A CATEGORY OF TOPOGICAL SPACES CLASSIFYING ACYCLIC SET THEORETIC DYNAMICS

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Abstract. For each set X and each self map $f : X \to X$, we construct an associated topology on X, called the induced topology, τ_f, which is an invariant of the conjugacy type of f, i.e. we show that for all functions $f : X \to X$, $g : Y \to Y$, f is set theoretically conjugate to g then the corresponding topological spaces (X, τ_f) and (Y, τ_g) are homeomorphic. If the only cyclic points of $f : X \to X$ are fixed points, we say f is acyclic, and show that acyclic maps f, g are conjugate if and only if the induced topological spaces are homeomorphic. As an application, we show that the well-known $3x + 1$ conjecture is true if and only if the induced topological space is connected. We also give sufficient conditions to prove or disprove the conjecture in terms of certain semiconjugacies between dynamical systems and their induced topologies.

1. Introduction

Let X, Y be sets and let $f : X \to X$ and $g : Y \to Y$ be any functions. The maps f, g are said to be *conjugate* (or *set theoretically conjugate*) if there exists a bijection $h : X \to Y$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h \downarrow & & \downarrow h \\ Y & \xrightarrow[g]{} & Y \end{array}$$

commutes. If, in addition, X and Y are topological spaces and h is a homeomorphism, then f and g are said to be *topologically conjugate* [Dev]. In either situation the map h is called a *conjugacy* between f and g.

Discrete dynamical systems theory generally studies those properties of continuous self maps on a topological space which are preserved by topological conjugacy. However, there are important situations where it is the set theoretical conjugacy of maps and not topological conjugacy that is of interest.

Consider for example, the famous $3x+1$ Conjecture (see [Lag] for a nice exposition). Let $T : \mathbb{Z} \to \mathbb{Z}$ by $T(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ (3x+1)/2 & \text{if } x \text{ is odd} \end{cases}$. The conjecture states that

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for every positive integer x, there exists k such that $T^k(x) = 1$ where T^k denotes the k-fold composition of T with itself (T^0 is the identity map).

For any $f : X \to X$ and $n > 0$ we say $x \in X$ is a *cyclic point of order* n if $f^n(x) = x$ and $f^j(x) \neq x$ for any $0 < j < n$. If x is a cyclic point of order n for f we say $\{x, f(x), \ldots, f^{n-1}(x)\}$ is a *cycle of order* n *for* f (or an f-*cycle of order* n). A cyclic point of order one is said to be a *fixed point of* f. The point x is *eventually cyclic with order* n *for* f if there exists $k \geq 0$ such that $f^k(x)$ is cyclic with order n for f. If x is eventually cyclic we say it is *eventually cyclic after* k *iterations* if $f^k(x)$ is cyclic and $f^{k-1}(x)$ is not.

Since $T(1) = 2$, $T(2) = 1$ and it is easy to check that this is the only 2-cycle for T, the conjecture can be restated in dynamical terms as asking whether or not every positive integer is eventually cyclic with order 2. The related Finite Cycles Conjecture states that T has only finitely many cycles on $\mathbb{Z}$.

The number of cycles and their orders are examples of a properties that are preserved by set theoretic conjugacy. For example, the Finite Cycles Conjecture is true if and only if any function which is conjugate to T also has finitely many cycles. If h is a conjugacy between T and some other function f, then the $3x+1$ conjecture is true if and only if for every positive integer n, $h(n)$ is eventually periodic for f with period 2. For more information on conjugacies of T we refer the reader to [Lag], [BerLag], and [MonYaz].

As expected, topological conjugacy preserves all of the properties which are preserved by set theoretic ones and additional properties, e.g. whether or not a cycle is attracting or repelling. But in a scenario such as the $3x+1$ problem, one is not necessarily interested in whether or not a positive integer is attracted to a given cycle under some metric. We are concerned instead with whether some iterate of a positive integer is eventually in the period 2 cycle itself. Thus set theoretic conjugacy can be of great interest in such situations.

From this perspective it is of interest to find invariants of a function which completely classify its set theoretic conjugacy type. Algebraic topologists have used category theory as an effective tool for understanding and organizing algebraic invariants of the homotopy type, homeomorphism type, and so on, of topological spaces. In this paper we do this in a different direction, using category theory to organize the use of topological invariants for categories of discrete dynamical systems. We also study the basic category theoretic properties of the categories we discuss.

2. Definitions of the Categories

2.1. **Two Categories of Set Theoretical Dynamical Systems.** In this section we define two categories of dynamical systems and give some basic definitions used in the rest of the paper.

Definition 2.1. *A* set theoretic discrete dynamical system *is a pair* (X, f) *where* X *is a set and* $f : X \to X$*.*

The usual definition of a discrete dynamical system requires that the set X be supplied with a metric or topology. The study of discrete dynamical systems focuses on properties of a self-map of a topological space under composition, i.e. the properties of the iterates f^k. Some of those properties do not depend on the topology of the space X, but only on X as a set and the map f as a map between sets (such as the number of fixed points, n cycles, etc.). For this reason we refer to these self maps as set theoretic discrete dynamical systems. Thus every discrete dynamical system can also be considered to be a set-theoretical one by simply ignoring the topology on its set X. In the remainder of this paper when we refer to a dynamical system we will mean a set theoretic dynamical system unless we explicitly state otherwise.

To distinguish such pairs from other ordered pairs and for notational convenience, we will sometimes write $\mathrm{Dyn}(X, f)$ for the dynamical system (X, f). When referring to the properties of a dynamical system we will often identify $\mathrm{Dyn}(X, f)$ with the map f referring to either one interchangeably. For example, saying that f has no fixed points is equivalent to saying that $\mathrm{Dyn}(X, f)$ has no fixed points.

As mentioned above, conjugacies preserve all of those properties that are of interest from a dynamical systems viewpoint. Dropping the requirement of bijectivity of conjugacies leads to the obvious

Definition 2.2. *Let X, Y be sets and let $f : X \to X$ and $g : Y \to Y$ be any functions. The maps f, g are said to be* semi-conjugate *if there exists a map $h : X \to Y$ such that*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h\downarrow & & \downarrow h \\ Y & \xrightarrow[g]{} & Y \end{array}$$

commutes.

Using these for our morphisms we are now ready to define the category of dynamical systems.

Definition 2.3. *Define a category* SetDyn *whose objects are dynamical systems and whose morphisms are semiconjugacies, with composition of morphisms defined to be ordinary composition of semiconjugacies considered as maps between sets.*

It is easy to verify that SetDyn is a category. Let $\mathrm{Dyn}(X, f)$, $\mathrm{Dyn}(Y, g)$, and $\mathrm{Dyn}(Z, h)$ be objects of SetDyn and let $i \in \mathrm{Hom}_{\mathrm{SetDyn}}(\mathrm{Dyn}(X, f), \mathrm{Dyn}(Y, g))$ and $j \in \mathrm{Hom}_{\mathrm{SetDyn}}(\mathrm{Dyn}(Y, g), \mathrm{Dyn}(Z, h))$. Then $g \circ i = i \circ f$ and $h \circ j = j \circ g$ so $h \circ (j \circ i) = (j \circ i) \circ f$. Thus $j \circ i \in \mathrm{Hom}_{\mathrm{SetDyn}}(\mathrm{Dyn}(X, f), \mathrm{Dyn}(Z, h))$. The identity map id_X on X is the required identity morphism in this category.

There are certain kinds of dynamical systems which will be of interest.

Definition 2.4. $\mathrm{Dyn}(X, f)$ *is* acyclic *if f has no cycles other than fixed points, i.e. if for all $x \in X$, $f^k(x) = x$ implies $k \leq 1$.*

This allows us to define a subcategory of SetDyn. If C is a category and B a subcategory of C, then B is a *full* subcategory of C if for every pair of objects X, Y of B, $\mathrm{Hom}_B(X,Y) = \mathrm{Hom}_C(X,Y)$.

Definition 2.5. *Let* ADyn *be the full subcategory of* SetDyn *whose objects are acyclic dynamical systems.*

2.2. **Two Categories of Topological Spaces.** In this section we define a category of topological spaces which will later be shown to be isomorphic to ADyn.

A key concept in the study of any discrete dynamical system is that of the *orbit* of a point.

Definition 2.6. *Let $f : X \to X$. The f–*orbit *of $x \in X$ is the set*

$$\mathcal{O}_f(x) = \left\{ f^k(x) : k \in \mathbb{N} \right\}$$

This is also called the *f-trajectory* of x. We will sometimes refer to the f–orbit of x as simply the orbit of x and denote it $\mathcal{O}(x)$ when the function is clear from context. Note that $\mathcal{O}(x)$ is called the forward orbit and denoted $\mathcal{O}^+(x)$ in [Dev].

Definition 2.7. *Let X be a set and $f : X \to X$ a function. Define*

$$\tau_f = \{ A \subseteq X : f(A) \subseteq A \}$$

τ_f will be called the topology induced by f. The open sets consist of all those subsets of X which are invariant under f in the sense that f maps any element in the set to another element in the same set. To see that it is deserving of being called a topology we have

Theorem 2.8. *τ_f is a topology.*

Proof: $f(X) \subseteq X$ and $f(\phi) \subseteq \phi$. Given any collection $\{A_\alpha\}_{\alpha \in I}$ of subsets of X, if $f(A_\alpha) \subseteq A_\alpha$ for all $\alpha \in I$, then $f\left(\bigcup_{\alpha\in I} A_\alpha\right) = \bigcup_{\alpha\in I} f(A_\alpha) \subseteq \bigcup_{\alpha\in I} A_\alpha$ and $f\left(\bigcap_{\alpha\in I} A_\alpha\right) \subseteq \bigcap_{\alpha\in I} f(A_\alpha) = \bigcap_{\alpha\in I} A_\alpha$. $\square$

Notice that we could also define the compliments of f-invariant subsets to be the open sets of a topology on X, which we will denote $\overline{\tau}_f$. Also notice that an arbitrary intersection of open sets is open in τ_f.

Just as with dynamical systems, we will sometimes find it useful to distinguish ordered pairs by writing $\mathrm{Top}(X,\tau)$ for the topological space (X,τ).

Definition 2.9. *Let (X,σ) be a topological space. We say that the topology σ is a* map induced topology *(or simply* induced*) if there is a map $f : X \to X$ such that $\sigma = \tau_f$. If the map f is acyclic we say in addition that σ is* acyclically induced topology *(or simply* acyclic*).*

Similarly we can say that a topological space is induced if its topology is induced and acyclic if its topology is acyclic. We can now form the categories of interest.

Definition 2.10. *Let* IndTop *be the category whose objects consist of induced topological spaces and whose morphisms are continuous maps, with composition defined as to be ordinary composition of continuous maps. Further let* ATop *be the full subcategory of* IndTop *whose objects are acyclic topological spaces.*

Clearly these define categories whose identity morphism is the usual identity map between sets.

3. Properties of the Categories

3.1. Properties of SetDyn and ADyn.

3.1.1. *Elementary Properties.* We begin by stating some well known elementary properties that are used quite often in the proofs that follow.

Lemma 3.1. *If* $h : \mathrm{Dyn}(X, f) \to \mathrm{Dyn}(Y, g)$ *is a semiconjugacy, then for any* $k \geq 0$, $h : \mathrm{Dyn}\left(X, f^k\right) \to \mathrm{Dyn}\left(Y, g^k\right)$ *is also a semiconjugacy.*

Proof: Let $h : \mathrm{Dyn}(X, f) \to \mathrm{Dyn}(Y, g)$ be a semiconjugacy. Then $h \circ f = g \circ h$. Let $k = 0$. Then $h \circ f^0 = h \circ id_X = h = id_X \circ h = g^0 \circ h$. Let $k \geq 0$. Assume $h \circ f^k = g^k \circ h$. Then $h \circ f^{k+1} = h \circ f^k \circ f = g^k \circ h \circ f = g^k \circ g \circ h = g^{k+1} \circ h$. So by induction on k, for any $k \geq 0$, $h \circ f^k = g^k \circ h$. Hence for any $k \geq 0$, $h : \mathrm{Dyn}\left(X, f^k\right) \to \mathrm{Dyn}\left(Y, g^k\right)$ is also a semiconjugacy. □

In the language of our categories, this says that if $h \in \mathrm{Hom}_{\mathrm{SetDyn}}(\mathrm{Dyn}(X, f), \mathrm{Dyn}(Y, g))$ then $h \in \mathrm{Hom}_{\mathrm{SetDyn}}\left(\mathrm{Dyn}\left(X, f^k\right), \mathrm{Dyn}\left(Y, g^k\right)\right)$ for any $k \geq 0$. What about ADyn?

Remark 3.2. *The composition of two acyclic maps is not always acyclic.*

Example 3.3. *Let* $X = \{a, b, c, d\}$, $f, g : X \to X$ *by*

x	$f(x)$	$g(x)$
a	b	a
b	b	c
c	d	c
d	d	a

Both f *and* g *are acyclic by inspection. But* $(g \circ f)(a) = c$ *and* $(g \circ f)^2(a) = a$. *So* $g \circ f$ *has a cycle of minimum period* 2, *and is therefore not acyclic.*

However the iterates of an acyclic map are acyclic.

Theorem 3.4. *If* $f : X \to X$ *is acyclic and* $k \geq 0$ *then* f^k *is acyclic.*

Proof: Let $f : X \to X$ be acyclic and $k \geq 0$.

If $k = 0$ then $f^k = id_X$ which is acyclic because every point is a fixed point.

If $k = 1$ then $f^k = f$, which is acyclic by assumption.

Let $k > 1$ and let $x \in X$. Assume x is cyclic of order j for f^k. Then $x = \left(f^k\right)^j(x) = f^{jk}(x)$. Thus x is cyclic for f. So x is a fixed point of f because f is acyclic. So $f^k(x) = x$ and x is a fixed point of f^k as well. Thus f^k is acyclic.

So for all $k \geq 0$, f^k is acyclic. □

To what extent does a semiconjugacy preserve set theoretic dynamics?

Theorem 3.5. *If $f : X \to X$ is acyclic, $g : Y \to Y$ any map, and $h : X \to Y$ a semiconjugacy between f and g. If x is cyclic of order k for f then $h(x)$ is cyclic for g and the order of $h(x)$ divides k.*

Proof: Let $f : X \to X$ be acyclic, $g : Y \to Y$ any map, and $h : X \to Y$ a semiconjugacy between them. Let $x \in X$ be cyclic of order k. Then $g^k(h(x)) = g^k \circ h(x) = h \circ f^k(x) = h\left(f^k(x)\right) = h(x)$. So $h(x)$ is cyclic for g. Let j be the order of $h(x)$. Since $g^k(h(x)) = h(x)$ we have $j \leq k$. Let $k = jq + r$ where $0 \leq r < j$ by the division algorithm. Then $h(x) = g^k(h(x)) = g^{jq+r}(h(x)) = g^r(g^j)^q(h(x)) = g^r(h(x))$. Thus $r = 0$ by definition of order. So $jq = k$ and so j divides k. □

In particular, semiconjugacies map fixed points to fixed points. Note that it is not the case that semiconjugacies map noncyclic points to noncyclic points as can be easily seen by the example $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = x + 1$, $g : \mathbb{Z}_2 \to \mathbb{Z}_2$ by $g(x) = x + 1 \bmod 2$ and $h : \mathbb{Z} \to \mathbb{Z}_2$ by $h(x) = x \bmod 2$. This example also shows that it is not the case that semiconjugacies map acyclic dynamical systems to acyclic dynamical systems since f is acyclic but g has a cycle of order 2.

Semiconjugacies do preserve orbits, however, in the following sense.

Theorem 3.6. *Let $f : X \to X$, $g : Y \to Y$ be any maps, and $h : X \to Y$ a semiconjugacy between f and g. For any $x \in X$, $h(\mathcal{O}_f(x)) = \mathcal{O}_g(h(x))$.*

Proof: Let $f : X \to X$, $g : Y \to Y$ be any maps, and $h : X \to Y$ a semiconjugacy between f and g. Let $x \in X$. Then by Lemma 3.1,

$$\begin{aligned} h(\mathcal{O}_f(x)) &= h\left(\left\{x, f(x), f^2(x), \ldots\right\}\right) \\ &= \left\{h(x), h(f(x)), h\left(f^2(x)\right), \ldots\right\} \\ &= \left\{h(x), g(h(x)), g^2(h(x)), \ldots\right\} \\ &= \mathcal{O}_g(h(x)) \end{aligned}$$

□

However, it is not surprising that conjugacies do preserve all of these properties.

Theorem 3.7. *Let $f : X \to X$, $g : Y \to Y$ be any maps and $h : X \to Y$ a conjugacy between f and g. Then h^{-1} is a conjugacy between g and f.*

Proof: Let $f : X \to X$, $g : Y \to Y$ be any maps and $h : X \to Y$ a conjugacy between f and g. Then h is a bijective semiconjugacy. So $h^{-1} : Y \to X$ exists and is a bijection. Let $y \in Y$. Then $y = h(x)$ for some $x \in X$. So $f \circ h^{-1}(y) = f \circ h^{-1}(h(x)) = f(x) = h^{-1} \circ h \circ f(x) = h^{-1} \circ g \circ h(x) = h^{-1} \circ g(y)$. Thus h^{-1} is a semiconjugacy. So h^{-1} is a conjugacy. □

Using this we can prove

Theorem 3.8. *Let $f : X \to X$, $g : Y \to Y$ be any maps, and $h : X \to Y$ a conjugacy between f and g. For any $x \in X$, and any $k \geq 0$, x is cyclic of order k for f if and only if $h(x)$ is cyclic of order k for g.*

Proof: Let $f : X \to X$, $g : Y \to Y$ be any maps and $h : X \to Y$ a conjugacy between f and g. Let $x \in X$ be cyclic of order k for f. By Theorem 3.5, $h(x)$ is cyclic for g with order d that divides k. By Theorem 3.7, h^{-1} is a conjugacy between g and f, so again by Theorem 3.5, $h^{-1}(h(x)) = x$ is cyclic of order k for f and k divides d. Since $k|d$ and $d|k$, we must have $k = d$ and so $h(x)$ is cyclic of order k. Reversing the roles of f and g and h and h^{-1} in the proof gives the other direction. □

Corollary 3.9. *If $f : X \to X$ and $g : Y \to Y$ are conjugate, then f is acyclic if and only if g is acyclic.*

3.1.2. *Categorical Properties.* In this section we describe the common categorical properties for the categories SetDyn and ADyn.

Monics, Epics, Injections, Surjections, Sections, and Retractions

Recall the following definitions from category theory (cf. [Bly]).

Definition 3.10. *Let C be a category, A, B objects of C and $f \in \mathrm{Hom}_C(A, B)$. The morphism f is:*
(a) monic *if for any object D and all morphisms $g, h \in \mathrm{Hom}_C(D, A)$, $f \circ g = f \circ h$ implies that $g = h$, i.e. if f is left cancelable with respect to composition.*
(b) epic *if for any object D and all morphisms $g, h \in \mathrm{Hom}_C(B, D)$, $g \circ f = h \circ f$ implies that $g = h$, i.e. if f is right cancelable with respect to composition.*
(c) a section *if there is a morphism $g \in \mathrm{Hom}_C(B, A)$ such that $g \circ f = id_A$.*
(d) a retraction *if there is a morphism $g \in \mathrm{Hom}_C(B, A)$ such that $f \circ g = id_B$.*

A category is *concrete* if its objects are sets with possibly some additional structure and its morphisms are structure preserving maps between the sets. In such a category we can define injectivity and surjectivity for the morphisms using the usual definitions of set theory. In any concrete category, every section is an injection, and every injection is monic, but the reverse implications do not hold in general. Similarly in any concrete category, every retraction is a surjection, and every surjection is epic, but the reverse implications do not hold in general (cf. [Bly]). Our categories SetDyn and ADyn are concrete categories, and so it would be of interest to know which of the reverse implications mentioned above hold.

Theorem 3.11. *Let h be a morphism of* SetDyn *or* ADyn.
(a) h is injective if and only if h is monic.
(b) h is surjective if and only if h is epic.

Proof: As mentioned above, the forward implications in (a) and (b) hold in any category . Thus we must only show the reverse implications.

Let $h \in \mathrm{Hom}_{\mathrm{SetDyn}}\left(\mathrm{Dyn}\left(X,f\right),\mathrm{Dyn}\left(Y,g\right)\right)$.

(a) Assume h is monic. Let $x,y \in X$ and assume $h\left(x\right) = h\left(y\right)$. For all $i \geq 0$ let $x_i = f^i\left(x\right)$, $y_i = f^i\left(y\right)$, and $Z = \mathcal{O}_f\left(x\right) = \{x_0, x_1, x_2, \ldots\}$. Let $s : Z \to X$ be the inclusion map $s\left(x_i\right) = x_i$ and let $t : Z \to X$ be the map $t\left(x_i\right) = y_i$. Then for any $i \geq 0$

$$s \circ f\left(x_i\right) = s\left(x_{i+1}\right) = x_{i+1} = f\left(x_i\right) = f \circ s\left(x_i\right)$$

and

$$t \circ f\left(x_i\right) = t\left(x_{i+1}\right) = y_{i+1} = f\left(y_i\right) = f \circ t\left(x_i\right)$$

Thus $s \circ f = f \circ s$ and $t \circ f = f \circ t$, so both s and t are semiconjugacies between $\mathrm{Dyn}\left(Z, f|Z\right)$ and $\mathrm{Dyn}\left(X,f\right)$, i.e. $s,t \in \mathrm{Hom}_{\mathrm{SetDyn}}\left(\mathrm{Dyn}\left(Z,f|Z\right),\mathrm{Dyn}\left(X,f\right)\right)$. Now

$$\begin{aligned} h \circ s\left(x_i\right) &= h\left(x_i\right) = h\left(f^i\left(x\right)\right) = g^i\left(h\left(x\right)\right) \\ &= g^i\left(h\left(y\right)\right) = h\left(f^i\left(y\right)\right) = h\left(y_i\right) = h \circ t\left(x_i\right) \end{aligned}$$

so that $h \circ s = h \circ t$. But h is monic, so $s = t$. Thus $x = s\left(x\right) = t\left(x\right) = y$. So h is injective.

If $h \in \mathrm{Hom}_{\mathrm{ADyn}}\left(\mathrm{Dyn}\left(X,f\right),\mathrm{Dyn}\left(Y,g\right)\right)$ instead, then since $\mathrm{Dyn}\left(X,f\right)$ is acyclic, so is $\mathrm{Dyn}\left(Z,f|Z\right)$ and so the same proof works for ADyn as well.

(b) Assume h is epic. Let

$$Z = h\left(X\right) \cup \left(Y - h\left(X\right)\right) \times \{0,1\}$$

(where without loss of generality the union is taken to be disjoint). Define $s : Y \to Z$ and $t : Y \to Z$ by

$$s\left(x\right) = \begin{cases} x & \text{if } x \in h\left(X\right) \\ \left(x,0\right) & \text{if } x \notin h\left(X\right) \end{cases}$$

$$t\left(x\right) = \begin{cases} x & \text{if } x \in h\left(X\right) \\ \left(x,1\right) & \text{if } x \notin h\left(X\right) \end{cases}$$

for any $x \in Y$. Define $j : Z \to Z$ by

$$j\left(x\right) = \begin{cases} g\left(x\right) & \text{if } x \in h\left(X\right) \\ g\left(x'\right) & \text{if } \left(x = \left(x',0\right) \text{ or } x = \left(x',1\right)\right) \text{ and } g\left(x'\right) \in h\left(X\right) \\ \left(g\left(x'\right),0\right) & \text{if } x = \left(x',0\right) \text{ and } g\left(x'\right) \notin h\left(X\right) \\ \left(g\left(x'\right),1\right) & \text{if } x = \left(x',1\right) \text{ and } g\left(x'\right) \notin h\left(X\right) \end{cases}$$

for any $x \in Y$. Then for any $y \in Y$,

$$\begin{aligned} j \circ s\left(y\right) &= \begin{cases} j\left(y\right) & \text{if } y \in h\left(X\right) \\ j\left(y,0\right) & \text{if } y \notin h\left(X\right) \end{cases} \\ &= \begin{cases} g\left(y\right) & \text{if } g\left(y\right) \in h\left(X\right) \\ \left(g\left(y\right),0\right) & \text{if } g\left(y\right) \notin h\left(X\right) \end{cases} \\ &= s \circ g\left(y\right) \end{aligned}$$

So $j \circ s = s \circ g$. Similarly,

$$j \circ t(y) = \begin{cases} j(y) & \text{if } y \in h(X) \\ j(y,1) & \text{if } y \notin h(X) \end{cases}$$
$$= \begin{cases} g(y) & \text{if } g(y) \in h(X) \\ (g(y),1) & \text{if } g(y) \notin h(X) \end{cases}$$
$$= t \circ g(y)$$

so $j \circ t = t \circ g$. Thus the maps s, t are semiconjugacies from $\mathrm{Dyn}(Y,g)$ to $\mathrm{Dyn}(Z,j)$, i.e. $s, t \in \mathrm{Hom}_{\mathrm{SetDyn}}(\mathrm{Dyn}(Y,g), \mathrm{Dyn}(Z,j))$.

Let $x \in X$. Then $s \circ h(x) = s(h(x)) = h(x) = t(h(x)) = t \circ h(x)$. So $s \circ h = t \circ h$. Since h is epic, $s = t$. Let $y \in Y$. Then $s(y) = t(y)$. Since $(y,0) \neq (y,1)$, we have $y \in h(X)$. So $Y = h(X)$ and h is onto.

If $h \in \mathrm{Hom}_{\mathrm{ADyn}}(\mathrm{Dyn}(X,f), \mathrm{Dyn}(Y,g))$ instead, then in the above proof, we need to verify that $\mathrm{Dyn}(Z,j)$ is acyclic. Let $x \in Z$. Assume $j^k(x) = x$. If $x \in h(X)$ then $x = j^k(x) = g^k(x)$ so $k \in \{0,1\}$ because g is acyclic. If $x \in (Y - h(X)) \times \{0,1\}$ then $x = (x_0, 0)$ or $x = (x_0, 1)$ for some $x_0 \in Y - h(X)$. Assume without loss of generality that $x = (x_0, 0)$. Then $(x_0, 0) = x = j^k(x) = j^k((x_0,0)) = (g^k(x_0), 0)$. So $g^k(x_0) = x_0$ and again $k \in \{0,1\}$. So in either case, $j^k(x) = x$ implies $k \in \{0,1\}$, so $\mathrm{Dyn}(Z,j)$ is acyclic and so the same proof works for ADyn as well. □

If h is a section then h is injective, but the converse does not hold in general.

Example 3.12. *Let $X = \mathbb{Z}, Y = \mathbb{Z} \cup \{a\}$, $f : X \to X$ by $f(x) = x + 1$, and $g : Y \to Y$ by*

$$g(x) = \begin{cases} x+1 & \text{if } x \in \mathbb{Z} \\ a & \text{if } x = a \end{cases}$$

and $h : X \to Y$ be the inclusion map $h(x) = x$. Then for any $x \in \mathbb{Z}$, $g \circ h(x) = x + 1 = h \circ f(x)$, so h is a semiconjugacy between $\mathrm{Dyn}(X,f)$ *and* $\mathrm{Dyn}(Y,g)$ *which are both acyclic, and thus h is an element of* $\mathrm{Hom}_{\mathrm{ADyn}}(\mathrm{Dyn}(X,f), \mathrm{Dyn}(Y,g))$ *(and* $\mathrm{Hom}_{\mathrm{SetDyn}}(\mathrm{Dyn}(X,f), \mathrm{Dyn}(Y,g))$ *as well). Since h is clearly injective, it is an injective in both* ADyn *and* SetDyn*.*

Suppose $j \in \mathrm{Hom}_{\mathrm{ADyn}}(\mathrm{Dyn}(Y,g), \mathrm{Dyn}(X,f)) (= \mathrm{Hom}_{\mathrm{SetDyn}}(\mathrm{Dyn}(Y,g), \mathrm{Dyn}(X,f)))$. Then j is a semiconjugacy from $\mathrm{Dyn}(Y,g)$ *to* $\mathrm{Dyn}(X,f)$*. By Theorem 3.5 j maps fixed points to fixed points. Since a is a fixed point of g, $j(a)$ is a fixed point of f. But f has no fixed points. Therefore there are no semiconjugacies from* $\mathrm{Dyn}(Y,g)$ *to* $\mathrm{Dyn}(X,f)$*. Thus h is not a section in either* ADyn *or* SetDyn*.*

Similarly if h is a retraction then h is surjective, but the converse does not hold in general.

Example 3.13. *Let $X = \mathbb{Z}, Y = \{a\}$, $f : X \to X$ by $f(x) = x+1$, $g : Y \to Y$ be the identity map id_Y, and $h : X \to Y$ be the constant map $h(x) = a$. Then for any $x \in \mathbb{Z}$,*

$g \circ h(x) = a = h \circ f(x)$, so h is a semiconjugacy between $\mathrm{Dyn}(X, f)$ *and* $\mathrm{Dyn}(Y, g)$ *which are both acyclic, and thus h is an element of* $\mathrm{Hom}_{\mathrm{ADyn}}(\mathrm{Dyn}(X, f), \mathrm{Dyn}(Y, g))$ *(and* $\mathrm{Hom}_{\mathrm{SetDyn}}(\mathrm{Dyn}(X, f), \mathrm{Dyn}(Y, g))$ *as well). Since h is clearly surjective, it is a surjective in both* ADyn *and* SetDyn*.*

Suppose $j \in \mathrm{Hom}_{\mathrm{ADyn}}(\mathrm{Dyn}(Y, g), \mathrm{Dyn}(X, f)) (= \mathrm{Hom}_{\mathrm{SetDyn}}(\mathrm{Dyn}(Y, g), \mathrm{Dyn}(X, f)))$. Then j is a semiconjugacy from $\mathrm{Dyn}(Y, g)$ *to* $\mathrm{Dyn}(X, f)$*. By Theorem 3.5 j maps fixed points to fixed points. Since a is a fixed point of g, $j(a)$ is a fixed point of f. But f has no fixed points. Therefore there are no semiconjugacies from* $\mathrm{Dyn}(Y, g)$ *to* $\mathrm{Dyn}(X, f)$*. Thus h is not a retraction in either* ADyn *or* SetDyn*.*

Recall the following definitions from category theory (cf. [Bly]).

Definition 3.14. *Let C be a category, A, B objects of C and $f \in \mathrm{Hom}_C(A, B)$.*
(a) f is a bimorphism *if it is both monic and epic.*
(b) f is an isomorphism *(also called an* equivalence*) if it is both a section and a retraction.*
(c) C is a balanced category *if every bimorphism of C is an isomorphism.*

In any concrete category, every isomorphism is a bijection, and every bijection is a bimorphism. In our categories we have

Theorem 3.15. SetDyn *and* ADyn *are balanced categories.*

Proof: Let $h \in \mathrm{Hom}_{\mathrm{SetDyn}}(\mathrm{Dyn}(X, f), \mathrm{Dyn}(Y, g))$ be a bimorphism. By the previous theorem every epic morphism is injective and every monic is surjective, therefore h is a bijection. Since h is a semiconjugacy, $h \circ f = g \circ h$. Thus

$$f \circ h^{-1} = h^{-1} \circ h \circ f \circ h^{-1} = h^{-1} \circ g \circ h \circ h^{-1} = h^{-1} \circ g$$

Thus h^{-1} is a semiconjugacy as well. So h is an isomorphism. The proof also holds if $h \in \mathrm{Hom}_{\mathrm{ADyn}}(\mathrm{Dyn}(X, f), \mathrm{Dyn}(Y, g))$. $\square$

3.2. **Properties of** IndTop **and** ATop**.** Clearly for any map $f : X \to X$ and any $x \in X$, $\mathcal{O}_f(x)$ is an open set in (X, τ_f). In fact we can say something stronger, namely

$$\begin{aligned} f(\mathcal{O}_f(x)) &= f\left(\left\{x, f(x), f^2(x), \ldots\right\}\right) \\ &= \left\{f(x), f^2(x), f^3(x), \ldots\right\} \\ &= \mathcal{O}_f(f(x)) \end{aligned}$$

We also have the following useful fact.

Theorem 3.16. *Let $f : X \to X$ and $\mathcal{U} \in \tau_f$. Then $\mathcal{U} = \bigcup_{x \in \mathcal{U}} \mathcal{O}_f(x)$. In other words, the set of orbits forms a basis for the topology τ_f.*

Proof. Let $x \in \mathcal{U}$. Then $f^0(x) \in \mathcal{U}$. Assume $f^k(x) \in \mathcal{U}$. Then $f^{k+1}(x) = f\left(f^k(x)\right) \in f(\mathcal{U}) \subseteq \mathcal{U}$. So by induction on k, $\mathcal{O}_f(x) \subseteq \mathcal{U}$. Since $x \in \mathcal{O}_f(x)$, we see that $\mathcal{U} = \bigcup_{x \in \mathcal{U}} \mathcal{O}_f(x)$, so that the set of orbits forms a basis for τ_f. $\square$

Thus any open set in an induced topology contains the complete orbit of each of its points. Suppose we are given an induced topology, but are not given the function f that induces it. The next result allows us to determine which open sets are orbits (for some map that induces the topology).

Corollary 3.17. *Let $f : X \to X$ and $x \in A$. Then*

$$\mathcal{O}_f(x) = \bigcap_{\substack{x \in \mathcal{U} \\ \mathcal{U} \in \tau_f}} \mathcal{U}$$

Proof: Since $\mathcal{O}_f(x)$ is open, $\bigcap_{\substack{x \in \mathcal{U} \\ \mathcal{U} \in \tau_f}} \mathcal{U} \subseteq \mathcal{O}_f(x)$. Let $\mathcal{U} \in \tau_f$ be an open set such that $x \in \mathcal{U}$. Then $\mathcal{O}_f(x) \subseteq \mathcal{U}$ by Theorem 3.16. Thus $\mathcal{O}_f(x) \subseteq \bigcap_{\substack{x \in \mathcal{U} \\ \mathcal{U} \in \tau_f}} \mathcal{U}$. So $\mathcal{O}_f(x) = \bigcap_{\substack{x \in \mathcal{U} \\ \mathcal{U} \in \tau_f}} \mathcal{U}$. $\square$

So the orbit of a point in an induced topology is just the smallest open set which contains the point.

In general, the map that induces a topology is not unique as can be seen by the following example.

Example 3.18. *Let $X = \{a, b, c\}$, $f = (a, b, c)$, and $g = (a, c, d)$ (in permutation notation). Then $\tau_f = \tau_g = \{\emptyset, X\}$ is the trivial topology on X, but $f \neq g$.*

However, a key fact about the acyclic topologies is that the map that induces them is indeed unique.

Theorem 3.19. *Let $f : X \to X$ be acyclic and $g : X \to X$. If $\tau_f = \tau_g$ then $f = g$.*

Proof: Let $f : X \to X$ be acyclic, $g : X \to X$, and assume $\tau_f = \tau_g$. Then by Corollary 3.17, for any $y \in X$,

$$\mathcal{O}_f(y) = \bigcap_{\substack{y \in \mathcal{U} \\ \mathcal{U} \in \tau_f}} \mathcal{U} = \bigcap_{\substack{y \in \mathcal{U} \\ \mathcal{U} \in \tau_g}} \mathcal{U} = \mathcal{O}_g(y) \tag{1}$$

If y is a fixed point of f then $\{y, g(y), g^2(y), \ldots\} = \mathcal{O}_g(y) = \mathcal{O}_f(y) = \{y\}$. So $g(y) = y$ and thus y is a fixed point of g also.

If y is not a fixed point of f then it is not a cyclic point for f. Thus $y \neq f^j(y)$ for any $j \geq 1$ so that $\mathcal{O}_f(f(y)) = \{f(y), f^2(y), f^3(y), \ldots\} = \mathcal{O}_f(y) - \{y\}$. This proves the useful fact that for any function $h : X \to X$ and any $z \in X$ which is non-cyclic for h,

$$\mathcal{O}_h(h(z)) = \mathcal{O}_h(z) - \{z\} \tag{2}$$

Suppose y is a cyclic point of g. Then $y = g^j(y)$ for some $j \geq 1$ and so $\mathcal{O}_g(y) = \{y, g(y), g^2(y), \ldots, g^{j-1}(y)\}$. Also $f(y) \in \mathcal{O}_f(y) = \mathcal{O}_g(y)$, so that $f(y) = g^k(y)$

for some $1 \leq k \leq j-1$. Thus

$$\begin{aligned}\mathcal{O}_f(y) - \{y\} &= \mathcal{O}_f(f(y)) \\ &= \mathcal{O}_g(f(y)) \\ &= \mathcal{O}_g\left(g^k(y)\right) \\ &= \left\{g^k(y), g^{k+1}(y), \ldots, g^{j-1}(y), y, g(y), \ldots, g^{k-1}(y)\right\} \\ &= \mathcal{O}_g(y) \\ &= \mathcal{O}_f(y)\end{aligned}$$

Which is a contradiction. So y is not a cyclic point of g.

So fixed points of f are fixed points of g and non-cyclic points of f are non-cyclic points of g as well. Thus g is also an acyclic map.

Now let $x \in X$. Since f is acyclic, either x is a fixed point of f or x is not a cyclic point of x.

If x is a fixed point of f then as shown above, x is a fixed point of g as well. So $f(x) = x = g(x)$.

If x is not a fixed point of f, then as shown above, x is non-cyclic for both f and g. Since $f(x) \in \mathcal{O}_f(x) = \mathcal{O}_g(x)$, we know that $f(x) = g^k(x)$ for some $k \geq 1$. But then

$$\begin{aligned}\mathcal{O}_g(g(x)) &= \mathcal{O}_g(x) - \{x\} \\ &= \mathcal{O}_f(x) - \{x\} \\ &= \mathcal{O}_f(f(x)) \\ &= \mathcal{O}_g(f(x)) \\ &= \mathcal{O}_g\left(g^k(x)\right)\end{aligned}$$

and so $g(x) \in \mathcal{O}_g\left(g^k(x)\right)$. Thus $g(x) = g^{k+i}(x)$ for some $k \geq 1$ and some $i \geq 0$. Since g is acyclic, $g(x)$ is either a fixed point or is non-cyclic for g. If $g(x)$ is non-cyclic then $k+i = 1$ and so $k = 1$ and $i = 0$ and consequently $g^k(x) = g(x)$. If $g(x)$ is a fixed point then $g^k(x) = g(x)$ for any $k \geq 1$. So in either case $g^k(x) = g(x)$ and so $f(x) = g(x)$.

Thus for every $x \in X$, $f(x) = g(x)$, and so $f = g$. $\square$

Suppose we are given an object $\mathrm{Top}(X, \tau)$ in ATop, but not the acyclic function f that induces it. How can we recover f from τ? To do this we first consider the following

Lemma 3.20. *Let $f : X \to X$ be acyclic and $x \in X$. Then for any $y \in \mathcal{O}_f(x)$, $\mathcal{O}_f(x) - \{y\} \in \tau_f$ if and only if $y = x$.*

Proof: Let $f : X \to X$ be acyclic and $x \in X$. Let $y \in \mathcal{O}_f(x)$. Then $y = f^k(x)$ for some $k \geq 0$.

($\Rightarrow$) Assume $\mathcal{O}_f(x) - \{y\} \in \tau_f$.

If x is a fixed point, then $y = f^k(x) = x$.

If x is not a fixed point and $k > 0$ then $f^k(x) \neq x$. Therefore the least integer j such that $f^k(x) = f^j(x)$ is greater than zero. Thus $y = f^k(x) = f^j(x) \neq f^{j-1}(x)$. Thus $f^{j-1}(x) \in \mathcal{O}_f(x) - \{f^j(x)\}$ and $f^j(x) \in f(\mathcal{O}_f(x) - \{f^j(x)\})$. But

$$\begin{aligned} f^j(x) &\in f\left(\mathcal{O}_f(x) - \left\{f^j(x)\right\}\right) \\ &= f(\mathcal{O}_f(x) - \{y\}) \\ &\subseteq \mathcal{O}_f(x) - \{y\} \\ &= \mathcal{O}_f(x) - \left\{f^j(x)\right\} \end{aligned}$$

so that $f^j(x) \in \mathcal{O}_f(x) - \{f^j(x)\}$ which is a contradiction. So $k = 0$.

But if x is not a fixed point and $k = 0$ then $y = f^k(x) = f^0(x) = x$. So in every case, $y = x$.

($\Leftarrow$) Assume $y = x$. If x is a fixed point of f then $\mathcal{O}_f(x) - \{y\} = \{x\} - \{x\} = \emptyset \in \tau_f$. If x is noncyclic for f then $\mathcal{O}_f(x) - \{y\} = \mathcal{O}_f(x) - \{x\} = \mathcal{O}_f(f(x)) \in \tau_f$. So in either case $\mathcal{O}_f(x) - \{y\} \in \tau_f$. □

Thus if a topology is induced by an acyclic map, every orbit contains a unique point, which when removed leaves an open set behind. We can use this to reconstruct the map f, given the topology τ_f.

Corollary 3.21. *Let* Top(X, τ) *be an object of* ATop. *Let $x \in X$. Define $f(x)$ as follows. If $\{x\} \in \tau$ let $f(x) = x$. If $\{x\} \notin \tau$, let $\mathcal{A} = \bigcap_{\substack{x \in \mathcal{U} \\ \mathcal{U} \in \tau_f}} \mathcal{U}$, let y be the unique point in $\mathcal{A} - \{x\}$ such that $(\mathcal{A} - \{x\}) - \{y\} \in \tau$, and define $f(x) = y$. Then $\tau = \tau_f$.*

Proof: By definition of ATop there is an acyclic map f inducing τ. By Theorem 3.19, there is only one such map. By corollary 3.17, $\mathcal{A} = \mathcal{O}_f(x)$. If $\{x\} \in \tau$ then $f(\{x\}) \subseteq \{x\}$ so $f(x) = x$. If $\{x\} \notin \tau$ then $\mathcal{A} - \{x\} = \mathcal{O}_f(x) - \{x\} = \mathcal{O}_f(f(x))$ and by Lemma 3.20, $y = f(x)$ if and only if $\mathcal{O}_f(f(x)) - \{y\} \in \tau$. So there is a unique y such that $\mathcal{O}_f(f(x)) - \{y\} \in \tau$, and thus $y = f(x)$. □

We end this section by noting that the induced topologies are almost never Hausdorff.

Theorem 3.22. *An induced topological space* Top(X, τ_f) *is Hausdorff if and only if $f = id_X$.*

Proof: ($\Rightarrow$) Let Top(X, τ_f) be an induced topological space. Suppose $f \neq id_X$. Then there exists $x \in X$ such that $f(x) \neq x$. Let $\mathcal{U}$ be any open set containing x. By Theorem 3.16, $\mathcal{O}_f(x) \subseteq \mathcal{U}$. So $f(x) \in \mathcal{U}$. Thus no open set contains x but not $f(x)$, so Top(X, τ_f) is not Hausdorff.

($\Leftarrow$) If $f = id_X$ then τ_f is the discrete topology, which is Hausdorff. □

4. Relationships between the Categories

In this section we discuss the relationship between the categories developed above. We begin with an important fact.

Theorem 4.1. *Let X, Y be sets, $f : X \to X$ and $g : Y \to Y$ any functions. If $h : X \to Y$ is a semiconjugacy then it is continuous with respect to the induced topologies τ_f and τ_g.*

Proof: Let X, Y be sets and $f : X \to X$ and $g : Y \to Y$ any functions.

Assume $h : X \to Y$ is a semiconjugacy between f and g. Let $\mathcal{U} \in \tau_g$. Let $x \in h^{-1}(\mathcal{U})$. Then $h(\mathcal{O}_f(x)) = \mathcal{O}_g(h(x))$ by Theorem 3.6. But $h(x) \in \mathcal{U}$, so $\mathcal{O}_g(h(x)) \subseteq \mathcal{U}$ by Theorem 3.16. Thus $h(\mathcal{O}_f(x)) \subseteq \mathcal{U}$. So $\mathcal{O}_f(x) \subseteq h^{-1}(h(\mathcal{O}_f(x))) \subseteq h^{-1}(\mathcal{U})$. Thus $h^{-1}(\mathcal{U}) = \bigcup_{x \in h^{-1}(\mathcal{U})} \mathcal{O}_f(x)$. Since each orbit is open by Theorem 3.16, $h^{-1}(\mathcal{U})$ is a union of open sets, and therefore open. Thus for any $\mathcal{U} \in \tau_g$, $h^{-1}(\mathcal{U}) \in \tau_f$, and so h is continuous. □

Using this fact we can prove that conjugacies are isomorphisms in IndTop and ATop.

Theorem 4.2. *Let X, Y be sets, $f : X \to X$, and $g : Y \to Y$ any functions. If f is conjugate to g then* $\operatorname{Top}(X, \tau_f)$ *is homeomorphic to* $\operatorname{Top}(Y, \tau_g)$. *In fact, if $h : X \to Y$ is a conjugacy then it is a homeomorphism.*

Proof: Let X, Y be sets, $f : X \to X$, and $g : Y \to Y$ any function . Suppose $h : X \to Y$ is a conjugacy. Then both h and h^{-1} are conjugacies by Theorem 3.7, so they are both continuous by Theorem 4.1. Thus h is a homeomorphism. □

It is not the case in general that continuous maps are semiconjugacies, even if f and g are acyclic as can be seen by the following example.

Example 4.3. *Let $X = \{a, b\}$, $Y = \{c, d\}$, $f = id_X$, $g : Y \to Y$ be the constant map $g(x) = d$, and $h : X \to Y$ the map $h(a) = c$ and $h(b) = d$. Then $\tau_f = \{\emptyset, \{a\}, \{b\}, X\}$ is the discrete topology on X, so h is continuous, but $h \circ f(a) = c$ and $g \circ h(a) = d$, so h is not a semiconjugacy, even though both f and g are acyclic.*

It is also not the case in general that homeomorphisms are conjugacies as can be seen by the following example.

Example 4.4. *Let $X = Y = \{a, b, c\}$, $f = (a, b, c)$, $g = (a, c, b)$ (in permutation notation) $h : X \to X$ the identity map id_X. Then $\tau_f = \tau_g = \{\emptyset, X\}$ is the trivial topology on X, so h is continuous, a bijection, and h^{-1} is also continuous. So h is homeomorphism, but $h \circ f(a) = b$ and $g \circ h(a) = c$, so h is not a conjugacy.*

But if one of our maps is acyclic, then we have the following nice result.

Theorem 4.5. *Let X, Y be sets, $f : X \to X$ an acyclic function, and $g : Y \to Y$ any function. Then f is conjugate to g if and only if (X, τ_f) is homeomorphic to (Y, τ_g). In fact, $h : X \to Y$ is a conjugacy if and only if it is a homeomorphism.*

Proof. ($\Rightarrow$) By Theorem 4.2.

($\Leftarrow$) Suppose h is a homeomorphism. Then h is bijective so it suffices to show that $g \circ h = h \circ f$. Let $x \in X$.

If x is a fixed point then $\mathcal{O}_f(x) = \{x\}$ is open and thus so is $h(\mathcal{O}_f(x)) = \{h(x)\}$. So $g(\{h(x)\}) \subseteq \{h(x)\}$ and so $g(h(x)) = h(x) = h(f(x))$.

Now suppose x is not a fixed point. Then $\{x\}$ is not open, so $\{h(x)\}$ is not open either and thus $h(x)$ is also not a fixed point.

Now $\mathcal{O}_f(x)$ is an open set, and h is a homeomorphism, so $h(\mathcal{O}_f(x))$ is an open set which contains $h(x)$ and thus by Theorem 3.16 $\mathcal{O}_g(h(x)) \subseteq h(\mathcal{O}_f(x))$. In particular, $g(h(x)) \in \mathcal{O}_g(h(x))$ and so $g(h(x)) \in h(\mathcal{O}_f(x))$. Thus $g(h(x)) = h\left(f^k(x)\right)$ for some k. Since $h(x)$ is not a fixed point, $k \geq 1$.

Similarly, $\mathcal{O}_g(h(x))$ is an open set, so $h^{-1}(\mathcal{O}_g(h(x)))$ is an open set which contains x and thus by Theorem 3.16 $\mathcal{O}_f(x) \subseteq h^{-1}(\mathcal{O}_g(h(x)))$. In particular, $f(x) \in h^{-1}(\mathcal{O}_g(h(x)))$ and so $h(f(x)) \in \mathcal{O}_g(h(x))$. Thus $h(f(x)) = g^m(h(x))$ for some m. Since x is not a fixed point, $m \geq 1$.

But $h\left(\mathcal{O}_f\left(f^k(x)\right)\right)$ is open, so $\mathcal{O}_g(g(h(x))) \subseteq h\left(\mathcal{O}_f\left(f^k(x)\right)\right)$. In particular, $g^m(h(x)) \in \mathcal{O}_g(g(h(x)))$ so $g^m(h(x)) \in h\left(\mathcal{O}_f\left(f^k(x)\right)\right)$. Thus $h(f(x)) \in h\left(\mathcal{O}_f\left(f^k(x)\right)\right)$ and so $h(f(x)) = h(f^j(x))$ for some $j \geq k$. But h is bijective, so $f(x) = f^j(x)$ for some $j \geq k$.

But f is acyclic, so either $f(x)$ is a fixed point or it is noncyclic. If $f(x)$ is a fixed point then $g(h(x)) = h\left(f^k(x)\right) = h(f(x))$. If $f(x)$ is noncyclic point then $j = 1$, so $k = 1$ and $g(h(x)) = h(f(x))$ once again.

Thus whether or not x is a fixed point we have shown that $g(h(x)) = h(f(x))$. So $g \circ h = h \circ f$ as desired. □

Thus the induced topology completely encodes acyclic set theoretic dynamics: two dynamical systems are conjugate if and only if their induced topological spaces are homeomorphic.

We can summarize some of these results in the language of category theory as follows.

Definition 4.6. *Let* $\mathcal{K}$ *be the functor from* SetDyn *to* IndTop *such that* $\mathcal{K}(\mathrm{Dyn}(X,f)) = \mathrm{Top}(X,\tau_f)$ *and for* $h \in \mathrm{Hom}_{\mathrm{SetDyn}}(\mathrm{Dyn}(X,f), \mathrm{Dyn}(Y,g))$,

$$\mathcal{K}(h) = h \in \mathrm{Hom}_{\mathrm{IndTop}}(\mathrm{Top}(X,\tau_f), \mathrm{Top}(Y,\tau_g))$$

The fact that $h \in \mathrm{Hom}_{\mathrm{SetDyn}}(\mathrm{Dyn}(X,f), \mathrm{Dyn}(Y,g))$ implies that

$$h \in \mathrm{Hom}_{\mathrm{IndTop}}(\mathrm{Top}(X,\tau_f), \mathrm{Top}(Y,\tau_g))$$

is an immediate consequence of Theorem 4.1. To verify that $\mathcal{K}$ is a functor we merely note that $\mathcal{K}(id_X) = id_X$ and this is the identity morphism in both categories, and for any morphisms α, β of SetDyn for which $\alpha \circ \beta$ is defined, $\mathcal{K}(\alpha \circ \beta) = \alpha \circ \beta = \mathcal{K}(\alpha) \circ \mathcal{K}(\beta)$.

By definition of ATop and the fact that the identity map is acyclic, the restriction of $\mathcal{K}$ to the full subcategory ADyn gives us a functor from ADyn to ATop.

Definition 4.7. *Let κ be the functor from* ADyn *to* ATop *such that* $\kappa(\mathrm{Dyn}(X,f)) = \mathrm{Top}(X,\tau_f)$ *and for* $h \in \mathrm{Hom}_{\mathrm{ADyn}}(\mathrm{Dyn}(X,f), \mathrm{Dyn}(Y,g))$,

$$\kappa(h) = h \in \mathrm{Hom}_{\mathrm{ATop}}(\mathrm{Top}(X,\tau_f), \mathrm{Top}(Y,\tau_g)).$$

Recall that a functor, F, from category A to B is *faithful*, if for all objects X, Y of A, the map $F : \mathrm{Hom}_A(X,Y) \to \mathrm{Hom}_B(FX,FY)$ is injective. It is *representative*, if for every object Y of B there is an object X in A such that Y is isomorphic to FX. It is *full*, if for all objects X, Y of A, the map $F : \mathrm{Hom}_A(X,Y) \to \mathrm{Hom}_B(FX,FY)$ is surjective. Example 4.3 shows that the functors $\mathcal{K}$ and κ are not full. But

Theorem 4.8. *Both $\mathcal{K}$ and κ are faithful, representative functors.*

Proof: By definition of IndTop, if $\mathrm{Top}(X,\tau)$ is an object of IndTop then there is $f : X \to X$ such that $\tau = \tau_f$, so $\mathrm{Top}(X,\tau) = \mathcal{K}(\mathrm{Dyn}(X,f))$. So $\mathcal{K}$ is representative. Let $\mathrm{Dyn}(X,f), \mathrm{Dyn}(Y,g)$ be objects of SetDyn and $h, h' \in \mathrm{Hom}_{\mathrm{SetDyn}}(\mathrm{Dyn}(X,f), \mathrm{Dyn}(Y,g))$. Then $\mathcal{K}(h) = \mathcal{K}(h')$ implies that $h = h'$, so $\mathcal{K}$ is faithful. The same proof holds if $\mathcal{K}$ is replaced by κ, SetDyn by ADyn, and IndTop by ATop. □

We end this section by noting a simple relationship between the induced topologies and the subobjects of ADyn and SetDyn. Recall that if C is a category and $A \in C$ then a *subobject of* A is a pair (S,i) where $S \in C$ and $i \in \mathrm{Hom}_C(S,A)$ is a monic morphism.

Theorem 4.9. *In both* ADyn *and* SetDyn*.*
(1) If $(\mathrm{Dyn}(S,t), h)$ *is a subobject of* $\mathrm{Dyn}(A,\alpha)$ *then* $h(S) \in \tau_\alpha$ *and* $\mathrm{Dyn}(S,t)$ *is isomorphic to* $\mathrm{Dyn}(h(S), \alpha|h(S))$*.*
(2) If $H \in \tau_\alpha$ *is an open subset of* $\mathrm{Dyn}(A,\alpha)$ *and* $H \xrightarrow{i} A$ *is the inclusion map, then* $\mathrm{Dyn}(H, \alpha|H)$ *is a subobject of* $\mathrm{Dyn}(A,\alpha)$*.*

In other words, the subobjects of a dynamical system correspond to the open sets in the induced topology.

Proof:

(1) Let $S \subseteq A$. Assume $(\mathrm{Dyn}(S,t), h)$ is a subobject of $\mathrm{Dyn}(A,\alpha)$. Since h is monic, it is injective by Theorem 3.11. Let $y \in h(S)$. Then $y = h(x)$ for some $x \in S$. Now

$$\alpha(y) = \alpha(h(x)) = \alpha \circ h(x) = h \circ t(x) = h(t(x))$$

But $t(x) \in S$, so $\alpha(y) \in h(S)$. Since y was arbitrary, $\alpha(h(S)) \subseteq h(S)$ and so $h(S) \in \tau_\alpha$ and $\mathrm{Dyn}(h(S), \alpha|h(S))$ is well defined.

We now show that h is an isomorphism from $\mathrm{Dyn}(S,t)$ to $\mathrm{Dyn}(h(S), \alpha|h(S))$. Clearly h is a bijection since it is injective and $S \xrightarrow{h} h(S)$ is surjective. Let $x \in S$. Then $(\alpha|h(S)) \circ h(x) = \alpha \circ h(x) = h \circ t(x)$. So $(\alpha|h(S)) \circ h = h \circ t$ and thus h is a conjugacy. By Theorem 3.11 h is a bimorphism and ADyn and SetDyn are balanced categories by Theorem 3.15, so h is an isomorphism.

(2) Let $H \in \tau_\alpha$ be an open subset of $\mathrm{Dyn}(A,\alpha)$ and $H \xrightarrow{i} A$ is the inclusion map. Since $\alpha(H) \subseteq H$ the map $\mathrm{Dyn}(H, \alpha|H)$ is well defined. Since $i \circ (\alpha|H) = \alpha \circ i$, we

have $i \in \operatorname{Hom}(\operatorname{Dyn}(H, \alpha|H), \operatorname{Dyn}(A, \alpha))$. Also i is injective so by Theorem 3.11 it is monic. Thus $(\operatorname{Dyn}(H, \alpha|H), i)$ is a subobject of $\operatorname{Dyn}(A, \alpha)$.
□

5. Applications to the $3x+1$ Problem

This investigation was originally motivated by an interest in studying problems like the $3x+1$ problem mentioned in the introduction. In this section we provide some applications.

5.1. **Collatz graphs.**

Definition 5.1. *(cf.* [Wir, pg. 36]*) Let* $\operatorname{Dyn}(X, f)$ *be a dynamical system. The* Collatz graph *of* f*,* Γ_f*, is a directed graph* $\Gamma(V_f, E_f)$ *where* $V_f = X$ *is the set of vertices and* $E_f = \{(x, f(x)) : x \in X\}$ *is the set of directed edges.*

Definition 5.2. *A directed graph is weakly connected if there is an undirected path between any two points.*

One might ask how the connectedness of the Collatz graph is related to the connectedness of the induced topological space.

Theorem 5.3. *Let* $\operatorname{Dyn}(X, f)$ *be a dynamical system. The Collatz graph of* f *is weakly connected if and only if the topological space* $\operatorname{Top}(X, \tau_f)$ *is connected.*

Proof: ($\Rightarrow$) Assume Γ_f is weakly connected. Let $U \in \tau_f$ with $U \neq \emptyset$ and $U \neq X$. Let $x \in U$ and $y \in X - U$. Since Γ_f is weakly connected there exists j, k such that $f^j(x) = f^k(y)$. Since $x \in U$ and U is open, $\mathcal{O}_f(x) \subseteq U$. So $f^j(x) \in U$. Thus $f^k(y) \in U$ and so $\mathcal{O}_f(y) \nsubseteq X - U$. So $X - U$ is not open, and hence $\operatorname{Top}(X, \tau_f)$ is connected.

($\Leftarrow$) Assume $\operatorname{Top}(X, \tau_f)$ is connected. Let $x \in X$ and let U be the weakly connected component of x. Let $z \in U$. Then $(z, f(z))$ is a path from z to $f(z)$ so that $f(z)$ is also in the weakly connected component of x. Thus $f(U) \subseteq U$. So U is open.

Suppose $X - U \neq \emptyset$. Let $y \in X - U$. Since y is not in U, there is no undirected path from x to y and so $f^k(y) \notin U$ for any k. Thus $\mathcal{O}_f(y) \subseteq X - U$. So $X - U = \bigcup_{y \in X - U} \mathcal{O}_f(y)$, and so $X - U$ is a union of open sets and thus open. So U is both open and closed. Since $\operatorname{Top}(X, \tau_f)$ is connected and U is nonempty, $U = X$. So $X - U = \emptyset$. This is a contradiction.

Thus $X - U = \emptyset$. So $U = X$. So X is weakly connected. □

This relationship gives us a new approach to solving the $3x+1$ problem

Corollary 5.4. *The* $3x+1$ *Conjecture is true if and only if* $\operatorname{Top}(\mathbb{Z}^+, \tau_{T|\mathbb{Z}^+})$ *is a connected topological space.*

Proof: It is well known (cf. [Wir]) that the $3x+1$ Conjecture is true if and only if the Collatz graph of $T|\mathbb{Z}^+$ is weakly connected. This is true if and only if $\operatorname{Top}\left(\mathbb{Z}^+, \tau_{T|\mathbb{Z}^+}\right)$ is a connected topological space by the previous theorem. □

Similarly the categorical and topological viewpoint can be used to find other approaches to either proving or disproving the conjecture, as indicated in the next two results.

Corollary 5.5. *If h is a semiconjugacy from* $\operatorname{Dyn}(X, f)$ *onto* $\operatorname{Dyn}(\mathbb{Z}^+, T|\mathbb{Z}^+)$ *and* $\operatorname{Top}(X, \tau_f)$ *is connected, then the* $3x+1$ *conjecture is true.*

Proof: Let h be a semiconjugacy from $\operatorname{Dyn}(X, f)$ onto $\operatorname{Dyn}(\mathbb{Z}^+, T|\mathbb{Z}^+)$ and assume $\operatorname{Top}(X, \tau_f)$ is connected. Then h is continuous with respect to the induced topologies and the continuous image of a connected topological space is connected. So $\operatorname{Top}\left(\mathbb{Z}^+, \tau_{T|\mathbb{Z}^+}\right)$ is a connected space and so the $3x+1$ conjecture is true by the corollary 5.4. □

Corollary 5.6. *If h is a semiconjugacy from* $\operatorname{Dyn}(\mathbb{Z}^+, T|\mathbb{Z}^+)$ *onto* $\operatorname{Dyn}(X, f)$ *and* $\operatorname{Top}(X, \tau_f)$ *is not connected, then the* $3x+1$ *conjecture is false.*

Proof: Let h be a semiconjugacy from $\operatorname{Dyn}(\mathbb{Z}^+, T|\mathbb{Z}^+)$ onto $\operatorname{Dyn}(X, f)$ and assume $\operatorname{Top}(X, \tau_f)$ is not connected. Then h is continuous with respect to the induced topologies and the continuous image of a connected topological space is connected. So $\operatorname{Top}\left(\mathbb{Z}^+, \tau_{T|\mathbb{Z}^+}\right)$ is not a connected space and so the $3x+1$ conjecture is false by the corollary 5.4. □

Thus the results in this paper allow us to restate the conjecture in terms of the induced topological space, and open the door to new approaches to solving the conjecture via topological methods and category theory.

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