A Chaotic Extension of the Collatz Function to $\mathbb{Z}_2[i]$

John A. Joseph

May 1995

Abstract

We construct a non-trivial extension of the Collatz function to the the 2-adic integers adjoined with i (where $i = \sqrt{-1}$). We show that our extension is non-trivial in that not only is it not the cross product of the original function with itself, but it is not even conjugate to it via a $\mathbb{Z}_2$-module isomorphism (though they are topologically conjugate). We then prove that most of the interesting properties of the original function are preserved by the extension. Finally, we prove that the extension is chaotic.

1 Introduction

The $3x + 1$ problem, also known as the Collatz problem, is traditionally credited to Lothar Collatz at the University of Hamburg in 1930's. It is a simple problem to state and it seems as though its solution should be trivial. Yet, it has proved to be intractable thus far. Perhaps that is why it is such a frustratingly addictive problem. Jeffrey C. Lagarius at AT&T Bell Laboratories has written an excellent exposition containing the history of the $3x + 1$ problem and a survey of the literature on the subject [Lag].

The $3x + 1$ problem has appeared in many forms over the years but is most elegantly expressed in terms of iteration of the function $T : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$T(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

T is known as the Collatz function. The conjecture made by Collatz is that for every positive integer n, $T^{(k)}(n) = 1$ for some k. Given recent claims that Fermat's conjecture has been resolved, one could make a strong case that the Collatz conjecture is among the most famous open questions in all of mathematics.

The function T can be extended in a natural manner to the 2-adic integers, $\mathbb{Z}_2$, and this extension has proven to be quite fruitful. In this paper, we further extend the domain of T to $\mathbb{Z}_2[i]$, hoping to increase our understanding of the problem.

2 Summary of Main Results

In this section we provide an overview of our main results. A detailed discussion of the definitions, theorems, and proofs can be found in the remainder of the paper.

We construct an extension, $\widetilde{T}$, of the Collatz function, $T : \mathbb{Z}_2 \to \mathbb{Z}_2$ to the metric space $(\mathbb{Z}_2[i], D)$.

Definition 1 *Let* $\widetilde{T} : \mathbb{Z}_2[i] \to \mathbb{Z}_2[i]$ *by*

$$\widetilde{T}(\alpha) = \begin{cases} \frac{\alpha}{2} & \text{if } \alpha \in [0] \\ \frac{3\alpha+1}{2} & \text{if } \alpha \in [1] \\ \frac{3\alpha+i}{2} & \text{if } \alpha \in [i] \\ \frac{3\alpha+1+i}{2} & \text{if } \alpha \in [1+i] \end{cases}$$

where $[x]$ *denotes the equivalence class of* x *in* $\mathbb{Z}_2[i]/2\mathbb{Z}_2[i]$.

Our main results separate naturally into roughly three areas.

First, $\widetilde{T}$ is an extension of the original function and is non-trivial in the following sense:

Theorem A

(a) $\widetilde{T}|\mathbb{Z}_2 = T$
(b) $\widetilde{T}$ *is not conjugate to* $T \times T$ *via a* $\mathbb{Z}_2$*-module isomorphism.*
(c) $\widetilde{T}$ *is, however, topologically conjugate to* $T \times T$.

Second, $\widetilde{T}$ preserves the salient qualities of the original function. In particular there is a "parity vector function", Q_∞, for T which has been extremely important in understanding the nature of the problem. We show that Q_∞ can also be extended in an analogous manner. The original parity vector function and the extended parity vector function share several important properties (c.f. [Lag: Theorem B]). We simply state the results here, saving the details for later in the paper.

Theorem B *The extended parity vector function $\widetilde{Q}_k$ is periodic with period 2^k. In addition, $\widetilde{Q}_\infty$ is a measure preserving homeomorphism.*

Finally, T and $T \times T$ are both chaotic functions (in the sense of [Dev]) and thus it follows from part (c) of Theorem A that

Theorem C *$\widetilde{T}$ is chaotic.*

We have constructed $\widetilde{T}$ in the hope that by applying the tools of chaos, complex analysis, and algebraic number theory the theorems presented above might provide future researchers with further insight into the $3x+1$ problem and others like it.

3 Background and Notation

In this section, we develop the notation and discuss the relevant background material. Most of this material can be found in [Lag].

The sequence

$$n, T(n), T^{(2)}(n), T^{(3)}(n), \dots$$

is called the *orbit* of n.

Example 1 *The orbit of* 3 *is* $3 \to 5 \to 8 \to 4 \to 2 \to 1 \to 2 \to 1 \to \dots$.

Another way to state the Collatz conjecture is that for all $n \in \mathbb{Z}^+$, the orbit of n enters the cycle

$$2 \to 1 \to 2 \to 1 \to \dots.$$

This statement of the $3x+1$ problem is also valid on the ring known as the 2-adic integers, written $\mathbb{Z}_2$. The 2-adic integers consist of all infinite sequences

$$s_0, s_1, s_2, \ldots$$

where

$$s_i \in \{0,1\} \text{ for all } i \geq 0.$$

For brevity, we shall often refer to a 2-adic integer as simply a “2-adic”.

Addition is defined on the 2-adic integers by taking

$$a_0, a_1, a_2, \ldots = \Sigma_{i=0}^{\infty} a_i 2^i$$

and applying the usual rules for manipulating formal power series. Formally:

$$a_0, a_1, a_2, \ldots + b_0, b_1, b_2, \ldots = d_0, d_1, d_2, \ldots$$

where

$$d_i = \begin{cases} 1 & \text{if } (a_i, b_i, c_i) \in \{(1,1,1),(1,0,0),(0,1,0),(0,0,1)\} \\ 0 & \text{otherwise} \end{cases}$$

where

$$c_i = \begin{cases} 1 & \text{if } (a_{i-1}, b_{i-1}, c_{i-1}) \in \{(1,1,1),(0,1,1),(1,0,1),(1,1,0)\} \\ 0 & \text{otherwise.} \end{cases}$$

This is essentially the same algorithm as for the addition of integers in base 2 except that the 2-adic sequences are infinite and written from left to right. Multiplication is defined in a similar manner. A few examples should make this clear:

Example 2

$101\overline{01}_2 + 010\overline{10}_2 = 111\overline{1}_2$

$10110\overline{111}_2 + 10010\overline{100}_2 = 01101\overline{001}_2$

$\overline{10}_2 \times 11\overline{0}_2 = \overline{1}_2$

$\overline{110}_2 \times \overline{01}_2 = 01\overline{110}_2.$

Note that we omit the commas and add the subscript 2 (to distinguish from base 10) when writing 2-adics. We also use an overbar to denote a repeating pattern.

Having defined the ring structure on $\mathbb{Z}_2$ we define the metric $d : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{R}$ by

$$d(a,b) = |a-b|_2$$

where $||_2$ is known as the *2-adic valuation* and is given by

$$|a|_2 = \begin{cases} 0 & \text{if a=0} \\ 2^{-\min\{i:a_i=1\}} & \text{otherwise.} \end{cases}$$

In other words, $\min\{i : a_i = 1\}$ is the index of the position in the sequence where the first 1 appears. So the more leading zeros in the 2-adic, the smaller its valuation. In a sense, the valuation is a measure of the 'magnitude' of a 2-adic. Note that there are infinitely many 2-adics with any given positive valuation. In terms of the metric, the more leading digits that two 2-adics have in common, the closer they are.

Example 3
$|\overline{10}_2|_2 = 1$
$|000\overline{1}_2|_2 = \frac{1}{16}$
$d(\overline{10}_2, 000\overline{1}_2) = 1$
$d(\overline{10}_2, 101\overline{1}_2) = \frac{1}{16}$.

We partition the usual integers, $\mathbb{Z}$, into even and odd by considering equivalence in $\mathbb{Z}/2\mathbb{Z} = \{[0],[1]\}$. Similarly, we define even and odd on $\mathbb{Z}_2$ by considering equivalence in $\mathbb{Z}_2/2\mathbb{Z}_2 = \{[0],[1]\}$. Thus, we define $a \in \mathbb{Z}_2$ to be even if and only if $a \equiv 0 \bmod 2$ and odd if and only if $a \equiv 1 \bmod 2$, just as in $\mathbb{Z}$. Equivalently, $a \in \mathbb{Z}_2$ is even if the first digit of a is 0 and odd if the first digit of a is 1.

Example 4 $01101\overline{01}_2$ *is even and* $11101\overline{01}_2$ *is odd.*

Since even and odd are well defined for $\mathbb{Z}_2$, T extends naturally to $\mathbb{Z}_2$ and will be referred to as such throughout the rest of this paper.

An interesting fact about $\mathbb{Z}_2$ is that it contains $\mathbb{Z}$ as a subring. By associating each positive integer with its base-2 expansion (written backwards) and completing the sequence with a $\overline{0}$ we obtain the 2-adic representation of that integer.

Example 5
$3 = 11\overline{0}_2$
$6 = 011\overline{0}_2$.

Notice that parity is preserved by conversion between $\mathbb{Z}$ and $\mathbb{Z}_2$.

Because $\overline{1}_2 + 1\overline{0}_2 = \overline{0}_2$ we can say that $\overline{1}_2 = -1$ and we can embed the entire ring of integers using addition.

Another interesting fact is that the rationals with odd denominators, are also a subring of $\mathbb{Z}_2$. It can be shown that an integer in $\mathbb{Z}_2$ is invertible if and only if it is odd. Hence if $\frac{a}{b}$ is a rational number and b is odd, then we can associate $\frac{a}{b}$ with $a_2 b_2^{-1}$ in $\mathbb{Z}_2$.

Example 6 $\frac{1}{3} = \frac{1\overline{0}_2}{11\overline{0}_2} = 11\overline{01}_2$

Notice that we have obtained a repeating sequence. It is, in fact, a theorem that a 2-adic integer is rational if and only if it has an eventually repeating sequence. The rationals with even denominators are not a subring of $\mathbb{Z}_2$ because 2 is not invertible in $\mathbb{Z}_2$. For clarity, we will frequently write an integer or rational number in place of its 2-adic representation.

The extension of T to $\mathbb{Z}_2$ has led to the development of many very useful tools for studying the $3x + 1$ problem. We shall define several of the more useful of these tools here. A more extensive collection may be found in Lagarius [Lag].

First, define the *parity vector of length k for T of a* to be the sequence given by the function $Q_k : \mathbb{Z}_2 \to \mathbb{Z}_2/2^k\mathbb{Z}_2$ by

$$Q_k(a) = x_0(a), x_1(a), \ldots, x_k(a)$$

where

$$x_i(n) \equiv T^{(i)}(n) \bmod 2$$

and

$$x_i(n) \in \{0, 1\}$$

for all $i \geq 0$.

Example 7 *Since the orbit of* 3 *is* $3 \to 5 \to 8 \to 4 \to 2 \to 1 \to 2 \to \ldots$, *the first ten digits of the parity vector of* 3 *are:*

$$Q_{10}(3) = 1100010101.$$

The parity vector, $Q_k(a)$, completely describes the behavior of the first k iterates of a under T.

$Q_\infty(a)$ is defined in a similar manner and completely describes all iterates of a under T.

Q_k and Q_∞ have several interesting properties: Q_k is periodic with period 2^k and induces a permutation of $\mathbb{Z}_2/2^k\mathbb{Z}_2$, denoted $\overline{Q}_k$; Q_∞ is a continuous bijection. The proofs of these properties of Q_k and Q_∞ may be found in [Lag]. Both have proven to be extremely useful in the study of the $3x+1$ problem. One approach to the $3x+1$ problem where T is defined on $\mathbb{Z}_2$ which is explored by [FLW] uses the inverse of the Q_∞ function. Their approach illustrates the power and usefulness of the Q_∞ function and the value of extending the function T to larger domains.

With these examples in the literature of the usefulness of the extension of T to the 2-adic integers in terms of new approaches to the $3x+1$ problem, one naturally might ask if T could be extended to a larger set in a non-trivial way which would yield new insight while preserving the important properties of T. As a result, we investigate the 2-adic integers adjoined with i, written $\mathbb{Z}_2[i]$. We choose to extend to $\mathbb{Z}_2[i]$ because many number theoretic problems in $\mathbb{Z}$ have been solved by generalizing to the Gaussian integers $\mathbb{Z}[i]$. It is our hope that the same might happen here and that by working in $\mathbb{Z}_2[i]$ we might understand the dynamics of a new, more general function, thus solving the original problem for $\mathbb{Z}^+$. In keeping with this theme, we shall refer to $\mathbb{Z}_2[i]$ as the set of *Gaussian 2-adic integers* or simply, the *Gaussian 2-adics*.

4 The Gaussian 2-adics

We are now in a position to construct the metric space $(\mathbb{Z}_2[i], D)$. Let

$$\mathbb{Z}_2[i] = \{a + bi : a, b \in \mathbb{Z}_2\}$$

and define the metric $D : \mathbb{Z}_2[i] \times \mathbb{Z}_2[i] \to \mathbb{R}$ by

$$D(\alpha, \beta) = D'((a, b), (c, d))$$

where $\alpha = a + bi, \beta = c + di$ and

$$D'((a, b), (c, d)) = \max\{d(a, c), d(b, d)\}$$

is the product metric on $\mathbb{Z}_2 \times \mathbb{Z}_2$. Hence $\mathbb{Z}_2[i]$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are isometric. For that reason, we are justified in freely associating $a + bi \in \mathbb{Z}_2[i]$ and $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ for clarity. This relationship will be formalized later.

Addition and multiplication in $\mathbb{Z}_2[i]$ are defined in the usual manner. It is important to note that $\mathbb{Z}_2[i]$ is a commutative ring with identity, but not a field (the equation $2x = 1$ has no solution x in $\mathbb{Z}_2[i]$). In addition, $\mathbb{Z}_2$ is a commutative subring of $\mathbb{Z}_2[i]$ with identity and is also not a field.

5 Extension to $\mathbb{Z}_2[i]$

Now we are ready to propose an extension of the Collatz function T to $\mathbb{Z}_2[i]$. Since T was piecewise defined depending on equivalence in $\mathbb{Z}/2\mathbb{Z}$, our proposed extension is piecewise defined depending on equivalence in $\mathbb{Z}_2[i]/2\mathbb{Z}_2[i] = \{[0], [1], [i], [1 + i]\}$.

Definition 1 *Let $\widetilde{T} : \mathbb{Z}_2[i] \to \mathbb{Z}_2[i]$ by*

$$\widetilde{T}(\alpha) = \begin{cases} \frac{\alpha}{2} & \textit{if } \alpha \in [0] \\ \frac{3\alpha+1}{2} & \textit{if } \alpha \in [1] \\ \frac{3\alpha+i}{2} & \textit{if } \alpha \in [i] \\ \frac{3\alpha+1+i}{2} & \textit{if } \alpha \in [1+i]. \end{cases}$$

$T(a)$ is defined by dividing by 2 if a is equivalent to 0 and multiplying by 3, adding a representative member of the equivalence class, and then dividing by 2 otherwise. Notice that $\widetilde{T}(\alpha)$ is defined in a similar manner and in fact resembles $T \times T$ to a great degree.

It is then natural to ask how $\widetilde{T}$ is different from T and $T \times T$; after all, we claim that $\widetilde{T}$ is a non-trivial extension of T. Our response may be surprising: not only is $\widetilde{T}$ not equal to $T \times T$, but $\widetilde{T}$ and $T \times T$ are not even conjugate via a $\mathbb{Z}_2$-module isomorphism (though they are topologically conjugate, as we shall see in a later section.)

We begin by noting that $\widetilde{T}$ restricted to $\mathbb{Z}_2$ is the original Collatz function, i.e. $\widetilde{T}|\mathbb{Z}_2 = T$. This is because for any $\alpha \in \mathbb{Z}_2$ we have $\alpha = a + 0i \in [0]$ or $[1]$. Hence,

$$\widetilde{T}(\alpha) = \begin{cases} \frac{\alpha}{2} & \text{if } \alpha \in [0] \\ \frac{3\alpha+1}{2} & \text{if } \alpha \in [1] \end{cases}$$

and therefore, $\widetilde{T}$ restricted to $\mathbb{Z}_2$ is equal to T.

It is also clear that $\widetilde{T}$ is not the trivial extension $T \times T : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2$, i.e. $\widetilde{T} \neq T \times T$. If $\alpha = (a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ then

$$(T \times T)(\alpha) = \begin{cases} (\frac{a}{2}, \frac{b}{2}) & \text{if } \alpha \in [(0,0)] \\ (\frac{3a+1}{2}, \frac{b}{2}) & \text{if } \alpha \in [(1,0)] \\ (\frac{a}{2}, \frac{3b+1}{2}) & \text{if } \alpha \in [(0,1)] \\ (\frac{3a+1}{2}, \frac{3b+1}{2}) & \text{if } \alpha \in [(1,1)] \end{cases}$$

and it is easy to see that in the cases when $\alpha \in [(1,0)]$ or $\alpha \in [(0,1)]$, the functions differ for most values of $\alpha = a + bi$.

But what is more surprising is that $\widetilde{T}$ and $T \times T$ are not conjugate via a $\mathbb{Z}_2$-module isomorphism. Consider that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a $\mathbb{Z}_2$-module (a *module* is an object which satisfies all of the axioms for a vector space with the exception that the ring of scalars need not be a field) having basis $\{(1,0),(0,1)\}$ and thus dimension 2. A *$\mathbb{Z}_2$-module isomorphism on $\mathbb{Z}_2 \times \mathbb{Z}_2$* is an invertible function $A : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ satisfying the following properties:

1. $A(\alpha + \beta) = A(\alpha) + A(\beta)$,
2. $A(a\alpha) = aA(\alpha)$

for all $\alpha, \beta \in \mathbb{Z}_2[i]$ and for all scalars $a \in \mathbb{Z}_2$ (here we are freely identifying elements of $\mathbb{Z}_2[i]$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ in the natural manner). Essentially, a $\mathbb{Z}_2$-module isomorphism is analogous to an invertible linear transformation where the scalars are members of the ring $\mathbb{Z}_2$ instead of a field. We say that two functions $F : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ and $G : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ are conjugate via a $\mathbb{Z}_2$-module isomorphism if there exists a $\mathbb{Z}_2$-module isomorphism $A : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $F = A^{-1} \circ G \circ A$. Because $\dim(\mathbb{Z}_2 \times \mathbb{Z}_2) = 2$, any such $\mathbb{Z}_2$-module isomorphism can be represented as a 2×2 matrix:

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

where $x_1, x_2, y_1, y_2 \in \mathbb{Z}_2$.

We must now formalize our association between elements of $\mathbb{Z}_2[i]$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. Define a continuous bijection $B : \mathbb{Z}_2[i] \to \mathbb{Z}_2 \times \mathbb{Z}_2$ where

$$B(a + bi) = (a, b).$$

This provides us with a way to convert between $\mathbb{Z}_2[i]$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ and also allows us to define $\widehat{T} : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ by $\widehat{T} = B \circ \widetilde{T} \circ B^{-1}$.

Theorem 1 *There is no $\mathbb{Z}_2$-module isomorphism $A : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $\widehat{T} = A^{-1} \circ T \times T \circ A$.*

Proof. Assume that such a $\mathbb{Z}_2$-module isomorphism A exists. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Then $Ae_1 = (x_1, y_1)$ and $Ae_2 = (x_2, y_2)$ where x_1, x_2, y_1 and y_2 are 2-adics and $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. Then for all $a, b \in \mathbb{Z}_2 \times \mathbb{Z}_2$, $A \circ \widehat{T}((a, b)) = T \times T \circ A((a, b))$. Let $(a, b) \in (1, 0)$.

$$\begin{aligned}
T \times T(A((a,b))) &= A(\widehat{T}((a,b))) \\
\Rightarrow T \times T(\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}) &= A((\tfrac{3a+1}{2}, \tfrac{3b}{2})) \\
\Rightarrow T \times T((ax_1 + bx_2, ay_1 + by_2)) &= \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} \frac{3a+1}{2} \\ \frac{3b}{2} \end{pmatrix} \\
\Rightarrow T \times T((ax_1 + bx_2, ay_1 + by_2)) &= (\tfrac{3ax_1+3bx_2+x_1}{2}, \tfrac{3ay_1+3by_2+y_1}{2})
\end{aligned}$$

Thus, we have $T \times T((ax_1 + bx_2, ay_1 + by_2)) = (\frac{3ax_1+3bx_2+x_1}{2}, \frac{3ay_1+3by_2+y_1}{2})$. In order to evaluate $T \times T((ax_1 + bx_2, ay_1 + by_2))$, we must determine the parity of $ax_1 + bx_2$ and $ay_1 + by_2$. Because b is even and a is odd, the parities are completely determined by, and equivalent to, the parities of x_1 and y_1. This yields the following four cases:

$$T \times T((ax_1 + bx_2, ay_1 + by_2)) = \begin{cases} (\frac{ax_1+bx_2}{2}, \frac{ay_1+by_2}{2}) & \text{if } x_1 \text{ is even, } y_1 \text{ is even} \\ (\frac{3ax_1+3bx_2+1}{2}, \frac{ay_1+by_2}{2}) & \text{if } x_1 \text{ is odd, } y_1 \text{ is even} \\ (\frac{ax_1+bx_2}{2}, \frac{3ay_1+3by_2+1}{2}) & \text{if } x_1 \text{ is even, } y_1 \text{ is odd} \\ (\frac{3ax_1+3bx_2+1}{2}, \frac{3ay_1+3by_2+1}{2}) & \text{if } x_1 \text{ is odd, } y_1 \text{ is odd} \end{cases}$$

From this it is easy to check that $T \times T((ax_1 + bx_2, ay_1 + by_2)) = (\frac{3ax_1+3bx_2+x_1}{2}, \frac{3ay_1+3by_2+y_1}{2})$ if and only if $x_1 = 1$ and $y_1 = 1$. Thus, A must be of the form $\begin{pmatrix} 1 & x_2 \\ 1 & y_2 \end{pmatrix}$. In a similar manner, Let $(a, b) \in (0, 1)$

$$\begin{aligned}
T \times T(A((a,b))) &= A(\widehat{T}((a,b))) \\
\Rightarrow T \times T(\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} &= A((\tfrac{3a}{2}, \tfrac{3b+1}{2})) \\
\Rightarrow T \times T((ax_1 + bx_2, ay_1 + by_2)) &= \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} \frac{3a}{2} \\ \frac{3b+1}{2} \end{pmatrix} \\
\Rightarrow T \times T((ax_1 + bx_2, ay_1 + by_2)) &= (\tfrac{3ax_1+3bx_2+x_2}{2}, \tfrac{3ay_1+3by_2+y_2}{2})
\end{aligned}$$

Now, we have $(\frac{3ax_1+3bx_2+x_2}{2}, \frac{3ay_1+3by_2+y_2}{2}) = T \times T((ax_1 + bx_2, ay_1 + by_2))$. In this case, the parities of $ax_1 + bx_2$ and $ay_1 + by_2$ are completely determined by, and equivalent to, the parities of x_2 and y_2 respectively and again this yields four cases:

$$T\times T((ax_1+bx_2, ay_1+by_2)) = \begin{cases} (\frac{ax_1+bx_2}{2}, \frac{ay_1+by_2}{2}) & \text{if } x_2 \text{ is even, } y_2 \text{ is even} \\ (\frac{3ax_1+3bx_2+1}{2}, \frac{ay_1+by_2}{2}) & \text{if } x_2 \text{ is odd, } y_2 \text{ is even} \\ (\frac{ax_1+bx_2}{2}, \frac{3ay_1+3by_2+1}{2}) & \text{if } x_2 \text{ is even, } y_2 \text{ is odd} \\ (\frac{3ax_1+3bx_2+1}{2}, \frac{3ay_1+3by_2+1}{2}) & \text{if } x_2 \text{ is odd, } y_2 \text{ is odd} \end{cases}$$

This time we see that $T\times T((ax_1+bx_2, ay_1+by_2)) = (\frac{3ax_1+3bx_2+x_2}{2}, \frac{3ay_1+3by_2+y_2}{2})$ if and only if $x_2 = 1$ and $y_2 = 1$. This means A is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, but $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is not invertible and thus not a $\mathbb{Z}_2$-module isomorphism.

Therefore by contradiction there is no $\mathbb{Z}_2$-module isomorphism between $\widehat{T}$ and $T \times T$.

QED

Corollary 1 *$\widetilde{T}$ is not conjugate to $T \times T$ by a $\mathbb{Z}_2$-module isomorphism.*

Proof. Since $\widehat{T}$ is conjugate to $\widetilde{T}$ via the $\mathbb{Z}_2$-module isomorphism B, if $\widetilde{T}$ was conjugate to $T\times T$ via a $\mathbb{Z}_2$-module isomorphism, C, then $C \circ B^{-1}$ would be a $\mathbb{Z}_2$-module isomorphism between $\widehat{T}$ and $T \times T$ contradicting Theorem 1.

QED

6 Extension of Q_k and Q_∞

One of our main reasons for extending to $\mathbb{Z}_2[i]$ was to add the tools associated with $\mathbb{Z}_2[i]$ to the current tools for studying the Collatz problem. With this in mind, we redefine Q_k in terms of $\widetilde{T}$. This can be accomplished in a natural manner by simply replacing the T in the definition of the parity vector with $\widetilde{T}$ and extending the domain of Q_k to include $\mathbb{Z}_2[i]$. Thus we obtain $\widetilde{Q}_k : \mathbb{Z}_2[i] \to \mathbb{Z}_2[i]/2^k\mathbb{Z}_2[i]$ by

$$\widetilde{Q}_k(\alpha) = \widetilde{x}_0(\alpha), \widetilde{x}_1(\alpha), \widetilde{x}_2(\alpha), \ldots \widetilde{x}_k(\alpha)$$

where

$$\widetilde{x}_i(\alpha) \equiv \widetilde{T}^{(i)}(\alpha) \bmod 2 \text{ for all } i \geq 0.$$

and

$$\widetilde{x}_i(\alpha) \in \{0, 1, i, 1+i\}.$$

Example 8 *Since the orbit of* $5+2i$ *is* $5+2i \to 8+3i \to 12+5i \to 18+8i \to 9+4i \to 14+6i \to 7+3i \to 11+5i \to 17+8i \to 26+12i \to \ldots$, *the first ten digits of the parity vector of* $5+2i$ *are:*

$$\widetilde{Q}_{10}(5+2i) = 1, i, i, 0, 1, 0, 1+i, 1+i, 1, 0$$

As with Q_k, $\widetilde{Q}_k$ completely describes the behavior of the first k iterates of α under $\widetilde{T}$.

We also define $\widetilde{Q}_\infty : \mathbb{Z}_2[i] \to \mathbb{Z}_2[i]$ in a similar manner and note that, as you would expect, $\widetilde{Q}_\infty$ completely describes the behavior of all iterates of α under $\widetilde{T}$.

$\widetilde{Q}_k$ and $\widetilde{Q}_\infty$ have properties similar to Q_k and Q_∞ as will be demonstrated in the following theorems which mirror analogous theorems for Q_k and Q_∞ found in [Lag]:

Theorem 2 *The function* $\widetilde{Q}_k : \mathbb{Z}_2[i] \to \mathbb{Z}_2[i]/2^k\mathbb{Z}_2[i]$ *is periodic with period* 2^k.

In order to show that $\widetilde{Q}_k$ is periodic, we begin by showing:

Lemma 1 $\widetilde{T}^k(\alpha + \omega 2^k) \equiv \widetilde{T}^k(\alpha) + \omega \bmod 2$, *for any* $\alpha, \omega \in \mathbb{Z}_2[i]$.

Proof. We will proceed by induction on k. Let $\alpha, \omega \in \mathbb{Z}_2[i]$.

Base Case: $(k = 1)$

In this case,

$$\widetilde{T}(\alpha + \omega 2) = \begin{cases} \frac{\alpha}{2} + \omega \equiv \widetilde{T}(\alpha) + \omega \bmod 2 & \text{if } \alpha \in [0] \\ \frac{3\alpha+1}{2} + 3\omega \equiv \widetilde{T}(\alpha) + \omega \bmod 2 & \text{if } \alpha \in [1] \\ \frac{3\alpha+i}{2} + 3\omega \equiv \widetilde{T}(\alpha) + \omega \bmod 2 & \text{if } \alpha \in [i] \\ \frac{3\alpha+1+i}{2} + 3\omega \equiv \widetilde{T}(\alpha) + \omega \bmod 2 & \text{if } \alpha \in [1+i]. \end{cases}$$

General Case: Assume $\widetilde{T}^{k-1}(\alpha + \omega 2^{k-1}) \equiv \widetilde{T}^{k-1}(\alpha) + \omega \bmod 2$ for all n (inductive hypothesis).

Case 1: $\alpha \in [0]$. Then

$$\begin{aligned}
\widetilde{T}^k(\alpha + \omega 2^k) &= \widetilde{T}^{k-1}(\widetilde{T}(\alpha + \omega 2^k)) \\
&= \widetilde{T}^{k-1}(\tfrac{\alpha+\omega 2^k}{2}) && \text{(since } \alpha \in [0]) \\
&= \widetilde{T}^{k-1}(\tfrac{\alpha}{2} + \omega 2^{k-1}) \\
&\equiv \widetilde{T}^{k-1}(\tfrac{\alpha}{2}) + \omega \bmod 2 && \text{(by ind hyp)} \\
&\equiv \widetilde{T}^{k-1}(\widetilde{T}(\alpha)) + \omega \bmod 2 && \text{(since } \alpha \in [0]) \\
&\equiv \widetilde{T}^k(\alpha) + \omega \bmod 2
\end{aligned}$$

Case 2: $\alpha \in [1]$. Then

$$\begin{aligned}
\widetilde{T}^k(\alpha + \omega 2^k) &= \widetilde{T}^{k-1}(\widetilde{T}(\alpha + \omega 2^k)) \\
&= \widetilde{T}^{k-1}(\tfrac{3(\alpha+\omega 2^k)+1}{2}) && \text{(since } \alpha \in [1]) \\
&= \widetilde{T}^{k-1}(\tfrac{3\alpha+1}{2} + 3\omega 2^{k-1}) \\
&= \widetilde{T}^{k-1}(\tfrac{3\alpha+1}{2} + \omega 2^k + \omega 2^{k-1}) \\
&\equiv \widetilde{T}^{k-1}(\tfrac{3\alpha+1}{2} + \omega 2^k) + \omega \bmod 2 && \text{(by ind hyp)} \\
&\equiv \widetilde{T}^{k-1}(\tfrac{3\alpha+1}{2} + \omega 2^{k-1}) \bmod 2 && \text{(by ind hyp)} \\
&\equiv \widetilde{T}^{k-1}(\tfrac{3\alpha+1}{2}) + \omega \bmod 2 && \text{(by ind hyp)} \\
&\equiv \widetilde{T}^{k-1}(\widetilde{T}(\alpha)) + \omega \bmod 2 && \text{(since } \alpha \in [1]) \\
&\equiv \widetilde{T}^k(\alpha) + \omega \bmod 2
\end{aligned}$$

Case 3 ($\alpha \in [i]$) and Case 4 ($\alpha \in [1+i]$) are very similar to this case. Therefore, $\widetilde{T}^k(\alpha + \omega 2^k) \equiv \widetilde{T}^k(\alpha) + \omega \bmod 2$ for all k by induction on k.
QED
It follows easily that $\tilde{x}_k$ is also periodic in the same sense.

Corollary 2 *For every* $\alpha, \omega \in \mathbb{Z}_2[i]$, $\tilde{x}_j(\alpha + \omega 2^j) \equiv \tilde{x}_j(\alpha) + \omega \bmod 2$ *for all* $0 \leq j \leq \infty$.

Proof. Let $\alpha, \omega \in \mathbb{Z}_2[i]$. Then

$$\begin{aligned}
\tilde{x}_j(\alpha + \omega 2^j) &\equiv \widetilde{T}^j(\alpha + \omega 2^j) \bmod 2 \\
&\equiv \widetilde{T}^j(\alpha) + \omega \bmod 2 && \text{(by Lemma 1)} \\
&\equiv \tilde{x}_j(\alpha) + \omega \bmod 2.
\end{aligned}$$

QED
From this we obtain Theorem 2.

Proof of Theorem 2. We shall proceed using induction on k.

Base Case :($k = 1$)

$$\begin{aligned}\widetilde{Q}_1(\alpha + 2\omega) &= \widetilde{x}_1(\alpha + 2\omega) \\ &= \widetilde{x}_1(\alpha) && \text{(by Corollary 2)} \\ &= \widetilde{Q}_1(\alpha)\end{aligned}$$

General Case: Assume $\widetilde{Q}_{k-1}(\alpha + \omega 2^{k-1}) = \widetilde{Q}_{k-1}(\alpha)$ (inductive hypothesis).

$$\begin{aligned}\widetilde{Q}_k(\alpha + \omega 2^k) &= \textstyle\sum_{j=0}^{k-1} \widetilde{x}_j(\alpha + \omega 2^k) 2^j \\ &= \textstyle\sum_{j=0}^{k-2} \widetilde{x}_j(\alpha + \omega 2^k) 2^j + \widetilde{x}_{k-1}(\alpha + \omega 2^k) 2^{k-1} \\ &= \widetilde{Q}_{k-1}(\alpha + \omega 2^k) + \widetilde{x}_{k-1}(\alpha) 2^{k-1} && \text{(by Corollary 2)} \\ &= \widetilde{Q}_{k-1}(\alpha) + \widetilde{x}_{k-1}(\alpha) 2^{k-1} && \text{(by ind hyp)} \\ &= \textstyle\sum_{j=0}^{k-2} \widetilde{x}_j(\alpha) 2^j + \widetilde{x}_{k-1}(\alpha) 2^{k-1} \\ &= \textstyle\sum_{j=0}^{k-1} \widetilde{x}_j(\alpha) 2^j \\ &= \widetilde{Q}_k(\alpha)\end{aligned}$$

QED

Theorem 3 *$\widetilde{Q}_\infty$ is a measure preserving homeomorphism.*

Proof. We begin by showing that $\widetilde{Q}_\infty$ is continuous.

Let $\epsilon > 0$. Choose n so that $2^{-n} < \epsilon$ and choose $\delta = 2^{-n}$. For any $\alpha, \beta \in \mathbb{Z}_2[i]$ if $D(\alpha - \beta) < \delta$ then $\alpha \equiv \beta \bmod 2^n$. This implies that $\widetilde{Q}_n(\alpha) = \widetilde{Q}_n(\beta)$ and, consequently, $\widetilde{Q}_\infty(\alpha) \equiv \widetilde{Q}_\infty(\beta) \bmod 2^n$. Thus $D(\widetilde{Q}_\infty(\alpha) - \widetilde{Q}_\infty(\beta)) \leq 2^{-n} < \epsilon$ and $\widetilde{Q}_\infty$ is continuous.

We now show that $\widetilde{Q}_\infty$ is one-to-one.

Let $\alpha, \beta \in \mathbb{Z}_2[i]$, $\alpha \neq \beta$. Then there exists $\omega \in \mathbb{Z}_2[i]$ such that $\alpha = \beta + \omega 2^k$ where $k = \min\{j : \alpha_j \neq \beta_j\}$, $\alpha = \alpha_0, \alpha_1, \ldots$, $\beta = \beta_0, \beta_1, \ldots$, and ω is not equivalent to $0 \bmod 2$. By Corollary 2, $\widetilde{x}_j(\alpha) \equiv \widetilde{x}_j(\beta + \omega 2^j) \equiv \widetilde{x}_j(\beta) + \omega \bmod 2$. Consequently, $\widetilde{x}_j(\alpha) - \widetilde{x}_j(\beta) \equiv \omega \bmod 2$. Since ω is not equivalent to $0 \bmod 2$, $\widetilde{x}_j(\alpha) \neq \widetilde{x}_j(\beta)$ and therefore by definition of $\widetilde{Q}_\infty$, $\widetilde{Q}_\infty(\alpha) \neq \widetilde{Q}_\infty(\beta)$. Thus, $\widetilde{Q}_\infty$ is one-to-one.

Next we show that $\widetilde{Q}_\infty$ is measure preserving. A map, $f : (X, d) \to (X, d)$, is said to be *measure preserving* if $d(a, b) = d(f(a), f(b))$.

Let $\alpha, \beta \in \mathbb{Z}_2[i]$ where $\alpha = \alpha_0, \alpha_1, \ldots$, $\beta = \beta_0, \beta_1, \ldots$. Choose k so that $D(\alpha, \beta) = 2^{-k}$. Then $\alpha \equiv \beta \bmod 2^k$. By Theorem 2, $\widetilde{Q}_k(\alpha) = \widetilde{Q}_k(\beta)$, so $\widetilde{Q}_\infty(\alpha) \equiv \widetilde{Q}_\infty(\beta) \bmod 2^k$. Thus, $D(\widetilde{Q}_\infty(\alpha), \widetilde{Q}_\infty(\beta)) \leq 2^{-k}$. However, because α is not equivalent to $\beta \bmod 2^{k+1}$, $\alpha = \beta + \omega 2^{k+1}$ for some $\omega \in \mathbb{Z}_2[i]$, where ω is not equivalent to $0 \bmod 2$. Hence, $\widetilde{x}_{k+1}(\alpha) = \widetilde{x}_{k+1}(\beta + \omega 2^{k+1}) \equiv \widetilde{x}_k(\beta) + \omega \bmod 2$ by Corollary 2. However, because ω is not equivalent to $0 \bmod 2$, $\widetilde{x}_{k+1}(\alpha) \neq \widetilde{x}_{k+1}(\beta)$. It follows that $\widetilde{Q}_\infty(\alpha)$ is not equivalent to $\widetilde{Q}_\infty(\beta) \bmod 2^{k+1}$. Therefore, $D(\widetilde{Q}_\infty(\alpha),\, \widetilde{Q}_\infty(\beta)) = 2^{-k}$ and $\widetilde{Q}_\infty$ is measure preserving.

Finally, we show that $\widetilde{Q}_\infty$ is onto.

Let $\alpha = \alpha_0, \alpha_1, \ldots \in \mathbb{Z}_2[i]$, $\alpha'_k = \alpha_0, \ldots, \alpha_k, \overline{0} \in \mathbb{Z}_2[i]$, and $\widehat{\alpha}_k \in \mathbb{Z}_2[i]/2^k\mathbb{Z}_2[i]$ such that $\alpha \in \widehat{\alpha}_k$. We first note that $\widetilde{Q}_k$ is onto as can be seen by induction on k using Corollary 2. There exists a β'_k such that $\widetilde{Q}_k(\beta'_k) = \widehat{\alpha}_k$. Let $\beta'_k = \beta_0, \ldots, \beta_k, \overline{0}$. We can see that $\widetilde{Q}_\infty(\beta'_k) = \alpha'_k \bmod 2^k$. Thus, $\lim_{k\to\infty} D(\widetilde{Q}_\infty(\beta), \alpha'_k)) = 0$. Consequently, $\lim_{k\to\infty} \widetilde{Q}_\infty(\beta'_k) = \lim_{k\to\infty} \alpha'_k = \alpha$. Now, because $\widetilde{Q}_\infty$ is continuous, $\lim_{k\to\infty} \widetilde{Q}_\infty(\beta'_k) = \widetilde{Q}_\infty(\lim_{k\to\infty} \beta'_k) = \alpha$. So all that remains is to show that $\lim_{k\to\infty} \beta'_k$ exists as a Gaussian 2-adic. Since the sequence $\{\widetilde{Q}_\infty(\beta'_k)\}$ converges to α it is Cauchy. Because $\widetilde{Q}_\infty$ is measure preserving, the sequence $\{\beta'_k\}$ in $\mathbb{Z}_2[i]$ is also a Cauchy sequence. Now, $\mathbb{Z}_2[i]$ is a compact metric space so by the Tychonoff theorem every Cauchy sequence in $\mathbb{Z}_2[i]$ has a limit in $\mathbb{Z}_2[i]$. Thus the sequence $\{\beta'_k\}$ converges to some $\beta \in \mathbb{Z}_2[i]$ and $\widetilde{Q}_\infty(\beta) = \alpha$. Therefore $\widetilde{Q}_\infty$ is onto.

$\widetilde{Q}_\infty^{-1}$ is continuous because $\widetilde{Q}_\infty$ is an isometry and therefore $\widetilde{Q}_\infty$ is a measure preserving homeomorphism.

QED

Now that we have shown $\widetilde{Q}_\infty$ is a continuous bijection, we shall see just how powerful a tool it is in our exploration of the dynamics of $\widetilde{T}$.

7 Chaos and the $3x + 1$ Problem

The field of dynamical systems can introduce a great number of tools to our study of the $3x+1$ problem. These include orbit analysis, symbolic dynamics and, most importantly, topological conjugacy to other well understood maps which can be studied more easily. We shall see in this section that not only is T chaotic, but $T \times T$ and $\widetilde{T}$ are as well. In discussing chaoticity, we will employ several definitions and theorems from Devaney [Dev] and use his notation wherever possible.

A *dynamical system* is essentially "a process in motion" [Dev]. Examples of dynamical systems are the weather, currents in the ocean, and the mixing patterns created by cream dribbled into hot coffee. Mathematically these can be described by a metric space (X, d) and a continuous function $F : (X, d) \rightarrow (X, d)$. We can construct abstract dynamical systems in this way.

Example 9 *Define the metric space known as the sequence space* (Σ, d) *where*

$$\Sigma = \{s_0, s_1, s_2, \ldots | s_j \in \{0, 1\}\}$$

and the metric d' *on* Σ *is*

$$d'(s, t) = \Sigma_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}.$$

Now define the shift map $\sigma : \Sigma \rightarrow \Sigma$ *by*

$$\sigma(s_0, s_1, s_2, \ldots) = s_1, s_2, s_3, \ldots.$$

This is a well defined and easily understood dynamical system (the behavior of every seed is completely known).

Example 10 $\widetilde{T} : (\mathbb{Z}_2[i], D) \rightarrow (\mathbb{Z}_2[i], D)$ *is a dynamical system.*

Devaney defines a *periodic point* to be a point that is a member of a cycle. i.e. x is a periodic point for F if there exists $n \geq 1$ such that $F^{(n)}(x) = x$.

Example 11 $101\overline{010} \in \Sigma$ *is a periodic point for* σ *because*

$$\sigma^{(2)}(101\overline{010}) = 101\overline{010}.$$

A dynamical system $F : (X, d) \to (X, d)$ is defined to be *transitive* if $\forall x, y \in X, \forall \epsilon > 0, \exists z \in X$ such that $d(x, z) < \epsilon$ and $d(y, F^{(k)}(z)) < \epsilon$ for some $k \geq 0$ [Dev]. Intuitively, F is transitive if given any two points we can find an orbit that comes arbitrarily close to both.

Example 12 *Let* $\widehat{s} \in \Sigma$ *where*

$$\widehat{s} = (1\ 0\ 00\ 01\ 10\ 11\ 000\ 001\ 010\ 100\ 110\ 101\ 011\ 111\ 0000\ \ldots).$$

In other words, $\widehat{s}$ *is the sequence of all possible sequences of length 1 followed by all possible sequences of length 2, then length 3, and so on. It is easy to see that the orbit of* $\widehat{s}$ *under* σ *passes arbitrarily close to every point in* Σ*. So for any two points in* Σ*, the orbit of* $\widehat{s}$ *passes arbitrarily close to both. Thus* σ *is a transitive dynamical system.*

An orbit such as the orbit of $\widehat{s}$ which passes arbitrarily close to all points in a dynamical system is called a *dense orbit*. It is, in fact, a theorem that a dynamical system is transitive if and only if it has a dense orbit [Dev].

Devaney defines a *chaotic dynamical system* to be any transitive dynamical system with dense periodic points.

Example 13 σ *is a chaotic dynamical system. We have already shown that* σ *is transitive and it is easy to see that it has dense periodic points. Given any* $s = s_1 s_2 s_3 \ldots \in \Sigma$ *and any* $\epsilon \geq 0$*, we can choose the periodic point* $\overline{s_0 s_1 s_2 \ldots s_k}$ *where* $k = min\{i : 2^{-i+1} < \epsilon\}$*. Then* $d'(\overline{s_0 s_1 s_2 \ldots s_i}, s) < \epsilon$ *and hence* σ *has dense periodic points.*

It is shown in Devaney that chaoticity is preserved by topological conjugacy, so we can show that a function is chaotic if it is topologically conjugate to a known chaotic map.

Two functions $F : X \to X$ and $G : Y \to Y$ are said to be *topologically conjugate* if there exists a continuous bijection, $h : X \to Y$, such that h^{-1} is continuous and $h \circ F = G \circ h$. Such a map, h, is called a *homeomorphism*.

Theorem 4 $T : \mathbb{Z}_2 \to \mathbb{Z}_2$ *is chaotic.*

Proof. We shall proceed by proving that T is topologically conjugate to $\sigma : \Sigma \to \Sigma$. Note that Σ and $\mathbb{Z}_2$ are homeomorphic topological spaces and equal as sets. Here we will use the homeomorphism Q_∞ between $\mathbb{Z}_2$ and Σ. We now must verify that $Q_\infty \circ T = \sigma \circ Q_\infty$.

Let $\alpha \in \mathbb{Z}_2$. Then

$$\begin{aligned}(Q_\infty \circ T)(\alpha) &= Q_\infty(T(\alpha)) \\ &= x_0(T(\alpha)), x_1(T(\alpha)), x_2(T(\alpha)), \ldots \\ &= x_1(\alpha), x_2(\alpha), x_3(\alpha) \ldots \\ &= \sigma(x_0(\alpha), x_1(\alpha), x_2(\alpha), \ldots) \\ &= \sigma(Q_\infty(\alpha)) \\ &= (\sigma \circ Q_\infty)(\alpha)\end{aligned}$$

So T is conjugate to σ and therefore chaotic.

QED

Corollary 3 *$T \times T$ is chaotic*

Proof. It follows directly that $T \times T$ is chaotic via conjugacy to $\Sigma \times \Sigma$ by $Q_\infty \times Q_\infty$ because the cross product of homeomorphisms is a homeomorphism between the product spaces, where we take the topology on $\Sigma \times \Sigma$ to be the product topology.

QED

Certainly if T is chaotic any reasonable extension of T should also be chaotic. Accordingly, our next task is to prove that $\widetilde{T}$ is chaotic using $\widetilde{Q}_\infty$.

We would like to show that $\widetilde{T}$ is chaotic by showing that it is conjugate to some known chaotic system. With this in mind we define $\sigma_4 : (\Sigma_4, d_\delta) \to (\Sigma_4, d_\delta)$ and show that it is chaotic:

Let $\sigma_4 : (\Sigma_4, d_\delta) \to (\Sigma_4, d_\delta)$ be the shift map on the sequence space with four elements $\{0, 1, i, 1+i\}$ where

$$d_\delta((s_0, s_1, \ldots), (t_0, t_1, \ldots)) = \Sigma_{k=0}^{\infty} \frac{\delta_k(s,t)}{4^k}$$

and

$$\delta_k = \begin{cases} 0 & \text{if } s_k = t_k \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 2 *The function $F : \Sigma_4 \to \Sigma \times \Sigma$ by*

$$F(s) = ((a_1(s), a_2(s), a_3(s), \ldots), (b_1(s), b_2(s), b_3(s), \ldots))$$

where each

$$a_i(s) = \begin{cases} 0 & \text{if } s_i = 0 \text{ or } i \\ 1 & \text{if } s_i = 1 \text{ or } 1+i \end{cases}$$

and each

$$b_i(s) = \begin{cases} 0 & \text{if } s_i = 0 \text{ or } 1 \\ 1 & \text{if } s_i = i \text{ or } 1+i. \end{cases}$$

is a homeomorphism.

Proof. It is clear that F is a bijection. We now show that F is continuous.

Let $\epsilon > 0$, $\delta = 4^{-k}$ where k is chosen to make $2^{-k} < \epsilon$, $s = s_0, s_1, \ldots \in \Sigma_4$, $t = t_0, t_1, \ldots \in \Sigma_4$. If $d_\delta(s,t) < \delta$ then $s_j = t_j$ for all $0 \le j \le k$. Consequently, if we consider that $F(s) = (x, y)$ and $F(t) = (z, w)$ for some $(x, y), (z, w) \in \Sigma \times \Sigma$ where $x = x_0, x_1, \ldots \in \Sigma$, $y = y_0, y_1, \ldots \in \Sigma$, $z = z_0, z_1, \ldots \in \Sigma$, and $w = w_0, w_1, \ldots \in \Sigma$ then $x_j = z_j$ and $y_j = w_j$ for all $0 \le j \le k$ by definition of d_δ. Thus, by definition of F, $d_x(F(s), F(t)) \le 2^{-k} < \epsilon$ where d_x is the product metric on Σ. So F is continuous.

By letting $\epsilon > 0$ and choosing $\delta = 2^{-k}$ where k is such that $4^{-k} < \epsilon$, we can apply a similar argument to show that F^{-1} is continuous. Therefore, F is a homeomorphism.

QED

Corollary 4 *σ_4 and $\sigma \times \sigma$ are conjugate via F.*

Proof. Let $s \in \Sigma_4$, $s = s_0, s_1, \ldots$. Then

$$\begin{aligned} (F \circ \sigma_4)(s) &= F(\sigma_4(s)) \\ &= F(s_2, s_3, s_4, \ldots) \\ &= ((a_2(s), a_3(s), a_4(s), \ldots), (b_2(s), b_3(s), b_4(s), \ldots)) \\ &= \sigma \times \sigma(((a_1(s), a_2(s), a_3(s), \ldots), (b_1(s), b_2(s), b_3(s), \ldots))) \\ &= \sigma \times \sigma(F(s)) \\ &= (\sigma \times \sigma \circ F)(s) \end{aligned}$$

QED

We are now prepared to show that $\widetilde{T}$ is chaotic.

Theorem 5 *$\widetilde{T}$ is chaotic.*

We see in Corollary 4 that σ_4 is conjugate to $\sigma \times \sigma$ and therefore chaotic. We now show that $\widetilde{T}$ is conjugate to σ_4 by the homeomorphism $\widetilde{Q}_\infty$. The argument is essentially the same as above: Let $\alpha \in \mathbb{Z}_2[i]$. Then

$$\begin{aligned}
\widetilde{Q}_\infty(\widetilde{T}(\alpha)) &= \widetilde{x}_1(\widetilde{T}(\alpha)), \widetilde{x}_2(\widetilde{T}(\alpha)), \widetilde{x}_3(\widetilde{T}(\alpha)), \ldots \\
&= \widetilde{x}_2(\alpha), \widetilde{x}_3(\alpha), \widetilde{x}_4(\alpha), \ldots \\
&= \sigma_4(\widetilde{x}_1(\alpha), \widetilde{x}_2(\alpha), \widetilde{x}_3(\alpha), \widetilde{x}_4(\alpha), \ldots) \\
&= \sigma_4(\widetilde{Q}_\infty(\alpha))
\end{aligned}$$

Therefore $\widetilde{Q}_\infty \circ \widetilde{T} = \sigma_4 \circ \widetilde{Q}_\infty$ and $\widetilde{T}$ is chaotic.
QED

It turns out that in proving the chaoticity of T, $\widetilde{T}$, and $T \times T$ we have defined some very useful conjugacies as we shall see in the next section.

8 Relationship between $\widetilde{T}$ and $T \times T$

Though $\widetilde{T}$ and $T \times T$ are not equal, they are topologically conjugate. We shall show this using the transitivity of topological conjugacy.

Theorem 6 *$\widetilde{T}$ and $T \times T$ are topologically conjugate (via $(Q_\infty \times Q_\infty)^{-1} \circ F \circ \widetilde{Q}_\infty$).*

Proof. Since topological conjugacy is transitive,

$$\widetilde{T} \cong \sigma_4 \cong \sigma \times \sigma \cong T \times T$$

implies that $\widetilde{T} \cong T \times T$ where $\cong$ denotes topological conjugacy.
QED

These theorems allow us to work in the system of our choice and then convert the results to any other system using the homeomorphisms defined above.

9 Acknowledgement

This paper was completed as a result of three years of work by the author as an undergraduate in the Faculty/Student Research Program at the University of Scranton under the direction of Dr. Kenneth Monks. I would like to dedicate this paper to Dr. Monks for all of his time, effort and assistance without which this paper could not have been completed.

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Department of Mathematics
University of Scranton
Scranton, PA 18510
E-mail address: jaj3@jaguar.uofs.edu