The Elimination of a Family of Periodic Parity Vectors in the 3x+1 Problem

Carolyn Farruggia
Michael Lawrence
Brian Waterhouse

January 27, 1996

Abstract

Let $s(k) = s_1 s_2 \ldots$ where $s_i = \begin{cases} 1, & \text{if } i \equiv 0 \bmod k \\ 0, & \text{otherwise} \end{cases}$. We prove that there is no positive integer whose parity vector is $s(k)$ for $k \geq 2$. In proving this result we discuss a general method for eliminating other families of periodic 0,1-sequences as parity vectors.

1 Introduction

The $3x + 1$ Problem, also known as the Collatz Conjecture, is traditionally credited to Lothar Collatz at the University of Hamburg in 1930's. Jeffrey C. Lagarius at AT&T Bell Laboratories has written an excellent expository paper on the subject [1], and we will use much of his notation here. Simply put the $3x + 1$ problem proposes that repeated iteration of the following function $T : \mathbb{Z}^+ \to \mathbb{Z}^+$ will eventually lead to the value 1 for any $n > 0$:

$$T(n) = \begin{cases} (3n+1)/2, & \text{if } n \equiv 1 \bmod 2 \\ n/2, & \text{if } n \equiv 0 \bmod 2. \end{cases}$$

Define the trajectory of n to be the sequence of iterates

$$n, T(n), T^{(2)}(n), T^{(3)}(n), \ldots$$

where $T^{(i)}(n)$ represents the i^{th} composition of T with itself. We can classify these trajectories into 3 types for $n > 0$:

(i) **Convergent trajectory**. $T^{(k)}(n) = 1$ for some k.

(ii) **Non-trivial cyclic trajectory**. The sequence $T^{(k)}(n)$ eventually becomes periodic and $T^{(k)}(n) \neq 1$ for any $k \geq 1$.

(iii) **Divergent trajectory**. The $\lim_{k\to\infty} T^{(k)}(n) = \infty$.

Define the *parity vector* of n to be the sequence of 0's and 1's

$$Q_\infty(n) = s_0(n)s_1(n)\ldots$$

satisfying $s_i(n) \equiv T^{(i)}(n) \bmod 2$ for all $i \geq 0$. The parity vector completely describes the result of k iterations of T, since $T^{(k)}(n) = \lambda_k(n)n + \rho_k(n)$ where $\lambda_k(n) = \frac{3^{s_0(n)+\ldots+s_{k-1}(n)}}{2^k}$ and $\rho_k(n) = \sum s_i(n)\frac{3^{s_{i+1}(n)+\ldots+s_{k-1}(n)}}{2^{k-i}}$ [1].

A non-trivial cyclic trajectory has a periodic parity vector. It has yet to be determined whether or not there are any non-trivial cycles. Thus in order to show that there are no non-trivial cycles, it suffices to show that any periodic sequence of 0's and 1's is not the parity vector of an integer ≥ 3.

2 Elimination of Parity Vectors

For a 0,1-sequence s, to *eliminate* s as a parity vector (or simply, to *eliminate* s), means to show that s is not the parity vector of a positive integer. Our main result is the elimination of a family of periodic 0,1-sequences as parity vectors.

Theorem 1 *Let* $s(k) = s_1 s_2 \ldots$ *where* $s_i = \begin{cases} 1, & \text{if } i \equiv 0 \bmod k \\ 0, & \text{otherwise} \end{cases}$. *There is no positive integer whose parity vector is* $s(k)$ *for* $k \geq 2$.

While this family of 0,1-sequences might easily be eliminated by other means, what is of interest in this paper is not only the result, but also the method used in the proof. In theory, this method can be used to eliminate any family of 0,1-sequences as parity vectors. Also, it gives a very good expository insight into the nature of the problem, especially the relationship with the 2-adics.

3 Relationship with the 2-adic Integers

Any parity vector is a sequence of 0's and 1's and thus can be interpreted as an element of the 2-adic integers, $\mathbb{Z}_{(2)} = \{ s_0 s_1 s_2 \ldots | s_i \in \{0,1\} \text{ for all } i \}$. One can define a ring structure on $\mathbb{Z}_{(2)}$ by the usual rules for manipulating formal power series where we identify the sequence $s_0 s_1 s_2 \ldots \in \mathbb{Z}_{(2)}$ with the formal power series $s_0 + s_1 2 + s_2 2^2 + s_s 2^3 + \ldots$ (see any standard text on p-adic numbers for details e.g. [2].) Note that the integers, $\mathbb{Z}$ (and in fact, the rationals with odd denominators $\mathbb{Q}_{odd}$) can be considered to be subrings of $\mathbb{Z}_{(2)}$ by associating each positive integer p with its base two expansion. That is, if $p = \sum_{i=0}^{\infty} b_i 2^i$ is the base two representation of $p \in \mathbb{N}$, then we associate p with the 2-adic integer $b_0 b_1 b_2 \ldots \in \mathbb{Z}_{(2)}$. This inclusion can be extended to an embedding of the rings $\mathbb{Z}$ and $\mathbb{Q}_{odd}$ into $\mathbb{Z}_{(2)}$ in a unique way (see [2]). In particular, $1/(1-r) = \sum_{i=0}^{\infty} r^i$ if $r = 2^k$ for some $k \in \mathbb{N}$.

Define the set of even 2-adics to be the set of all sequences, $s_0 s_1 s_2 \ldots$, such that $s_0 = 0$ and the set of odd 2-adics to be the complement of this set in $\mathbb{Z}_{(2)}$. Thus, we can extend T to the 2-adics in the obvious manner, that is $T : \mathbb{Z}_{(2)} \to \mathbb{Z}_{(2)}$ by

$$T(s) = \begin{cases} (3s+1)/2, & \text{if } s \text{ is odd} \\ s/2, & \text{otherwise} \end{cases}.$$

Similarly, we can define the parity vector $Q_\infty(s)$ for any $s \in \mathbb{Z}_{(2)}$ just as was done in the integer case. The map $Q_\infty : \mathbb{Z}_{(2)} \to \mathbb{Z}_{(2)}$ is a continuous, one-to-one onto, and measure-preserving map on the 2-adic integers $\mathbb{Z}_{(2)}$ [1, Theorem L]. Since the Q_∞ function is one-to-one, the value of n is uniquely determined by its parity vector. Since Q_∞ is onto, every 2-adic is the parity vector of some other parity vector. Therefore we will use the terms *parity vector* and *2-adic* interchangeably to mean any sequence of 0's and 1's.

4 Method

A natural question to ask when one first encounters the Collatz problem is whether or not there is a trajectory whose entries are all odd. In terms of parity vectors this is equivalent to asking if we can eliminate the parity vector $\overline{1}$ consisting of all 1's. (We will sometimes denote the repeating part of a periodic sequence by an over bar.)

Example 1 *There is no positive integer n such that $Q_\infty(n) = 111\ldots$.*

A straightforward argument that there cannot be such a trajectory might proceed as follows:

First proof of Example 1: We begin by stating some number theoretic Lemmas. The following is a standard result whose proof will be omitted.

Lemma 1 *There is no positive integer n such that $n \equiv -1 \bmod 2^k$, for all $k \geq 1$.*

If $Q_\infty(n) = s_0 s_1 s_2 \ldots$ then define $Q_k(n) = s_0 s_1 \ldots s_{k-1}$.

Lemma 2 *If $Q_k(n) = \underbrace{111\ldots1}_{k\ ones}$ then $n \equiv -1 \bmod 2^k$.*

Proof. If $k = 1$ then $Q_1(n) = 1 \Leftrightarrow n$ is odd $\Leftrightarrow n \equiv -1 \bmod 2$. Assume the lemma is true for $k-1$. Suppose $Q_k(n) = \underbrace{11\ldots1}_{k\ ones}$. Then $Q_{k-1}(T(n)) = \underbrace{11\ldots1}_{k-1\ ones}$ by definition of Q_{k-1}. Hence, $T(n) \equiv -1 \bmod 2^{k-1}$. So, by the definition of T, $T(n) = \frac{(3n+1)}{2}$. Therefore, $\frac{3n+1}{2} + 1 = q2^{k-1}$, for some $q \in \mathbb{Z}^+$. Therefore $3(n+1) = q2^k$. Since $3(n+1)$ is divisible by 3, $q2^k$ is also divisible by 3. But 2^k is not divisible by 3. So q must be divisible by 3. That is, $q = 3x$ for some x. Therefore $3(n+1) = 3x2^k$. Therefore $n + 1 = x2^k$, for some x. Therefore $n \equiv -1 \bmod 2^k$. *QED*

To complete the proof of Example 1, assume that there is a positive integer, n, such that $Q_\infty(n) = 111\ldots$. Then $Q_k(n) = \underbrace{11\ldots1}_{k\ ones}$, for all $k \geq 1$. Therefore by Lemma 2, $n \equiv -1 \bmod 2^k$ for all $k \geq 1$. This contradicts Lemma 1. Therefore, there is no positive integer n such that $Q_\infty(n) = 11\ldots$. *QED*

This elementary method is straightforward, but cannot easily be generalized to eliminate other parity vectors. Let us consider another approach. Since Q_∞ is one-to-one, we can eliminate a parity vector, s, by showing that $Q_\infty^{-1}(s)$ is not a positive integer.

Second Proof of Example 1: Since $T(-1) = -1$, the trajectory of -1 is $-1, -1, -1, \ldots$. So $Q_\infty(-1) = 111\ldots$. Since Q_∞ is one-to-one, there is no positive integer n whose parity vector is $111\ldots$. *QED*

Thus in order to generalize this technique to eliminate other sequences, it is necessary to have a method for computing Q_∞^{-1}.

Let $s = s_0 s_1 \ldots$ be a periodic parity vector. By [1, Theorem B], $Q_k(n) = Q_k(n + 2^k)$ for all integers n, and there is a unique non-negative integer $t < 2^k$ such that $Q_k(t) = s_0 s_1 \ldots s_{k-1}$. Thus if $Q_k(t) = s_0 s_1 \ldots s_{k-1}$, then either $Q_{k+1}(t) = s_0 s_1 \ldots s_k$ or $Q_{k+1}(t + 2^k) = s_0 s_1 \ldots s_k$ since t and $t + 2^k$ are the only numbers $< 2^{k+1}$ that are congruent to t mod 2^k. Thus, we can recursively define a set of integers t_k as follows: Let $t_0 = s_0$, and let

$$t_k = \begin{cases} t_{k-1}, & \text{if } T^k(t_{k-1}) \equiv s_k \bmod 2 \\ t_{k-1} + 2^k, & \text{otherwise} \end{cases}.$$

Then $Q_{k+1}(t_k) = s_0 s_1 \ldots s_k$ for all k. So, the sequence of integers t_k converges to $p = Q_\infty^{-1}(s)$ in $\mathbb{Z}_{(2)}$.

Thus, by looking at the binary expansion of t_k for sufficiently large k, one can conjecture what the 2-adic digits of p might be. (For example, if p is rational, its digits will be eventually repeating.) It is then a simple matter of verifying that the conjectured value of p is, in fact, $Q_\infty^{-1}(s)$, by directly computing the parity vector of p. If p is not a positive integer, we have successfully eliminated the vector s.

Example 2 Let us eliminate the parity vector $s = \overline{001}$. By definition, $t_0 = 0$, $t_1 = 0$, $t_2 = 4$, etc. We continue computing t in a similar manner, until we reach $t_{14} = 13108$. In (reversed) binary form this number is 00101100110011, and we see a pattern developing in the binary expansion. We then conjecture that $p = 001\overline{0110} = 4/5$. To verify, we check the parity vector of $4/5$. Since $T(4/5) = 2/5$, $T(2/5) = 1/5$, and $T(1/5) = 4/5$, the parity vector of $4/5$ is $\overline{001}$ and we have eliminated this parity vector.

If $\{ s(k) | k \in \mathbb{N} \}$ is a family of parity vectors, one can use this technique to determine $p(k) = Q_\infty^{-1}(s(k))$ for the first few values of k. Using these values we can conjecture what $p(k)$ might be for any k. Verification that $p(k) = Q_\infty^{-1}(s(k))$ for all k again can be obtained by direct computation of the parity vector of $p(k)$. This is the method used in the proof of Theorem 1.

Example 3 Let $s(k) = \overline{s_0 s_1 \ldots s_k}$ where $s_i = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{otherwise} \end{cases}$. Then a calculation similar to the one used in Example 2 yields the following results:

k	$s(k)$	$p(k)$ (2-adic expansion)	$p(k)$ (in base ten)
0	$\overline{1}$	$\overline{1}$	-1
1	$\overline{01}$	$\overline{01}$	2
2	$\overline{001}$	$001\overline{0110}$	4/5
3	$\overline{0001}$	$0001\overline{010001101110}$	8/13
4	$\overline{00001}$	$00001\overline{0101100010000110100111011110}$	16/29

By looking at the values of $p(k)$ in base ten, we are led to the conjecture:

$$p(k) = 2^k/(2^{k+1} - 3) \text{ for } k \in \mathbb{N}.$$

5 Proof of the main theorem

Having conjectured the values of the $p(k)$, we are now ready to prove the main theorem.

Proof of Theorem 1. Let $p(k) = 2^k/(2^{k+1} - 3)$. Then $T^{(k)}(p(k)) = T^{(k)}\left(\frac{2^k}{2^{k+1}-3}\right) = \frac{1}{2^{k+1}-3}$. So,

$$\begin{aligned} T^{(k+1)}(p(k)) &= T(T^{(k)}(p(k))) \\ &= T\left(\frac{1}{2^{k+1}-3}\right) \\ &= \frac{3\left(\frac{1}{2^{k+1}-3}\right)+1}{2} \\ &= \frac{2^k}{2^{k+1}-3} \\ &= p(k). \end{aligned}$$

Therefore, the trajectory of $p(k)$ is cyclic and the parity vector of $p(k)$ is equal to $\overline{\underbrace{00\ldots0}_{k\ zeros}1} = s(k)$. It is clear that this will always lead to a fraction for $k > 1$, since the numerator is a power of 2 and the denominator is an odd number > 1. Thus $s(k)$ is not the parity vector of a positive integer for $k > 1$. *QED*

6 Acknowledgment

We would like to dedicate this paper to Dr. Kenneth Monks for all of his time, effort, and assistance.

References

[1] J. C. Lagarias. *The 3x+1 Problem and Its Generalizations.* American Mathematics Monthly 92 (1985), 3-23.

[2] G. Bachman. *Introduction to p-adic Numbers and Valuation Theory.* Academic Press, 1964.