Fractal Themes at Every Level

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OK I admit it. I love fractals. Fractal programs, fractal tee-shirts, fractal notebooks, fractal screen savers... What other mathematical phenomenon has made it into mass marketing? I have had students walk into my classroom carrying folders with fractal designs on them, unaware that they were fractals. This is a phenomenon we rarely encounter in mathematics, and one which we should capitalize on.

Teaching fractals and chaos often necessitates the use of special texts, software, hardware and other instructional materials to varying degrees. In a small institution on a limited budget, an instructor is often forced to improvise or make due with the equipment at hand. However, with a little bit of planning and forethought a math department can make some wise decisions that can make the necessary tools available. One purpose of this note is to provide a review of the instructional materials I have used, and to provide the reader with some of the software I have used or produced for my students use in these courses.

I have been incorporating fractal geometry and chaos theory into my teaching at the University of Scranton (a Jesuit undergraduate liberal arts university in northeastern Pennsylvania) to varying degrees since 1990. The degree has ranged from minor bonus projects in linear algebra and complex variables to entire courses, including a freshmen level general education course for non-technical liberal arts students, and junior level course for students who have had the second calculus course. The teaching of fractals has also touched a wide range of audiences outside of my normal courses ranging from a presentation in front of a group of second graders, to mentoring award-winning undergraduate research projects in mathematics, to teaching two undergraduate courses Yale. Like the fractals themselves, the teaching of fractals seems to have something of beauty and value to impart at all levels.

1 Which Courses can benefit?

1.1 Second Grade Elementary School Students

Without a doubt, the most enthusiastic audience I have ever talked to about fractals was my daughter's second grade class. I have yet to see a group of college students leaning on the edge of their seats waving their hands wildly ("Ooh, ooh, call on meeee!!!") as these students did in the hope of being allowed to answer a question about math! I had been invited to give a talk on mathematics to this wonderful group of kids and decided that an introduction to fractal geometry would be a perfect topic.

I brought with me the university's LCD panel which allows me to project the images from my notebook computer using a standard overhead projection unit (which the elementary school provided). The night before I had loaded many fractal images (generated mostly by FRACTINT (see below)) in a directory on the notebook for easy recall. After a brief fractal slide show (which they received quite enthusiastically) I began the main topic of the talk, namely an introduction to the Sierpinski Triangle (a.k.a. gasket).

There are many age-appropriate activities involved with this idea. We began with a large triangle on the blackboard with the children taking turns subdividing it in the usual manner (we put a "safety dot" in the center triangle formed after each subdivision to "protect" it from further subdivision). This involves the concept of finding the midpoint of line segments in a increasingly complicated figures, the concept of a line segment connecting two points, and the idea of a recursive geometric construction. At each stage of the construction we also counted the number of new triangles which did not contain "safety dots", and made conjectures as the construction progressed about how many "non-safety dot" triangles would be formed at each stage (a nice illustration of a geometric sequence and pattern recognition).

After this I showed them the Chaos Game (using the students themselves as points on the floor and dice to choose between the tree vertices) and returned to the computer to show how the Chaos game generates the Sierpinski triangle.

At the end of the presentation, I handed out large paper triangles and straight edges. They then constructed paper Sierpinski triangles of their own. They are not very competent with rulers at that age, but well more than half of the students were able to make a quite respectable looking Sierpinski triangle.

1.2 Freshmen General Education Course

In the Fall 1992 and Spring 1993 terms I taught a sequence of two special topics courses designed to meet the science general education requirements for non-technical freshmen liberal arts majors (English, history, etc.). General education requirements were under extensive review at the time and I felt that the traditional topics covered in a course of this nature needed to be reconsidered.

I had recently finished producing the color plates and some other figures for Don Davis' new book: *The Nature and Power of Mathematics* [4]. Don was my thesis advisor at Lehigh University, and while our area of research is in algebraic topology, we had both been bitten by the bug to learn more about the wonders of fractal geometry while I was still a graduate student. I had known about fractals from a computer hobbyist standpoint long before becoming a mathematician, but my first actual mathematical course in fractal geometry was in a once per week seminar on iterated function systems taught by Don at Lehigh during the Spring of 1989. As a result, I naturally decided to teach the freshman courses out of photocopies of Don's book (which had not yet been published at the time).

The first course was on fractal geometry and number theory covering roughly Chapters 5 and 3 in Davis' book as well as some supplementary material of my own. In order to provide a more unifying theme to the course, I tried to center the course around the concept of recursion in both geometry and number theory, using the connection between Sierpinski's triangle in fractal geometry and Pascal's triangle in number theory (see, for example, [14, Section 8.3]) as the link to make a smooth transition from the fractal geometry section of the course to the number theory section of the course. This approach seemed to work very well and gave the course a unified "feel".

The theme of the second course was non-Euclidean geometry, and concomitantly a lot of time was spent on formal axiom systems and the nature of mathematical proof. This course covered roughly Chapters 1 and 2 in Davis's book.

As it turned out, the first course was a far greater "success" than the second in terms of student enthusiasm. As a mathematician, I feel that the material covered in the second course is one of the most beneficial topics we can teach to non-technical liberal arts majors for their general education exposure to mathematics. It reveals the very nature of mathematical thought, and illustrates the profound impact that the non-Euclidean revolution has had on our perception of mathematical "truth". Yet, the students did not seem to share in my enthusiasm for the topic.

I attribute this to the fact that it is simply more enjoyable and accessible for the typical liberal arts student to make color blow-ups of the Mandelbrot set or construct Pascal's triangle than it is to work on proofs or compare the axioms of non-Euclidean geometries with those of Euclidean geometry. The fractal geometry and number theory students had a good deal of enthusiasm for the material throughout the course while I got the sense that the non-Euclidean geometry students viewed it as a chore; something to be mastered even though they had no interest in it.

Looking back, I should have known this would be so. It takes several years before even our math majors have a decent idea of what the concept of a mathematical proof is. To expect that non-technical liberal arts freshmen would be able to obtain a deep enough understanding in only a few short weeks to appreciate the difference between the geometries is now, in my opinion, unrealistic. That is not to say that they did not benefit by struggling with the material. I feel that they did, but that my course goal of imparting on them a general understanding of the impact of the non-Euclidean revolution on the nature of mathematical thought was certainly not attained, and perhaps unattainable with that particular audience in the limited time frame available.

With fractal geometry and number theory however, one can teach the material at many different levels, starting from the level of the second grade presentation if necessary and gradually adding rigor. Students can produce zooms of the Mandelbrot set, for example, even before they know its definition or any of it's properties. This keeps them connected to the material in a fundamental way when you *do* proceed to discuss the definition and properties. It is the "cover design" that entices the students to open the book. With the non-Euclidean geometry course it is much more difficult to find a suitable carrot.

The software requirements of the freshman fractal course were very minimal and are discussed below. There were no software requirements for the non-Euclidean geometry course.

1.3 Chaos course

During the Spring 1994 and again in Fall 1995, I taught a three credit junior level course entitled "Chaos and Fractals" geared toward students who had at least a Calculus II background and a basic familiarity with computers. I circulated an advertising "flyer" of sorts at registration time and there was a lot of student interest in the course.

The class was primarily composed of math and computer science majors, but I also had chemists, a physical therapy student, an economics major and a philosophy major who audited (the physical therapist changed his major to mathematics after taking the course!). In addition, one of my colleagues in the math department sat in on the 1994 course.

The course met on three days per week throughout the term. My original intention for the course was that it would be primarily about chaos theory, using Devaney's book, *A First Course in Chaotic Dynamical Systems,* [5] as the primary text (required) and *Chaos and Fractals* by Peitgen, Jurgens, and Saupe [14] as a secondary text (optional, but strongly recommended to the students). Both books contain an excellent description of the topics of chaos theory, but the lack of exercises in the latter precluded its use as a primary textbook. Devaney's book does not go into nearly as much depth on the topic of fractal geometry so that it was necessary to have the secondary book available to the students.

The original plan was to cover nearly all of Devaney's book, pausing every second Tuesday to interject some topic from fractal geometry. By the end of the course, the connections between these fractal digressions and our study of dynamical systems became apparent.

After the course was completed, several students mentioned that they had signed up for the course hoping to learn more about fractals than they did. They were disappointed at the low fractal-to-chaos course content ratio. I must agree. There is not much you can accomplish in only 6 or 7 lectures on fractal geometry. As a result, when I offered the course again in the Fall 1995 term I devoted a much larger percentage of the course to fractal geometry, covering topics from fractal geometry once per week on what was affectionately dubbed "Fractal Fridays". To compensate for the extra fractal material in the course, I covered the first five chapters of Devaney's text (which discuss iteration, orbits, graphical analysis, and periodic points) at a much more rapid pace, as most students had no difficulty in this part of the course the first time it was taught.

The course went quite well once again. But the students had two suggestions for improvement at the end of the course. Just as with the first course, they had hoped there would be a larger fractal geometry content to the course rather than the roughly 2/3 chaos to 1/3 fractal content that the once per week fractal lectures provided. Secondly, they felt that it would be more coherent to do all the fractal geometry lectures together and all of the chaos lectures in another contiguous block. Often from one Fractal Friday to the next they would forget definitions and topics discussed the week before, making it hard to follow the theme of the fractal lectures.

As a result of the overwhelming success of this course it has now been added as a permanent listing in our catalog, to be offered once every two years. The prerequisite for the course is now one mathematics course beyond Calculus II. Thus it is being offered as a junior level course for students who have had Calculus II and one other math course. The decision to strengthen the prerequisite was prompted mainly because of the difficulties that the students had understanding the proofs in the course, so they could better appreciate the theory as well as the applications and results.

1.4 Fractals and Chaos at Yale

During the Spring 1998 semester I had the honor of being invited by Benoit Mandelbrot to teach two courses on fractals and chaos at Yale University. It was a very rewarding experience to have worked with Mandelbrot, the father of modern fractal geometry. My conversations and interaction with this legendary innovator and genius has expanded and altered my perspective on fractal geometry and all of mathematics in a many ways which will impact on my teaching of the subject in the future, and I am very grateful to have had this opportunity.

The two courses I taught at Yale were geared towards completely different audiences. One was a freshman level course designed for liberal arts majors (similar to the course I taught at Scranton using Davis's book). The second was a course designed for sophomore-junior level students who have had calculus (similar to the course I taught at Scranton using Devaney's book). In the rest of this article I will refer to these two different types of courses as the "liberal arts course" and the "junior level course".

In the liberal arts course at Yale, I used the excellent textbook by Peak and Frame [13]. We were able to cover nearly all the material in their book in a single semester. The course focused on the major topics in fractal geometry and chaos theory from a mathematical standpoint, but in addition had many illustrations and references to applications in other areas of interest to liberal arts students as well. Students were required to produce a term project which related fractal geometry and chaos theory to their own interests, and some of these projects were quite well done and interesting (fractal patterns in dance, music, art, literature, chaos in baseball statistics, and CD player track changers just to name a few).

In the junior level course I decided to used Crownover's book [3] as this book has a much higher fractal to chaos content ratio and covers the topics of fractals and chaos in contiguous blocks of lectures rather than trying to intersperse them with each other as I had done in the past. I also had [10] and [14] as recommended texts in the course that were not required. The choice and level of topics covered in Crownover is very good for a course like this. However, the exposition leaves much to be desired and within a few weeks both the students and I had become so disenchanted with the text that we stopped using it altogether, instead preparing my lectures using material from [14] and [2].

In the end, the course focused on the study of iterated functions systems and their mathematical applications, and the basic theory of chaos. The students were also required to produce a term project, most of which were quite impressive. Both courses went quite well and student feedback was overwhelmingly positive.

1.5 Fractals in Other Courses

In addition to the courses discussed above, many other traditional courses can benefit from a healthy dose of fractal geometry.

In linear algebra, I assign a project on iterated function systems at the point in the course where we study linear and affine transformations of the plane. The project consists of a handout which explains the iterated function system concept, and then leads the students through the computation of a Sierpinski Carpet. The students are then asked to produce a Sierpinski Triangle via iterated function systems. The project does not usually generate the amount of enthusiasm that I would expect, perhaps because the students feel it is just additional work on top of the main topics of linear algebra many of them are struggling to learn.

Naturally, I cannot resist giving the definition and a short project on the Mandelbrot set in the complex analysis course, where it serves as an illustration of complex arithmetic, the geometry of subsets of $\mathbb{C}$, and the complicated action of an iterated quadratic polynomial as a mapping of the complex plane. This is also given as an outside assignment. As in linear algebra, almost no lecture time is taken away from the primary subject matter to devote to fractals.

In general, I feel that these projects did not add (or detract) substantially to the courses for most students, and I now believe that the correct place for such material is in special topics courses devoted to them such as those discussed above.

1.6 Undergraduate Research Projects

In 1991 the University of Scranton instituted a Faculty Student Research program. In this program, students receive transcript recognition for no fee or credit upon the approval of the mentor. At the start of each term in which the student participates, the mentor and student sign a contract of sorts, stating what will be required of the student during that term. If the contract is met then the student gets transcript recognition stating that he participated in the program, otherwise nothing happens.

Most of the student projects to date involved topics related chaos and fractals [7], [6], [8], and [9]. All of these undergraduate student papers would make interesting supplementary reading in an undergraduate chaos and fractals course and would provide the students with an example of the kind of research that an undergraduate is capable of doing in mathematics. Nearly all of my student researchers were former students from my chaos and fractals course, indicating the mathematical enthusiasm that this course is capable of generating. For a more detailed description of the content of these papers, see the papers themselves or [11]. The papers themselves are available at [12].

2 Textbooks

In this section I would like to give brief informal reviews of the textbooks I have used in the above mentioned courses from a pedagogical perspective.

2.1 The Nature and Power of Mathematics

As mentioned above, I have taught a two-course sequence using this book [4], covering most of the material contained in it. I must also warn the reader that I might not be the most objective reviewer of this book; first, because the author, Don Davis, is my Ph.D. thesis advisor and friend, and second, because I produced all of the color plates as well as many of the black-and white diagrams for the book and was a major proofreader of the original drafts.

That having been said, I must say that the book is a very well written exposition, which does a masterful job of explaining extremely difficult mathematical concepts, at a very low level, while maintaining the mathematical integrity of the subject matter. This is not an easy balance to achieve. Yet Davis succeeds in doing this extremely well. As he says in his introduction:

> This book was written for the liberal arts student, but it could also be read with benefit by a good high school student, ... or by an educated general reader who is willing to do a little work while reading.

The book succeeds in this regard.

The exercises range from very basic problems to those which are far too difficult for a normal liberal arts student. The book is not a standard text in the sense of having a lot of worked examples for the students to imitate when working the homeworks, but rather expects the students to work the exercises in order to develop a deeper understanding of the concepts introduced in the text.

In addition to the material on fractal geometry, the major topics covered in the text are formal axiom systems, non-Euclidean geometry, and number theory and cryptography. Each of these can provide excellent supplementary material in a course designed around the fractal geometry "core" material. Alternatively, the instructor may wish to supplement the fractal material in the book with additional topics in fractal geometry if a fractals-only course is desired.

2.2 A First Course in Chaotic Dynamical Systems

Devaney's book [5] is a perfect text for teaching an undergraduate course on chaos. The book can easily be read by the students, the exercises are just the right level of difficulty, and the book comes with an excellent set of "laboratory experiments" which are intended to be assigned as projects to the students. I used several of these projects with a great deal of success.

The pacing of the book is a bit slow in the first five chapters (which discuss iteration, orbits, graphical analysis, and periodic points), and I recommend that you cover this material rather quickly, as the students had very little difficulty with it. The meat of the course is contained in chapters 6 through 12 (bifurcations, quadratic maps, symbolic dynamics, chaotic maps, Sarkovskii's theorem, etc.). This material should be covered carefully by the instructor, as the students do have difficulty with the amount of analysis and topology required to go through this material. This is unavoidable, but Devaney does an excellent job of teaching only those concepts from these areas which the students require to understand the key concepts and results of chaos. The book is certainly appropriate for students who only have a Calculus II background.

If no outside topics are introduced, one should easily be able to cover the entire book in a three credit course. If a substantial amount of fractal geometry from other sources or other material is to be covered early on in the course, then the material in chapters 13 through 16 on Julia and Mandelbrot sets and Newton's method fractals can be omitted without much loss, as these topics would have already been covered.

This book is almost entirely a book on chaos, so if you want to teach a course which thoroughly covers the key ideas in fractal geometry, you might want to use a different text. For chaos theory and discrete dynamical systems, Devaney's book is excellent.

2.3 Chaos and Fractals

This book [14] is a true work of art in every sense. It is organized in such a way that it a non-technical reader can get the basic ideas covered, while interjecting mathematical rigor for the more sophisticated reader in a very unobtrusive way. In addition to covering many topics from fractal geometry, the book also presents the material of chaos covered in Devaney's text very well. The illustrations are simply fantastic.

This might make an excellent text for a course on fractal geometry and chaos theory and provide the instructor with a lot of leeway for selecting topics (as there is far too much material to be covered in a 3 credit course). It has been either a recommended or required text in nearly all of my chaos and fractals courses to date.

The only serious drawback this book has as a text is the lack of exercises. I strongly hope that someone will produce a companion exercise manual for this text in the future, or that exercises will be incorporated into future editions. Even still, the extra work required to make up problem sets is somewhat compensated for by the sheer joy of being able to cover such a wide variety of topics. It is also the kind of book many students might think twice about selling back to the bookstore at the end of the term.

2.4 Chaos Under Control

This book *Chaos Under Control* [13] by Frame and Peak is the perfect choice for a freshman level liberal arts course. It is written at a level which assumes almost no mathematics on the part of the reader and yet gently nudges the student into an understanding of very difficult mathematical topics without sacrificing mathematical rigor. The author's go to great pains to describe the mathematics accurately, but with a minimum of mathematical jargon.

The material is constantly illustrated with applications and implications to areas outside of mathematics that might be of interest to the typical liberal arts student. The book is very up to date, and discusses many of the recent developments in the areas under discussion. It covers the major topics from fractal geometry and chaos theory, and also a lot of applications of both to the study of randomness and data analysis (something not found in most of the other texts on chaos and fractals). This gives the course a decidedly applied flavor which appeals to the liberal arts students without compromising the mathematical integrity of the material. The pace of the book is perfect and I was able to cover almost the entire book in a single course which met three hours per week for one semester. I will use this book again when I teach the course during the Fall 1998 term at Scranton.

The only minor flaw in the book is a lack of included exercises at the end of each chapter (there are a few in the back of the book). But this was not a problem as one of the authors, Michael Frame, has a complete exercise set available at his web site. The author was also very helpful to me during the course via email and provided other instructional materials such as solutions to the homework sets, some nice fractal animations, lecture notes, and so on. With the easy availability of the exercise sets via internet, there is no reason not to use this text in a freshman liberal arts course on fractals and chaos, or even in a freshmen general education course designed to give students the flavor of modern mathematics and mathematical research. I think the topics are more exciting and relevant than some of the other traditional texts which cover more classical mathematics, and I intend to use Peak and Frame's book for our general education math course at Scranton well. I give their text my highest recommendation for this kind of course.

2.5 Fractals and Chaos

Crownover's textbook *Fractals and Chaos* [3] has all the markings of what should be an ideal textbook for a sophomore-junior level undergraduate text on fractal geometry and chaos. It has an excellent choice and order of topics, covering fractal geometry first and then leading into several extensive chapters on chaos theory. It has slightly more material on fractal geometry than chaos theory, but the topics and order chosen blend well into a cohesive syllabus for such a course. It appears to be written at the right level of difficulty for students at first glance, and contains many exercises. All of this led me to select it as the primary textbook for the advanced undergraduate course at Yale.

However, after only a few weeks into the course, both the students and I decided to abandon the book in frustration. The exposition is very uneven and confusing in general. The students found the material confusing to read. I found the coverage difficult to prepare lectures from, not because the level of presentation is too difficult for such a course, but rather because of the lack of care in defining terms and symbols, proving theorems, and mathematical errors in the material beyond simple typos. I ended up correcting and revising the material in the text so much in my lectures that it became easier to simply abandon the text altogether and have the students learn everything from my lectures.

The author seems to be making an honest and sincere effort to create the perfect textbook, and I have no doubt that in a future edition it might be an excellent text to use in such a course, but much revision will be required before that can be attained. I cannot recommend it for others use in its current form. However, it is possible that I simply have a different perspective on how the material should be presented than the author does, and thus you might want to check out the text to decide for yourself.

2.6 Other Texts

No discussion of textbooks on fractals and chaos theory would be complete without mentioning Mandelbrot's seminal treatise *The Fractal Geometry of Nature* [10]. This book is the *de facto* recommended reading in my courses as it is the original source of inspiration from which the subject has developed. It provides both historical background for the subject and is a wonderful source of applications and ideas to inspire student projects, especially in an advanced undergraduate course.

The book itself a written as a classic work documenting the ideas and discoveries of Mandelbrot, not in the expository style of a textbook intended for classroom use. It is very dense, each chapter being the equivalent of one or more traditional research articles. Thus the book is not well suited for use as a primary textbook in an undergraduate course. It is too difficult for most students, who would find the densely packed collection of abstract ideas and concepts very difficult going without the careful detailed hand-holding that exemplifies a typical textbook. However, as a reference work and source of inspiration, it is a true classic that I highly encourage my students to read, and often refer to as a source of applications and historical development when developing my lectures. I highly recommend it to any instructor in a course on fractals and their applications.

During my recent advanced undergraduate course at Yale, I had an opportunity to prepare many of my lectures using Barnsley's book, *Fractals Everywhere [2]*. While I have not used his text as the primary required textbook in any course thus far, I was very impressed with both the depth and variety of the topics covered, the consistency and rigor of the mathematical exposition, a the truly excellent collection of wonderful exercises at the end of each section. The book is in some sense complementary to Devaney's in that Barnsley's book is almost entirely about fractal geometry with very little chaos discussed instead of the other way around. I was very impressed and I would consider using this as a primary textbook in the future in a course which was primarily about fractal geometry for undergraduates with a calculus prerequisite.

3 Software Requirements

Finding and/or developing the appropriate software to teach courses on fractal geometry or chaos requires a bit of planning. In a small undergraduate institution it is a good idea to incorporate software and hardware planning into a larger picture that considers the department's needs as a whole.

For this reason our department has centered our computational needs around *Maple*. The flexibility of *Maple* makes it possible to use it in almost all of our undergraduate courses which require (or can be enhanced by) the use of a computer. We currently have a mathematics computer lab housing nineteen Pentium II PC's running *Maple V R5* under *Windows 95*. In addition, *Maple* is available on some public labs and also under VMS (text only) on our academic mainframe and on other WIN95 machines around campus. However, students primarily use it in our math lab setting.

3.1 Maple

Maple is extremely flexible for the needs of most courses and provides the students with a uniform tool that they can use as they move through our curriculum. In my Chaos course, in the beginning of the term I make available a *Maple* notebook [12] containing many of the routines the students would require to complete their assignments (I did not want to spend a lot of time teaching *Maple* syntax).

The routines consist of the usual array of tools for studying chaos and fractals. There are routines to compute orbits, plot graphical analysis diagrams, make bifurcation (orbit) diagrams, and one to illustrate the chaos game. They are listed in the notebook with examples showing their use.

The orbit routine comes in two flavors: exact and approximate. The approximate orbit gives the values of the orbit floating point form with its concomitant roundoff error, while the exact version uses *Maple's* exact arithmetic, which is quite useful for studying periodic points.

Also available is an animated graphical analysis routine which produces an animation of the graphical analysis diagram so that students can actually see the construction of the diagram in real time, rather than just the finished product.

I decided to provide these routines to the students in the form of a *Maple* notebook rather than putting them in the *Maple* library so that they would have access to the source code which they could easily modify or learn from when creating their own *Maple* routines for their projects.

All of the projects which were assigned from Devaney's book were completed on *Maple*. A particularly interesting and difficult project is the determination of Feigenbaum's constant [5, Experiment 10.4]. This is a particularly tricky calculation because of the proximity of the roots to each other as you iterate the function. Students learned very quickly that one cannot always rely on *Maple's* built-in fsolve routine to magically come up with the answer they desire. There were many different approaches to this problem, some good, and some not so good. I used Maple to come up with a solution using a simple bisection method for finding the roots of and using the graph to determine an interval containing the desired root.

However, even *Maple,* despite its sophistication, has some limitations. For example, *Maple* is somewhat slow in producing even the crudest plots of the Mandelbrot or Julia sets. It can be helpful in analyzing the fate of individual points, but its graphics capabilities and slow computation speed limit its ability to produce fractals.

However, some of these limitations have been overcome recently with the introduction of hardware floating arrays in Release 5 of Maple. By doing floating point computations directly in the host's hardware, the user can avoid the slow arbitrary precision floating point routines of Maple. Recently I discovered the excellent Maple fractal software at

`http : //www.math.utsa.edu/mirrors/maple/frame03.htm`

which overcomes many of these limitations and I will certainly use this software and modify my own routines to incorporate hardware floats for use in my future courses.

3.2 FRACTINT

This is the mother of all fractal programs, and what's best, its free! This DOS program (there is a Windows version, but the DOS version is more robust) is a public domain fractal generation program that is very fast and very complete. It is in version 19.6 at the time of this writing and is being revised to add new features constantly. I have not encountered any program in its league for doing Mandelbrot set zooms. It is blazingly fast and runs on a variety of video hardware to produce stunning pictures in the highest resolutions your system supports. Almost every conceivable fractal type is supported. I have had students in my courses use it to produce Mandelbrot and Julia type fractals, IFS fractals, and to generate L-systems. It is also available by anonymous ftp on internet at many different sites(e.g. [12]). We always keep the latest version installed on all relevant machines on campus, since it costs us nothing and benefits the students. I have walked into a computer lab to find students who know nothing about fractals and who are not even in a course requiring its use, playing with the program. This program is a must for every math department.

3.3 Calculators

In the classroom, we do not currently have available an easy method for displaying computer graphics. The classroom I usually teach in is equipped with a computer and overhead projection system that is time consuming to hook up and activate (see discussion of projection units below). Also when working the homework problems, it is often overkill to be chained to a PC running *Maple* and many students in our school still do not have a personal computer. A solution to some of these problems comes in the form of the programmable graphics calculator. Several of the students in my chaos course purchased one of these calculators for use in the course.

When I taught the chaos courses in 1994 and 1995 the calculator I was using at that time was a TI-85 programmable calculator. The TI-85 is an affordable, and sophisticated programmable graphics calculator. In the appendix you will find a scaled down version of the *Maple* utility discussed above written for the TI-85 [12] which I wrote for my own personal use in the course (doing homework, making test problems, answering spontaneous questions that arise in the classroom). It basically is a menu driven program that allows the calculator to compute orbits and draw graphical iteration diagrams.

One nice feature of the TI-85 calculator is the ability to transfer programs from one calculator to another via a cable that comes with the calculator. Several of the students in my class also purchased the TI-85 and I gave them this chaos program for their use. In most cases, the calculator is sufficiently powerful to work some of the projects in Devaney's book, and I think that it would be possible in theory (though less desirable) to run an entire chaos and fractals course using only this calculator as a computing platform (although tiny black and white fractals produced on a calculator after literally days of computation are not exactly awe inspiring).

I have written for the TI-85 to compute bifurcation (orbit) diagrams [12]. This program allows the user to move the TI-85's cursor on the finished drawing to get numerical information when the diagram is complete. Many other programs, including ones to draw the Mandelbrot set are available for the TI-85 on internet.

Since teaching the 1994 and 1995 courses I have purchased a TI-92 calculator. This is a more expensive ($165-$200) calculator but is *very* powerful. It has essentially the entire computer algebra system *Derive* built into it, providing arbitrary precision arithmetic and symbolic computation capability, with a standard alphanumeric keyboard.

Thus I did not have to write *any* software in order to use the TI-92 for use in a chaos course as it already has the capability to compute orbits (using the Table feature, set the Mode to Sequence) and it also has the ability to do graphical analysis built-in (from the y= screen, set Axes to Web). So this calculator is ready to use out of the box in a chaos course!

It is an easy to use calculator with more power than rightly ought to be in a calculator! The only bad feature of both the TI-92 and the TI-85 is the poor display. They are *extremely* difficult to read in incandescent lighting like you might find at home. I have to prop my calculator at a certain angle and position all of the lights in the room just right in order to be able to use it for any length of time. It is really an unbearable feature on otherwise fine pieces of equipment and I hope that TI will fix this problem in future models.

3.4 Other Software

There are probably hundreds of fractal and chaos programs currently available (see [12]). Most deal with a specific set of examples or are companion software to certain textbooks. I do not think that anything beyond *Maple* and *Fractint* is required for serious study of fractals and chaos at an undergraduate level. Since *Maple* does double duty by providing superb CAS support to other courses in our department, it is a very cost effective approach to satisfying the computational needs of the department as a whole (as opposed to buying individual programs that service only one particular course).

4 Hardware Requirements

4.1 Classroom Displays

The classroom that Yale provided for the liberal arts course was outstanding and this influenced the way I taught the course. The room was a medium sized auditorium holding about 100 students with a sloped seating arrangement. There was a ceiling mounted projector which could display video on a screen that lowered above the blackboard. On the wall was a control box containing a VCR, audio system, and a video port which I could plug my laptop into to display it through the ceiling projector. The screen could be lowered to just above the blackboard, and the lighting was on a dimmer switch, allowing the simultaneous use of the blackboard and computer display.

This was an ideal setup for such a course, and as a result I found myself making the course a lot more multimedia intensive. My course directory grew during the term into a collection of 35 different programs, graphics, animations, and Power Point slide shows Of course, this took a lot of work to develop the first time, but I now have these available for teaching the course in the future (as I will be doing during the Fall 1998 semester at Scranton).

I cannot emphasize how much such a multimedia approach enhanced the course and aided these non-technical students in learning the material, and I cannot imagine teaching the course exclusively at the blackboard. The ability to discuss the theory at the blackboard and illustrate the results on the video display simultaneously was very effective at getting across otherwise difficult and abstract mathematical ideas.

As a result I am now in the process of working with the administration at Scranton to upgrade the computer display capabilities of the mathematics classrooms at Scranton, so that other courses might benefit from this multimedia approach as well. One such classroom should be outfitted to support my laptop by the start of the Fall 1998 term.

We currently have an overhead projector with a color VGA LCD display connected to a Pentium computer in two of our math classrooms. But I rarely use these because of the time involved in setting them up and the fact that the projection cart and screen obstruct the student's view of the blackboard, making it impossible to use both at the same time. If you are trying to outfit your classrooms with computer displays, be sure to outfit the rooms with the ability to control lighting (blackout curtains to eliminate daytime light pollution, and a dimmer to allow the taking of notes while viewing the projection). Also if you can install the computer in such a way that it does not require extensive setup and tear-down time it will make it more practical for your already-time-constrained classes. Finally, it is desirable to be able to use the blackboard at the same time as the computer without darkening the room.

I also recommend the use of a laptop for each instructor over the use of a dedicated in class PC. It simply takes too much time to prepare multimedia lectures and get them working on your own computer at home or in the office and then have to transfer the files to the classroom PC (which may be not be the same classroom each term) and test it to be sure the required software is installed, there is no version incompatibilities, the hardware difference doesn't cause a problem, etc.etc. etc. With a laptop the instructor can prepare his presentation before class and use that same machine for the presentation, eliminating all the transfer and debugging time and assuring the presentation will work as expected.

4.2 Lab Computers

As mentioned above we have nineteen Pentium II machines in a very nice mathematics lab. The lab is not limited to computer use, but is rather intended to be a generic student study area for mathematics of every kind. We have a small library of math books available as well as a discussion area with white boards for the students to talk about mathematics. This room is located in close proximity to the math department faculty offices, making the faculty very accessible to the students working in the lab.

4.3 Programmable Calculators

As mentioned above, I have used the TI-85 and currently use the TI-92. These are both wonderful calculators that are capable of streamlining most numerical processes and do a very nice job of graphing (for a calculator). Many students in my numerical analysis course had purchased one of these and seem to be very pleased. It would be nice to have every student own one of these, but we have not required this in any course so far.

There is also a computer link available that works with either calculator which allows you to write your calculator programs on a PC and then transfer them to the calculator. The software supplied is elementary and the link is very expensive (almost as much as the calculator itself), but it is absolutely necessary if you want to write programs or use those you find on internet. You can enter programs from the tiny calculator keyboard, but it is very awkward to do so (especially on the TI-85), making the link very necessary for programmers.

5 Conclusions

Fractal geometry and chaos theory have the magical ability to attract students to mathematics, to engage their attention on the material, and to keep it there while teaching them some very difficult and beautiful mathematics. I am certainly not advocating teaching a subject simply because it pleases the students. What makes students happy and what is best for their mathematical education is certainly not always the same thing. However, a happy interested student is much more likely to devote the effort and time needed to master difficult material than an alienated one. If fractal geometry and chaos can make students *eager* to learn about metric spaces and topological conjugacy, dense subsets, and fractal dimension, then I am all for it.

For a one semester liberal arts course, I highly recommend the book by Peak and Frame [13]. If your university has a two semester liberal arts general education math sequence (Scranton does), I recommend the text by Davis [4] for the second semester as it has a wide variety of wonderful topics beyond fractal geometry and chaos and excellent exercises. For a junior level course, there is no ideal textbook which covers both fractals and chaos. Devaney's book [5] is an excellent text for a course that emphasizes chaos theory and discrete dynamics but does little fractal geometry. Barnsley's book [2] emphasizes fractal geometry iterated function systems. Peitgen's book [14] covers both fractals and chaos very well, but lacks exercises. Crownover's book has the right mix of topics, but the exposition is lacking.

However, as a result of my experiences, I no longer believe that the topics of chaos and fractals should be taught in a single advanced undergraduate course. There is simply too much important material in both areas to fit both topics into a single course. Instead, I am currently suggesting to our department that we break up the chaos and fractals course at the University of Scranton into two courses. The first course would cover the topics of chaos in discrete dynamical systems that appear in Devaney, and in addition would cover continuous dynamical systems and chaos in differential equations instead of discussing fractal geometry. The second course would be on fractal geometry, and would cover most of the material in Barnsley's book including applications of iterated function system fractals to fractal data interpolation. So it is my hope that our existing course will split into the two courses, one on fractals and one on dynamical systems. (The liberal arts course can still cover both topics in one semester, of course, since it is designed as a survey course anyway.) In all of these courses, Mandelbrot's classic text [**?**] should be recommended reading and referred to frequently by both students and faculty.

In planning for your software requirements a program like *Maple* can satisfy almost all of the software needs of such courses, and does double duty by providing software support in most cases across the mathematics curriculum. Thus this provides and efficient strategy both economically and pedagogically. Programmable calculators provide the means for students to have substantial computing power available at the cost of a good textbook, and can be a nice complement to the power of Maple, but their use is not essential and need not be required in any of these courses if Maple is available to the students on campus.

The topics of chaos and fractal geometry are very timely. It is one of the rare undergraduate mathematics subjects that deals with theorems and definitions that are recent and constantly being refined. During the 1994 chaos course we had been discussing Devaney's definition of chaos, namely that a dynamical system is chaotic if:

1. it has dense periodic points,

2. it is transitive, and

3. it has sensitive dependence on initial conditions.

We then discussed the recent article of Banks, et. al. [1] in which they show that conditions #1 and #2 imply condition #3 for infinite sets. As the course went on, I received the current issue of the *Monthly* which contained an article by Vellekoop and Berglund [16] in which they prove that condition #2 implies #1 for functions defined on intervals. I then presented these new results in the 1995 chaos course. Since then Touhey has published yet another definition of chaos [15], namely that a continuous function on a set is chaotic if every two disjoint open subsets share a periodic orbit.

It is very exciting to be able to watch the definitions of the topic evolve and develop before our eyes, sometimes changing from the start of the course until it is finished. In my chaos courses, other articles were discussed which included simple-to-state open problems in these areas. The opportunity for undergraduate students to experience mathematics as a fresh, living art, in which there are more questions than answers, in which definitions and theories are refined and developed over a long period of time by a group of real living people, is a rare one indeed... and one which can do a lot of good for our subject and students.

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