1}{2}x^2 + C\right) \\ &\equiv \frac{1}{6}x^3 + Cx. \end{aligned}$$

This situation is made even worse if we use the heuristic of abbreviating $\equiv$ by $=$ since $f = g$ certainly does imply ${}^{\backprime}f \equiv {}^{\backprime}g$. Note that we cannot iterate the Magic Formula, since ${}^{\backprime\backprime}f'' \equiv f$ is not true if we interpret ${}^{\backprime\backprime}f''$ as ${}^{\backprime\backprime}(f'')$, for example. So we must take care to use, but not abuse, the notation with respect to higher order antiderivatives.

For functions of more than one variable, the natural notation to use would be ${}_{x}f$ for the antipartial of f with respect to x as a generalization of the notation f_x for the partial derivative of f with respect to x. Thus the magic formula still holds

$${}_{x}f_x \equiv f.$$

although now the definition of $\equiv$ must be modified so that the constant functions are replaced by functions of the other variables besides x. But the same warnings apply about misusing this notation that apply in the single variable case.

Finally, while we are reviewing calculus notation, there is one other minor bit of notation that deserves mention. Instead of using (a, b) for the open interval from a to b, why not use the more *Maple*-like notation $(a..b)$ (and similar notation for half open and closed intervals)? As it stands, the notation for open intervals in $\mathbb{R}$ and points in $\mathbb{R}^2$ is exactly the same. With $(2, 3)$ and $(2..3)$ it is clear which is the point and which is the interval.

I have also seen some students using the absolutely abhorrent notation $]a, b[$ for the interval $(a..b)$. This causes severe mental discomfort on my part when grading exams because I always try to match brackets in pairs when reading. Thus an answer like

$$f \text{ is increasing on }]-\infty, -1[\cup]-1, 0[\cup[1, \infty[$$

immediately causes my right arm to reach for the Tylenol bottle.

4.5. **Concluding remarks.** The notation ${}^{\backprime}$ is so simple and natural that one must wonder why we have been laboring under the traditional indefinite integral notation for so long. Pedagogically the antiprime notation clarifies the meaning of the Fundamental Theorem and simplifies the exposition in lecture. Computationally it provides the student with clean computational algorithms with which to evaluate antiderivatives. Given the current interest in calculus reform, perhaps it is time to simplify our sometimes cumbersome traditional notation in order to help our students see the meaning behind the symbols and more easily master the computations we ask of them. Notation does not change the content of the material being presented, but can enhance the exposition and simplify computations immensely. It has a concrete tangible and direct impact on our students, for which they may be thankful... or even applaud.

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