GROEBNER BASES AND THE COHOMOLOGY OF GRASSMANN MANIFOLDS WITH APPLICATION TO IMMERSION

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Abstract. Let $G_{k,n}$ be the Grassmann manifold of k-planes in $\mathbb{R}^{n+k}$. Borel showed that $H^*(G_{k,n};\mathbb{Z}_2) = \mathbb{Z}_2[w_1,\ldots,w_k]/I_{k,n}$ where $I_{k,n}$ is the ideal generated by the dual Stiefel-Whitney classes $\overline{w}_{n+1},\ldots,\overline{w}_{n+k}$. We compute Groebner bases for the ideals $I_{2,2^i-3}$ and $I_{2,2^i-4}$ and use these results along with the theory of modified Postnikov towers to prove immersion results, namely that $G_{2,2^i-3}$ immerses in $\mathbb{R}^{2^{i+2}-15}$. As a benefit of the Groebner basis theory we also obtain a simple description of $H^*\left(G_{2,2^i-3};\mathbb{Z}_2\right)$ and $H^*\left(G_{2,2^i-4};\mathbb{Z}_2\right)$ and use these results to give a simple proof of some non-immersion results of Oproui.

1. Introduction

Let $G_{k,n}$ denote the Grassmann manifold of unoriented k-planes in $\mathbb{R}^{n+k}$. Define the immersion dimension of $G_{k,n}$ to be the smallest j such that $G_{k,n}$ immerses in $\mathbb{R}^j$. There have been many results in the literature which obtain upper and lower bounds on the immersion dimension of $G_{k,n}$. Lower bounds are obtained for certain families of $G_{k,n}$ in [6], [9], [12], [13] through the use of Stiefel-Whitney classes. Upper bounds have been obtained by Lam [7] who showed $G_{k,n}$ has immersion dimension less than or equal to $\binom{n+k}{2}$. For many values of n, k Lam's result improves on the standard upper bound of $2nk-1$ given by Whitney's theorem [14] and the stronger upper bound of $2nk-\alpha(nk)$ (where $\alpha(nk)$ is the number of ones in the binary expansion of nk) given by Cohen's theorem [3]. However, except for the results of Hiller-Stong [6] who showed the immersion dimension $G_{k,k}$ and $G_{k,k+1}$ is equal to Lam's upper bound, the exact immersion dimension of $G_{k,n}$ is largely not known. There are also many results on the immersion and non-immersion of projective spaces which is the case $G_{1,n}$ (or since $G_{k,n}$ is diffeomorphic to $G_{n,k}$, we could also say $G_{k,1}$). As we are not interested in the projective space case in this article, we shall henceforth assume that $1 < k \leq n$ when discussing $G_{k,n}$.

Many common techniques for computing bounds on the immersion dimension of $G_{k,n}$ require a good working understanding of the structure of the cohomology ring

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$H^*(G_{k,n};\mathbb{Z}_2)$. Borel [1] gave the following description of this ring. It is well known that

$$H^*(G_{k,\infty};\mathbb{Z}_2) = H^*(BO(k);\mathbb{Z}_2) = \mathbb{Z}_2[w_1,\dots,w_k]$$

where w_i is the i^{th} Stiefel-Whitney class of the canonical k-bundle γ over $BO(k)$. Let $\overline{w} = 1 + \overline{w}_1 + \overline{w}_2 + \cdots$ and $w = 1 + w_1 + w_2 + \cdots$ satisfy the relation $w\overline{w} = 1$. Borel showed that

$$H^*(G_{k,n};\mathbb{Z}_2) = \mathbb{Z}_2[w_1,\dots,w_k]/I_{k,n}$$

where $I_{k,n} = \langle \bar{w}_{1+n},\dots,\bar{w}_{k+n}\rangle$. Chern [2] also gave a complete description of this ring involving Schubert symbols.

However both descriptions are often quite unwieldy to work with in practice. This lack of a simple workable description has been one of the major obstacles to computing immersion bounds (e.g. when computing the Stiefel-Whitney classes of an arbitrary Grassmann). In this paper we apply the methods of Groebner bases (c.f. [4]) to Borel's description of the cohomology to give a very tractable and simple description of the cohomology of certain Grassmanns and then use this to obtain new immersion results. We also illustrate how the same techniques can be used to reprove in a straightforward manner some nonimmersion results obtained by Oproui in [12].

2. Main Results

Throughout the paper we fix a monomial ordering on the monomials in $\mathbb{Z}_2[w_1,w_2]$ by defining $w_2^b w_1^a \prec w_2^d w_1^c$ if either $b < d$ or else $b = d$ and $a < c$. We prove the following.

Theorem 2.1. *With respect to the ordering $\prec$, and for $i \geq 3$,*

1. *$\{\overline{w}_{2^i-2},\overline{w}_{2^i-1}\}$ is the reduced Groebner basis for $I_{2,2^i-3}$.*
2. *$\{\overline{w}_{2^i-3},\overline{w}_{2^i-2},\overline{w}_{2^i-1}\}$ is the reduced Groebner basis for $I_{2,2^i-4}$.*

From this we obtain as a corollary the following simple descriptions of the cohomology in these cases.

Corollary 2.2. *For $i \geq 3$,*

1. *A vector space basis for $H^*(G_{2,2^i-3};\mathbb{Z}_2)$ is the set of all monomials $w_2^b w_1^a$ such that $a < 2^i - 1$ and $b < 2^{i-1} - 1$. The product structure is completely determined by the relations $w_1^{2^i-1} = 0$ and $w_2^{2^{i-1}-1} = \sum_{j=0}^{i-2} w_2^{2^j-1} w_1^{2^i-2^{j+1}}$.*
2. *A vector space basis for $H^*(G_{2,2^i-4};\mathbb{Z}_2)$ is the set of all monomials $w_2^b w_1^a$ such that $a < 2^i - 1$ and $b < 2^{i-1} - 2$ union with $\left\{w_2^{2^{i-1}-2}\right\}$. The product structure is completely determined by the relations $w_1^{2^i-1} = 0$, $w_2^{2^{i-1}-1} = \sum_{j=0}^{i-2} w_2^{2^j-1} w_1^{2^i-2^{j+1}}$, and $w_2^{2^{i-1}-2} w_1 = \sum_{j=1}^{i-2} w_2^{2^j-2} w_1^{2^i-2^{j+1}+1}$.*

To illustrate these results, consider the case $i = 3$. By the corollary, $H^*(G_{2,5};\mathbb{Z}_2)$ has a basis consisting of all $w_2^b w_1^a$ with $b < 3$ and $a < 7$ subject to the relations $w_1^7 = 0$ and $w_2^3 = w_2 w_1^4 + w_1^6$. Since these relations are obtained from the Groebner basis we can easily compute arbitrary products in this ring by the standard techniques described in [4]. For example, suppose we wanted to know if $(w_2)^4$ is zero in $H^*(G_{2,5};\mathbb{Z}_2)$. We simply compute

$$\begin{aligned}(w_2)^4 &= \left(w_2^3\right) w_2 \\ &= \left(w_2 w_1^4 + w_1^6\right) w_2 \\ &= w_2^2 w_1^4 + w_2 w_1^6.\end{aligned}$$

Since $w_2^2 w_1^4 + w_2 w_1^6$ is expressed in terms of our vector space basis, we see that $(w_2)^4$ is not zero, and we have also determined its unique representation in our basis.

As a further application of these results, we obtain a new immersion result. Cohen's result shows $G_{2,2^i-3}$ immerses in $\mathbb{R}^{2^{i+2}-13-i}$. A scan of the literature indicates this is the smallest previously known upper bound. Using the description of the cohomology given above, and the method of modified Postnikov towers as described by Gitler and Mahowald [5] and extended by Nussbaum [11], we improve on this bound with the following new immersion result.

Theorem 2.3. *For $i \geq 3$, $G_{2,2^i-3}$ immerses in $\mathbb{R}^{2^{i+2}-15}$.*

Comparing with Whitney's result, we see that our result improves on his upper bound by two dimensions for all $i \geq 3$. Comparing with Cohen's result, we see that our result improves on his upper bound by one dimension when $i = 3$. Theorem 2.3 shows $G_{2,5}$ immerses in $\mathbb{R}^{17}$, whereas Cohen's result shows it immerses in $\mathbb{R}^{18}$.

As another application of these results we can easily obtain lower bounds on immersion using Stiefel-Whitney classes.

Theorem 2.4 (Oproui)**.** *For $i \geq 3$,*

(1) *$G_{2,2^i-3}$ does not immerse in $\mathbb{R}^{2^{i+1}-3}$.*
(2) *$G_{2,2^i-4}$ does not immerse in $\mathbb{R}^{2^{i+1}-3}$.*

These results were originally proven by Oproui [12], and later improved on for $i > 3$ by Tang [15] using completely different methods. But our use of Groebner bases allows us to give a proof which is quite straightforward.

3. Background

3.1. **Background on Groebner bases.** All of the information in this section can be found in [4].

A monomial ordering on $\mathbb{Z}_2[x_1,\ldots,x_k]$ is a total well-ordering $\prec$ on the set of monomials such that $x \prec y$ implies $xz \prec yz$ for any monomials x, y, z. For example, lexicographic ordering with respect to some ordering of the variables is a monomial ordering. Let $\prec$ be a monomial ordering on $\mathbb{Z}_2[x_1,\ldots,x_k]$ and $f \in \mathbb{Z}_2[x_1,\ldots,x_k]$. Then $\mathrm{LT}(f)$ denotes the leading term of f with respect to $\prec$, i.e. the largest monomial summand of f with respect to $\prec$. Let $F = (f_1,\ldots,f_s)$ be an ordered s-tuple of polynomials in $\mathbb{Z}_2[x_1, \ldots, x_k]$. There is a division algorithm which allows us to write every $f \in \mathbb{Z}_2[x_1, \ldots, x_k]$ as

$$f = a_1 f_1 + a_2 f_2 + \cdots a_s f_s + r$$

where $a_i, r \in \mathbb{Z}_2[x_1, \ldots, x_k]$ and r is a $\mathbb{Z}_2$-linear combination of monomials, none of which is divisible by any of $\mathrm{LT}(f_1), \ldots, \mathrm{LT}(f_s)$. This remainder r is sometimes denoted $\overline{f}^F$. It is not necessarily unique in the sense that changing the order of the functions in F can produce a different remainder. Even worse is the fact that $f \in I = \langle f_1, \ldots, f_s \rangle$ does not imply the remainder, r, on dividing f by F is necessarily 0.

Let $I \subseteq \mathbb{Z}_2[x_1, \ldots, x_k]$ be a non-zero ideal. Let $\mathrm{LT}(I)$ denote the set of all monomials which are leading terms of elements of I. Fix a monomial order. A finite subset $G = \{g_1, \ldots, g_t\}$ of an ideal I is said to be a *Groebner basis for I* if $\langle \mathrm{LT}(g_1), \mathrm{LT}(g_2), \ldots, \mathrm{LT}(g_t) \rangle = \langle \mathrm{LT}(I) \rangle$. A Groebner basis is *reduced* if for every $g_j \in G$, no summand of g_j is divisible by any $\mathrm{LT}(g_i)$ with $i \neq j$.

Let $G = \{g_1, \ldots, g_t\}$ be a Groebner basis for I and $f \in \mathbb{Z}_2[x_1, \ldots, x_k]$. Then there is a unique $r \in \mathbb{Z}_2[x_1, \ldots, x_k]$ such that no term of r is divisible by one of $\mathrm{LT}(g_1)$, $\mathrm{LT}(g_2)$, $\ldots$, $\mathrm{LT}(g_t)$ and $f = g + r$ for some $g \in I$. In addition, $f \in I$ if and only if $r = 0$. Then $\mathbb{Z}_2[x_1, \ldots, x_k]/I$ is isomorphic as a $\mathbb{Z}_2$ vector space to

$$\mathrm{Span}\{x : x \text{ is a monomial and } x \notin \langle \mathrm{LT}(I) \rangle\},$$

hence the set of monomials which are not divisible by any of $\mathrm{LT}(g_1)$, $\mathrm{LT}(g_2)$, $\ldots$, $\mathrm{LT}(g_t)$ is a basis for the quotient ring (i.e. it is a set of canonical equivalence class representatives).

Let $f, g \in \mathbb{Z}_2[x_1, \ldots, x_k]$ be nonzero polynomials. The *S-polynomial* of f and g is

$$S(f,g) = \frac{L}{\mathrm{LT}(f)} \cdot f + \frac{L}{\mathrm{LT}(g)} \cdot g$$

where $L = \mathrm{LCM}(\mathrm{LT}(f), \mathrm{LT}(g))$. Buchberger's algorithm states that a basis $G = \{g_1, \ldots, g_t\}$ for I is a Groebner basis if and only if for all pairs $i \neq j$ the remainder on division of $S(g_i, g_j)$ by G is zero.

3.2. **Background on Stiefel-Whitney classes and non-immersion.** Most of the following background information can be found in [10] and [7].

Let M^m denote an m-dimensional paracompact Hausdorff smooth connected manifold. Define

$$\mathrm{imm}(M^m) = \min\left\{j : M^m \text{ immerses in } \mathbb{R}^j\right\}.$$

Let $w_j(M^m)$ denote the j^{th} Stiefel-Whitney class of the manifold, i.e. the j^{th} Stiefel-Whitney class of the tangent bundle τ. Let $\overline{w}_j(M^m)$ denote the j^{th} dual Stiefel-Whitney class of M^m, i.e. the j^{th} Stiefel-Whitney class of the stable normal bundle ν. If we let $w(M^m) = 1 + w_1(M^m) + w_2(M^m) + \cdots \in H^*(M^m; \mathbb{Z}_2)$ denote the total Stiefel-Whitney class and $\overline{w}(M^m) = 1 + \overline{w}_1(M^m) + \overline{w}_2(M^m) + \cdots \in H^*(M^m; \mathbb{Z}_2)$

denote the total dual Stiefel-Whitney of M^m then $w(M^m)\,\overline{w}(M^m) = 1$. If $\bar{w}_j(M^m) \neq 0$ then $m + j \leq \operatorname{imm}(M^m)$. Thus to apply this result to Grassmann manifolds we need information regarding ν.

Let γ denote the canonical vector bundle of dimension k over $G_{k,n}$ consisting of those points (x, v) such that $x \in G_{k,n}$ and $v \in x$. Let $\gamma^{\perp}$ denote the orthogonal bundle of dimension n over $G_{k,n}$ consisting of those points (x, v) such that $x \in G_{k,n}$ and $v \perp x$. Then $\gamma \oplus \gamma^{\perp} = (n + k)\,\varepsilon$ where ε denotes the trivial line bundle over $G_{k,n}$. Let τ denote the tangent bundle of $G_{k,n}$. It is well known (c.f. [6]) that $\tau = \gamma \otimes \gamma^{\perp}$ and thus

$$\begin{aligned}
\tau \oplus \gamma \otimes \gamma &= \gamma \otimes \gamma^{\perp} \oplus \gamma \otimes \gamma \\
&= \gamma \otimes \left(\gamma^{\perp} \oplus \gamma\right) \\
&= \gamma \otimes (n + k)\,\varepsilon \\
&= (n + k)\,\gamma.
\end{aligned}$$

Computing the Stiefel-Whitney classes of both sides of the equation and solving for $1/w(G_{k,n})$ shows that the total dual Stiefel Whitney class of $G_{k,n}$ is given by

$$\overline{w}(G_{k,n}) = \frac{w(\gamma \otimes \gamma)}{w(\gamma)^{n+k}}.$$

Now $w(\gamma) = 1 + w_1 + w_2 + \cdots + w_k$. Using the formula for the Stiefel-Whitney class of a tensor product shows that for $k = 2$, $w(\gamma \otimes \gamma) = 1 + w_1^2$. Thus

$$\overline{w}(G_{2,n}) = \frac{1 + w_1^2}{(1 + w_1 + w_2)^{n+2}}.$$

Thus we can obtain lower bounds on the immersion dimension of $G_{2,n}$ by computing the highest grading in which $\frac{1+w_1^2}{(1+w_1+w_2)^{n+2}}$ is nonzero in the cohomology of $G_{2,n}$.

3.3. **Background on Modified Postnikov Towers and immersion.** Most of the following information can be found in [5].

The method of Gitler and Mahowald begins with an important theorem of Hirsch

Theorem 3.1 (Hirsch 1959)**.** *The following are equivalent.*

1. *M^m immerses in $\mathbb{R}^{m+p}$.*
2. *M^m has a normal bundle ν which is a p-plane bundle.*

(3) *There is a lifting* ν_p

$$\begin{tikzcd} & BO(p) \arrow[d, "q"] \\ M^m \arrow[ur, "\nu_p"] \arrow[r, "\nu"'] & BO \end{tikzcd}$$

of the classifying map ν *of the stable normal bundle of* M^m*.*

The obstructions to such a lifting are classes in $H^*(M^m;\pi_{*-1}(F))$ where F is the fibre of q. It is known that $F = V_p$, the Stiefel manifold of codimension p frames in $\mathbb{R}^\infty$. This has the same mod 2 cohomology ring as the stunted projective space P_p thru grading $2p$. Gitler and Mahowald's method tries to construct such liftings by handling the obstructions one Adams filtration of $\pi_*(F)$ at a time. Nussbaum [11] showed that their method could be applied to the fibration $BO(n) \to BO$ when n is odd.

Thus by Hirsch's theorem one can show $\operatorname{imm}(G_{k,n}) \leq nk + j$ if we can lift the classifying map for the normal bundle to $BO(j)$. In particular we will lift the normal bundle map for $G_{2,2^i-3}$ to $BO(2^{i+1}-9)$ to obtain our new results.

Finally, we will require Wu's formula for the action of the Steenrod algebra on $H^*(BO;\mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2, \ldots]$, namely

$$Sq^k w_m = \sum_{i=0}^{k} \binom{m+i-k-1}{i} w_{k-i} w_{m+i}.$$

4. Proofs

4.1. **Groebner bases.**

4.1.1. *Formulas for* $\overline{w}_i$*.* The elements $\overline{w}_i$ are defined in the Borel description above, by the relation $w\overline{w} = 1$. It is a simple matter using this relation to prove the following formulas for $\overline{w}_i$ which can also be found in [10]. From now on we will restrict to the case $k = 2$. In $\mathbb{Z}_2[w_1, w_2]$ we have the closed formula

(1)
$$\overline{w}_i = \sum_{a+2b=i} \binom{a+b}{a} w_2^b w_1^a$$

where the binomial coefficients are considered mod 2. Alternatively, we have a recursive formula $\overline{w}_0 = 1$, $\overline{w}_1 = w_1$, and

(2)
$$\overline{w}_i = w_1\overline{w}_{i-1} + w_2\overline{w}_{i-2} \qquad \text{for } i \geq 1.$$

Using these results we can prove the following technical lemmas. However, we first need to review some facts about binomial coefficients mod 2.

For any positive integer a, let $\alpha_i(a) \in \{0,1\}$ be the coefficient of 2^i in the binary expansion of a, i.e. $a = \sum_{i=0}^{\infty} \alpha_i(a)2^i$. We say that a and b are *disjoint* if $\alpha_i(a) + \alpha_i(b) \leq 1$ for all i. By a famous result of Lucas [8], a and b are disjoint if and only if the binomial coefficient $\binom{a+b}{b}$ is odd . We will also write $2^i \in a$ or say a contains 2^i as synonyms for $\alpha_i(a) = 1$ since the meaning will be clear from the context.

Note that since $2^i - 1 = \sum_{j=0}^{i-1} 2^j$, any integer of the form $2^i - 1$ contains all 2^j with $j < i$. Additionally if $s + t = 2^i - 1$ then s and t must be disjoint. (To see this let 2^m be the smallest two power they have in common. Then by the algorithm for base two addition, 2^m will not be in their sum, $2^i - 1$. But since $2^m \leq s \leq s + t = 2^i - 1$, we see that 2^m must be in $2^i - 1$ giving a contradiction.) Moreover, if $s + t = 2^i - 1$ then since s and t are disjoint, there can be no carries in the base two addition of s and t, so that for every $j < i$ either $2^j \in s$ or $2^j \in t$. Notice also that 2^j is in r if and only if 2^{j+1} is in $2r$ so that a and b are disjoint if and only if $2^j a$ and $2^j b$ are disjoint for any j. We will use such facts frequently in what follows.

Lemma 4.1. *In $\mathbb{Z}_2[w_1, w_2]$ for $i \geq 0$ we have*

$$\overline{w}_{2^i-1} = w_1^{2^i-1}.$$

Proof. By (1),

$$\overline{w}_{2^i-1} = \sum_{a+2b=2^i-1} \binom{a+b}{a} w_1^a w_2^b.$$

Suppose $a + 2b = 2^i - 1$. Then a and $2b$ are disjoint and if $j < i$ then 2^j is contained in either a or $2b$. Suppose $b \neq 0$. Then let 2^m be the smallest two power in $2b$. Since $2b$ is even, we have $m > 0$. Thus 2^{m-1} is not in $2b$ since m is the smallest. So 2^{m-1} is in a. But then 2^{m-1} is in b because 2^m is in $2b$. So a and b are not disjoint. So $\binom{a+b}{a}$ is zero mod 2. Thus the only summand on the right hand side of (1) which has nonzero coefficient occurs when $b = 0$, i.e. $w_1^{2^i-1}$. □

Lemma 4.2. *In $\mathbb{Z}_2[w_1, w_2]$ for $i \geq 1$ we have*

$$\overline{w}_{2^i-2} = \sum_{j=0}^{i-1} w_2^{2^j-1} w_1^{2^i-2^{j+1}}.$$

Proof: By (1) and Lucas theorem it suffices to show that if $a + 2b = 2^i - 2$ then a and b are disjoint if and only if for some $0 \leq j < i$, $b = 2^j - 1$.

Suppose $a + 2b = 2^i - 2$ and for some $0 \leq j < i$, $b = 2^j - 1$. Then solving for a we get $a = 2^{j+1}\left(2^{i-j-1} - 1\right)$. So $2^k \in a$ implies $k > j$ while $2^k \in b$ implies $k < j$, so a and b are disjoint.

Now suppose $a + 2b = 2^i - 2$ and a and b are disjoint. If $b = 0$ then $b = 2^0 - 1$. So assume $b \neq 0$. Let 2^{j-1} be the largest 2 power in b. Suppose there is $h < j - 1$ such that $2^h \notin b$ and let m be the largest such h. Then $2^{m+1} \in b$. Since $2b$ and $2^i - 2$ are even so is a. Let $A = a/2$. Then $A + b = 2^{i-1} - 1$, so A and b are disjoint and for all $l < i - 1$, $2^l \in A$ or $2^l \in b$. Thus $2^m \in b$ or $2^m \in A$. But $2^m \notin b$ by assumption so that $2^m \in A$. But $a = 2A$ so that $2^{m+1} \in a$. So $2^{m+1} \in a$ and $2^{m+1} \in b$, but this contradicts the fact that a and b are disjoint. Thus our supposition is false and so for all $h \leq j-1$, $2^h \in b$. Thus $b = 2^j - 1$. $\square$

Lemma 4.3. *In $\mathbb{Z}_2[w_1, w_2]$ for $i \geq 2$ we have*

$$\overline{w}_{2^i-3} = \sum_{j=1}^{i-1} w_2^{2^j-2} w_1^{2^i-2^{j+1}+1}.$$

Proof: By (2) we have

$$\overline{w}_{2^i-1} = w_1 \overline{w}_{2^i-2} + w_2 \overline{w}_{2^i-3}.$$

Solving for $\overline{w}_{2^i-3}$ and substituting for $\overline{w}_{2^i-2}$ and $\overline{w}_{2^i-1}$ via Lemma 4.2 and Lemma 4.1 gives us

$$\begin{aligned}
\overline{w}_{2^i-3} &= \frac{w_1 \overline{w}_{2^i-2} + \overline{w}_{2^i-1}}{w_2} \\
&= \frac{w_1 \sum_{j=0}^{i-1} w_2^{2^j-1} w_1^{2^i-2^{j+1}} + w_1^{2^i-1}}{w_2} \\
&= \frac{w_1 \left(w_1^{2^i-2} + \sum_{j=1}^{i-1} w_2^{2^j-1} w_1^{2^i-2^{j+1}} \right) + w_1^{2^i-1}}{w_2} \\
&= \sum_{j=1}^{i-1} w_2^{2^j-2} w_1^{2^i-2^{j+1}+1}.
\end{aligned}$$

$\square$

4.1.2. *Bases for $I_{2,2^i-3}$ and $I_{2,2^i-4}$.* Before proving Theorem 2.1, we prove a technical lemma.

Lemma 4.4. *Let $x, y \in G \subset \mathbb{Z}_2[x_1, \ldots, x_k]$. If x is a monomial and* $\gcd(x, \mathrm{LT}(y)) = 1$ *then* $\overline{S(x,y)}^G = 0$.

Proof. If $\gcd(x, \mathrm{LT}(y)) = 1$ then $S(x,y) = x\,\mathrm{LT}(y) + xy$. Since every summand of $x\,\mathrm{LT}(y) + xy$ is divisible by the monomial x, the remainder on division by G is zero. $\square$

We are now ready to prove Theorem 2.1 which we restate here for convenience.

Theorem 2.1. *With respect to the ordering $\prec$, and for $i \geq 3$,*

(1) $\{\overline{w}_{2^i-2}, \overline{w}_{2^i-1}\}$ *is the reduced Groebner basis for $I_{2,2^i-3}$.*
(2) $\{\overline{w}_{2^i-3}, \overline{w}_{2^i-2}, \overline{w}_{2^i-1}\}$ *is the reduced Groebner basis for $I_{2,2^i-4}$.*

Proof: First we consider $I_{2,2^i-3}$. By Borel's result $G = \{\overline{w}_{2^i-2}, \overline{w}_{2^i-1}\}$ is a basis for $I_{2,2^i-3}$. To show it is Groebner we use Buchberger's algorithm and compute $\overline{S(\overline{w}_{2^i-2}, \overline{w}_{2^i-1})}^G$. By Lemma 4.2 and Lemma 4.1 we see that $\mathrm{LT}(\overline{w}_{2^i-2}) = w_2^{2^{i-1}-1}$

and $\mathrm{LT}\left(\overline{w}_{2^i-1}\right)=w_1^{2^i-1}$. Then $\gcd\left(w_1^{2^i-1}, w_2^{2^{i-1}-1}\right)=1$ and $\overline{w}_{2^i-1}$ is a monomial, so by Lemma 4.4, $\overline{S\left(\overline{w}_{2^i-2}, \overline{w}_{2^i-1}\right)}^G=0$. So by Buchberger's algorithm G is Groebner.

To see that it is the reduced Groebner basis, it is a simple matter to check by Lemma 4.1 and Lemma 4.2 that no summand of $\overline{w}_{2^i-2}$ is divisible by $\mathrm{LT}\left(\overline{w}_{2^i-1}\right)=w_1^{2^i-1}$.

This completes the proof of the first part of the theorem. Note that this proof works for $i=2$ as well.

Now let us consider $I_{2,2^i-4}$. By Borel's result $\{\overline{w}_{2^i-3}, \overline{w}_{2^i-2}\}$ is a basis for $I_{2,2^i-4}$. By Lemmas 4.2 and 4.3 above we have $\mathrm{LT}\left(\overline{w}_{2^i-3}\right)=w_2^{2^{i-1}-2}w_1$ and $\mathrm{LT}\left(\overline{w}_{2^i-2}\right)=w_2^{2^{i-1}-1}$. Computing the S-polynomial of these two elements gives

$$\begin{aligned}
S\left(\overline{w}_{2^i-3}, \overline{w}_{2^i-2}\right) &= w_2\overline{w}_{2^i-3}+w_1\overline{w}_{2^i-2}\\
&= \overline{w}_{2^i-1}\\
&= w_1^{2^i-1}
\end{aligned}$$

by (2) and Lemma 4.1. Since neither leading term of $\overline{w}_{2^i-3}, \overline{w}_{2^i-2}$ divides into $w_1^{2^i-1}$ we must add this to our basis. Thus we let $G=\{\overline{w}_{2^i-3}, \overline{w}_{2^i-2}, \overline{w}_{2^i-1}\}$ and we want to show this is Groebner. So we check the remaining S-polynomials.

Since $\gcd\left(w_1^{2^i-1}, w_2^{2^{i-1}-1}\right)=1$ we can apply Lemma 4.4 to conclude

$$\overline{S\left(\overline{w}_{2^i-2}, \overline{w}_{2^i-1}\right)}^G=0.$$

Next we need to consider

$$\begin{aligned}
S\left(\overline{w}_{2^i-3}, \overline{w}_{2^i-1}\right) &= w_2^{2^{i-1}-2}\overline{w}_{2^i-1}+w_1^{2^i-2}\overline{w}_{2^i-3}\\
&= w_2^{2^{i-1}-2}w_1^{2^i-1}+w_1^{2^i-2}\sum_{j=1}^{i-1}w_2^{2^j-2}w_1^{2^i-2^{j+1}+1}\\
&= \sum_{j=1}^{i-2}w_2^{2^j-2}w_1^{2^i-2^{j+1}+2^i-1}\\
&= w_1^{2^i-1}\sum_{j=1}^{i-2}w_2^{2^j-2}w_1^{2^i-2^{j+1}}.
\end{aligned}$$

Thus every term of this S-polynomial is divisible by $\overline{w}_{2^i-1}=w_1^{2^i-1}$, so

$$\overline{S\left(\overline{w}_{2^i-3}, \overline{w}_{2^i-1}\right)}^G=0$$

. So G is a Groebner basis.

To see it is reduced, it is again a simple matter of checking that for $i\geq 3$ that no term of any element of G is divisible by the leading term of any other element of G. $\square$

Note that Corollary 2.2 follows immediately from Theorem 2.1 by solving the elements of the Groebner bases we found in each case for their leading terms (which is how they are used in practice).

4.2. **Non-immersion results.** In this section we illustrate how we can use the cohomology information given in Corollary 2.2 to easily reprove the non-immersion results of Oproui given in Theorem 2.4.

Theorem 2.4 (Oproui)**.** *For $i \geq 3$,*

1. *$G_{2,2^i-3}$ does not immerse in $\mathbb{R}^{2^{i+1}-3}$.*
2. *$G_{2,2^i-4}$ does not immerse in $\mathbb{R}^{2^{i+1}-3}$.*

Proof. As $G_{2,2^i-3}$ is a submanifold of $G_{2,2^i-4}$, part 1 of the theorem follows immediately from part 2. However we prove both directly to illustrate the method of using Groebner bases in each case.

We first consider $G_{2,2^i-3}$. By the discussion in §3.2 we want to determine the highest grading in $H^*\left(G_{2,2^i-3};\mathbb{Z}_2\right)$ for which

$$\overline{w}\left(G_{2,2^i-3}\right) = \frac{1+w_1^2}{\left(1+w_1+w_2\right)^{2^i-1}}$$

is nonzero. But $H^j\left(G_{2,2^i-3};\mathbb{Z}_2\right) = 0$ for $j > 2^{i+1}-6$ so $\left(1+w_1+w_2\right)^{2^{i+1}} = 1$ in the cohomology ring. So we have

$$\begin{aligned}
\frac{1+w_1^2}{\left(1+w_1+w_2\right)^{2^i-1}} &= \frac{1+w_1^2}{\left(1+w_1+w_2\right)^{2^i-1}}\left(1+w_1+w_2\right)^{2^{i+1}} \\
&= \left(1+w_1^2\right)\left(1+w_1+w_2\right)^{2^i+1} \\
&= \left(1+w_1^2\right)\left(1+w_1^{2^i}+w_2^{2^i}\right)\left(1+w_1+w_2\right).
\end{aligned}$$

Now the terms $w_1^{2^i}$ and $w_2^{2^i}$ are not in our vector space basis, so we convert them using our Groebner basis.

$$w_1^{2^i} = \left(w_1^{2^i-1}\right)w_1 = 0\cdot w_1 = 0,$$

and

$$\begin{aligned}
w_2^{2^i} &= \left(w_2^{2^{i-1}-1}\right)^2 w_2^2 \\
&= \left(\sum_{j=0}^{i-2} w_2^{2^j-1} w_1^{2^i-2^{j+1}}\right)^2 w_2^2 \\
&= \sum_{j=0}^{i-2} w_2^{2^{j+1}} w_1^{2^{i+1}-2^{j+2}} \\
&= \left(w_1^{2^i-1}\right) \sum_{j=0}^{i-2} w_2^{2^{j+1}} w_1^{2^i-2^{j+2}+1} \\
&= 0 \cdot \sum_{j=0}^{i-2} w_2^{2^{j+1}} w_1^{2^i-2^{j+2}+1} \\
&= 0.
\end{aligned}$$

Thus we see that

$$\begin{aligned}
\overline{w}\left(G_{2,2^i-3}\right) &= \left(1+w_1^2\right)\left(1+w_1+w_2\right) \\
&= 1+w_1+w_2+w_1^2+w_1^3+w_2w_1^2.
\end{aligned}$$

Since $i > 2$, these are all elements of the vector space basis, so that the highest grading nonzero term is $w_2w_1^2$ in grading 4. Thus since the dimension of $G_{2,2^i-3}$ is $2^{i+1}-6$, we see that $\operatorname{imm}\left(G_{2,2^i-3}\right) \geq 2^{i+1}-2$.

Now we consider $G_{2,2^i-4}$. We want to determine the highest grading in $H^*\left(G_{2,2^i-4};\mathbb{Z}_2\right)$ for which

$$\overline{w}\left(G_{2,2^i-4}\right) = \frac{1+w_1^2}{\left(1+w_1+w_2\right)^{2^i-2}}$$

is nonzero. But $H^j\left(G_{2,2^i-4};\mathbb{Z}_2\right) = 0$ for $j > 2^{i+1}-8$ so again $\left(1+w_1+w_2\right)^{2^{i+1}} = 1$ in the cohomology ring. So we have

$$\begin{aligned}
\frac{1+w_1^2}{\left(1+w_1+w_2\right)^{2^i-2}} &= \frac{1+w_1^2}{\left(1+w_1+w_2\right)^{2^i-2}}\left(1+w_1+w_2\right)^{2^{i+1}} \\
&= \left(1+w_1^2\right)\left(1+w_1+w_2\right)^{2^i+2} \\
&= \left(1+w_1^2\right)\left(1+w_1^{2^i}+w_2^{2^i}\right)\left(1+w_1^2+w_2^2\right).
\end{aligned}$$

Now the terms $w_1^{2^i}$ and $w_2^{2^i}$ are not in our vector space basis, so we convert them using our Groebner basis. The calculation goes through exactly as before and we find that $w_1^{2^i} = w_2^{2^i} = 0$. Thus we see that

$$\begin{aligned}\overline{w}\left(G_{2,2^i-4}\right) &= \left(1+w_1^2\right)\left(1+w_1^2+w_2^2\right)\\ &= 1+w_2^2+w_1^4+w_2^2w_1^2.\end{aligned}$$

For $i > 3$, these are all elements of the basis, so that the highest grading term is $w_2^2w_1^2$ in grading 6. Thus since the dimension of $G_{2,2^i-4}$ is $2^{i+1}-8$, we see that $\operatorname{imm}\left(G_{2,2^i-4}\right) \geq 2^{i+1}-2$.

For $i = 3$ the term $w_2^2w_1^2$ is no longer in the basis. Dividing $w_2^2w_1^2$ by our Groebner basis gives

$$w_2^2w_1^2 = \left(w_2^2w_1\right)w_1 = \left(w_1^5\right)w_1 = w_1^6$$

which is in the basis, so that the result holds for $i = 3$ as well. □

4.3. **Immersion Results.** In this section we use the technique of modified Postnikov towers described in Gitler-Mahowald [5] as extended by Nussbaum [11] to prove Theorem 2.3. Before proving the theorem, we require the following technical lemmas which are consequences of Wu's formula.

Lemma 4.5. *Let $m \geq 0$. Then in $H^*\left(BO\left(2\right);\mathbb{Z}_2\right) = \mathbb{Z}_2\left[w_1,w_2\right]$ we have*

(1) $Sq^1w_1^m = \begin{cases} 0 & \text{if } m \text{ is even} \\ w_1^{m+1} & \text{if } m \text{ is odd} \end{cases}$.

(2) $Sq^1w_2^m = \begin{cases} 0 & \text{if } m \text{ is even} \\ w_2^mw_1 & \text{if } m \text{ is odd} \end{cases}$.

(3) $Sq^2w_1^m = \begin{cases} 0 & \text{if } m \equiv 0,1 \mod 4 \\ w_1^{m+2} & \text{if } m \equiv 2,3 \mod 4 \end{cases}$.

(4) $Sq^2w_2^m = \begin{cases} 0 & \text{if } m \equiv 0 \mod 4 \\ w_2^{m+1} & \text{if } m \equiv 1 \mod 4 \\ w_2^mw_1^2 & \text{if } m \equiv 2 \mod 4 \\ w_2^{m+1}+w_2^mw_1^2 & \text{if } m \equiv 3 \mod 4 \end{cases}$.

Each formula can be proven by induction on m using the Wu formula and the Cartan formula. The details are elementary and straightforward, so we omit them here.

Lemma 4.6. *For $i \geq 3$, $Sq^2Sq^1\left(w_2^{2^{i-1}-2}w_1^{2^i-5}\right) = w_2^{2^{i-1}-2}w_1^{2^i-2}$.*

Proof: Applying the previous lemma and the Cartan formula,

$$\begin{aligned}Sq^1\left(w_2^{2^{i-1}-2}w_1^{2^i-5}\right) &= Sq^1w_2^{2^{i-1}-2}\cdot w_1^{2^i-5} + w_2^{2^{i-1}-2}\cdot Sq^1w_1^{2^i-5}\\ &= 0\cdot w_1^{2^i-5} + w_2^{2^{i-1}-2}\cdot w_1^{2^i-4}\\ &= w_2^{2^{i-1}-2}w_1^{2^i-4}.\end{aligned}$$

Thus

$$\begin{aligned}
Sq^2Sq^1\left(w_2^{2^{i-1}-2}w_1^{2^i-5}\right) &= Sq^2\left(w_2^{2^{i-1}-2}w_1^{2^i-4}\right)\\
&= Sq^2w_2^{2^{i-1}-2}\cdot w_1^{2^i-4} + Sq^1w_2^{2^{i-1}-2}\cdot Sq^1w_1^{2^i-4}\\
&+ w_2^{2^{i-1}-2}\cdot Sq^2w_1^{2^i-4}\\
&= w_2^{2^{i-1}-2}w_1^2\cdot w_1^{2^i-4} + 0\cdot 0 + w_2^{2^{i-1}-2}\cdot 0\\
&= w_2^{2^{i-1}-2}w_1^{2^i-2}.
\end{aligned}$$

□

Theorem 2.3. *For $i \geq 3$, $G_{2,2^i-3}$ immerses in $\mathbb{R}^{2^{i+2}-15}$.*

Proof: To simplify the notation we let $m = 2^{i-2}-2$. As discussed in the background section, it suffices to show that the classifying map for the normal bundle lifts to $BO\left(8m+7\right)$. Let A be the Steenrod algebra. Following the method of Gitler-Mahowald we begin by computing a minimal resolution of the A-module $M = H^j\left(V_{8m+7};\mathbb{Z}_2\right)$ for $j \leq 8m+10$ (the top nonzero grading of $H^*\left(G_{2,4m+5};\mathbb{Z}_2\right)$). In this range the resolution is

$$M \longleftarrow C_0 \longleftarrow C_1 \longleftarrow \cdots$$

where M has a generator m_7 in $H^{8m+7}\left(V_{8m+7};\mathbb{Z}_2\right)$, C_0 is a free A-module on a generator g_7 in grading $8m+7$ mapping into m_7 and C_1 is a free A-module on a generator h_{10} in grading $8m+10$ which maps into $Sq^2Sq^1g_7$. From this information we obtain the modified Postnikov tower

$$\begin{tikzcd}
 & BO(8m+7) \arrow[d] & \\
 & E_1 \arrow[r, "k_{8m+10}"'] \arrow[d] & K\left(Z_2, 8m+10\right) \\
G_{2,4m+5} \arrow[r, "\nu"'] & BO \arrow[r, "w_{8m+8}"'] & K\left(Z_2, 8m+8\right)
\end{tikzcd}$$

and it is our goal to lift the map ν up to $BO\left(8m+7\right)$.

To see that it lifts to E_1 we notice that the map labeled w_{8m+8} is the classifying map for the cohomology class $w_{8m+8} \in H^{8m+8}\left(BO;\mathbb{Z}_2\right)$. But we have shown in the proof of Theorem 2.4 that $w\left(\nu\right) = 1 + w_1 + w_2 + w_1^2 + w_1^3 + w_2w_1^2$ thus $\nu^*\left(w_{8m+8}\right) = 0$, so the map $w_{8m+8}^* \circ \nu^*$ is zero. Thus we have a lifting $l : G_{2,4m+5} \rightarrow E_1$.

To prove that we can lift this map to $BO\left(8m+7\right)$ it suffices to show that $l^{*}\left(k_{8m+10}\right)$ is zero. But the lifting l is not unique. We can vary $l^{*}\left(k_{8m+10}\right)$ through $H^{8m+7}\left(G_{2,4m+5};\mathbb{Z}_2\right)$ via the relation that produces k_{8m+10}, namely $\left(Sq^2+w_2+w_1^2\right)Sq^1w_{8m+8}=0$ in $H^{*}\left(BO;\mathbb{Z}_2\right)$ (which can be verified by the Wu formula). Thus it suffices to show that

$$l^{*}\left(k_{8m+10}\right)=\left(Sq^2+w_2\left(\nu\right)+w_1\left(\nu\right)^2\right)Sq^1x$$

for some $x\in H^{8m+7}\left(G_{2,4m+5};\mathbb{Z}_2\right)$. But since $w\left(\nu\right)=1+w_1+w_2+w_1^2+w_1^3+w_2w_1^2$ we see that $w_1\left(\nu\right)=w_1$ and $w_2\left(\nu\right)=w_2+w_1^2$. Thus we want to show $l^{*}\left(k_{8m+10}\right)=\left(Sq^2+w_2\right)Sq^1x$ for some $x\in H^{8m+7}\left(G_{2,4m+5};\mathbb{Z}_2\right)$.

The top nonzero class in $H^{*}\left(G_{2,4m+5};\mathbb{Z}_2\right)$ is in grading $8m+10$. In the description of cohomology we found in Corollary 2.2, this top class is $w_2^{2^{i-1}-2}w_1^{2^i-2}$. So either $l^{*}\left(k_{8m+10}\right)=0$ or else $l^{*}\left(k_{8m+10}\right)=w_2^{2^{i-1}-2}w_1^{2^i-2}$. By Lemma 4.6, we see that $w_2^{2^{i-1}-2}w_1^{2^i-2}=Sq^2Sq^1\left(w_2^{2^{i-1}-2}w_1^{2^i-5}\right)$. Also by Corollary 2.2

$$\begin{aligned}
w_2Sq^1\left(w_2^{2^{i-1}-2}w_1^{2^i-5}\right)&=w_2\left(w_2^{2^{i-1}-2}w_1^{2^i-4}\right)\\
&=\left(w_2^{2^{i-1}-1}\right)w_1^{2^i-4}\\
&=\left(\sum_{j=0}^{i-2}w_2^{2^j-1}w_1^{2^i-2^{j+1}}\right)w_1^{2^i-4}\\
&=\left(w_1^{2^i-1}\right)\sum_{j=0}^{i-2}w_2^{2^j-1}w_1^{2^i-2^{j+1}-3}\\
&=0\cdot\sum_{j=0}^{i-2}w_2^{2^j-1}w_1^{2^i-2^{j+1}-3}\\
&=0.
\end{aligned}$$

Hence, $\left(Sq^2+w_2\right)Sq^1\cdot\left(w_2^{2^{i-1}-2}w_1^{2^i-5}\right)=w_2^{2^{i-1}-2}w_1^{2^i-2}$ as required. Thus we can lift to $BO\left(8m+7\right)$ which completes the proof. □

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