PSEUDOPERIODICITY AND THE $3x+1$ CONJUGACY FUNCTION

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Abstract. The $3x+1$ function $T:\mathbb{N}\to\mathbb{N}$ is defined by $T(x)=\frac{3x+1}{2}$ for x odd and $T(x)=\frac{x}{2}$ for x even. The function T has a natural extension to the 2-adic integers $\mathbb{Z}_2$ and there is a continuous function Φ which conjugates T to the 2-adic shift map σ. Bernstein and Lagarias conjectured that -1 and $\frac{1}{3}$ are the only odd fixed points of Φ. In this paper we investigate periodicity associated with Φ, a property of the map which is a natural extention of solenoidality. We use it to show that there are nontrivial infinite families of 2-adics that are not fixed points of Φ. In particular, we prove that three sequences of farPoints of 2-adics are finitely pseudoperiodic, providing more evidence supporting the Φ Fixed Point Conjecture.

1. Introduction

Let $\mathbb{Z}_2$ denote the ring of 2-adic integers. If $x\in\mathbb{Z}_2$ with $x=\sum_{i=0}^{\infty}d_i2^i$ with $d_i\in\{0,1\}$, we write $x=d_0d_1d_2\cdots$. Define the $3x+1$ function $T:\mathbb{Z}_2\to\mathbb{Z}_2$ as follows.

$$T(x)=\begin{cases}\frac{3x+1}{2} & \text{if } x \text{ is odd}\\ \frac{x}{2} & \text{if } x \text{ is even}\end{cases}$$

The famous $3x+1$ conjecture states that for every $x\in\mathbb{N}^+$ there is some $k\in\mathbb{N}^+$ such that the k-th iterate $T^k(x)=1$, i.e. the conjecture asserts that the T-orbit of every positive integer will eventually enter the cycle $\{1,2\}$. Define the 2-adic shift map $\sigma:\mathbb{Z}_2\to\mathbb{Z}_2$ by

$$\sigma(x)=\begin{cases}\frac{x-1}{2} & \text{if } x \text{ is odd}\\ \frac{x}{2} & \text{if } x \text{ is even}\end{cases}$$

so that $\sigma(d_0d_1\cdots)=d_1d_2\cdots$. Hence σ just removes the first digit of a 2-adic integer[1]. There is a continuous bijective map $\Phi:\mathbb{Z}_2\to\mathbb{Z}_2$ such that the following diagram commutes:

$$\begin{array}{ccc}\mathbb{Z}_2 & \xrightarrow{\sigma} & \mathbb{Z}_2\\ \Phi\downarrow & & \Phi\downarrow\\ \mathbb{Z}_2 & \xrightarrow{T} & \mathbb{Z}_2\end{array} \tag{1.1}$$

so that $\Phi\circ\sigma=T\circ\Phi$ [2]. The map Φ is **solenoidal**, that is, $x\underset{2^n}{\equiv}y\Rightarrow\Phi(x)\underset{2^n}{\equiv}\Phi(y)$ [1] (where $\underset{2^n}{\equiv}$ denotes congruence $\bmod 2^n$). In fact, Φ induces permutations on $\mathbb{Z}/2^n\mathbb{Z}$.

An explicit formula for Φ was discovered by Bernstein [3], but a convenient one is not known for its inverse. If $x=\sum_i 2^{d_i}$, with $0\le d_0<d_1<\cdots$, then

$$\Phi(x)=-\sum_i 2^{d_i}3^{-i} \tag{1.2}$$

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[1] σ is called S in [1]

It follows from the conjugacy in (1.1) that the inverse map is given by

$$\Phi^{-1}(x) = \sum_{i=0}^{\infty} \left(T^i(x)\ Mod\ 2\right) 2^i$$

and we say that Φ^{-1} maps a 2-adic to its *parity vector*[2].

Many of the interesting properties of the $3x+1$ conjugacy function do not lead to an apparent solution to the $3x+1$ problem [1]. The map Φ seems to inherit much of the problematic nature of T, and the authors of [1] have proposed the seemingly equally intractable

Φ **Fixed Point Conjecture:** The only odd fixed points of Φ are -1 and $\frac{1}{3}$.

From (1.2), we see that $\Phi\left(2^k x\right) = 2^k \Phi(x)$ so that every even fixed point of Φ is either 0 or $2^k x$, where x is an odd fixed point. There is another more general conjecture proposed in [1] stating that for any given 2 power, Φ will have finitely many odd periodic points whose period is that 2 power.

2. Main Results

We now provide evidence supporting the Φ Fixed Point Conjecture by demonstrating a periodicity associated with Φ and a related property which we will call pseudoperiodicity. Solenoidality allows us to study the action of Φ on the first k digits of a 2-adic; however, periodicity will allow us to look at beginning of the infinite "tail" end. First, we define what it means for a sequence to be pseudoperiodic.

Definition 2.1. *Let X be a set, and let $\{a_n\}$ be a sequence (where $a : \mathbb{N} \to X$). For all $n \in \mathbb{N}$ we say that the n-th term a_n is* ***pseudoperiodic*** *if*

$$\exists m \in \mathbb{N}\ \forall r \in \mathbb{N}^+\ a_{n+mr} = a_n$$

We say that the sequence $\{a_n\}$ is pseudoperiodic if all of its terms are pseudoperiodic.

We will also say that the sequence $\{a_n\}_{n \geq k}$ is pseudoperiodic if we replace "For all $n \in \mathbb{N}$" in the above definition with "For all $n \geq k$". It will be convenient in some applications to take $X \cup \{\infty\}$, where $X \subseteq \mathbb{R}$ to be the codomain of a and say that in this case $\{a_n\}$ is **finitely pseudoperiodic** when all terms $a_n \neq \infty$ are pseudoperiodic.

Hence a sequence is pseudoperiodic if for every term in the sequence, there is some period m with which that term will repeat in the sequence. For example, every periodic sequence is pseudoperiodic. For a somewhat less trivial example, consider $b : \mathbb{N} \to \mathbb{N}$, where b_n is defined to be the smallest positive prime dividing $n+2$. The pseudoperiodicity of this sequence is reminiscent of the sieve of Eratosthenes.

Because Φ is solenoidal, "the action of Φ on the first k digits of a 2-adic" is well defined. That is, the first k digits of $\Phi(x)$ are completely determined by the first k digits of x. Therefore we can classify non-fixed points by their congruence class $\bmod 2^k$. In fact, we will say that x **is a fixed point of** $\Phi \bmod 2^k$ when $\Phi(x) \underset{2^k}{\equiv} x$. Consequently, whenever x is a fixed point $\bmod 2^k$, we can ask how many more digits r are needed so that there is no fixed point $\bmod 2^{k+r}$ having the same first k digits as x. This motivates the following definition.

[2] As given in [1], the formula for $\Phi^{-1}(x)$ uses mod instead of Mod, but throughout the course of this paper we will keep the convention that for $z \in \mathbb{Z}_2$ and $k \in \mathbb{N}$, $z\ Mod\ 2^k$ will be the least $m \in \mathbb{N}$ such that $m \underset{2^k}{\equiv} z$.

Definition 2.2. *For $k \in \mathbb{N}$ and $x \in \mathbb{N} \subseteq \mathbb{Z}_2$ with $x < 2^k$, define*

$$\mathrm{fP}\,(x,k) = \min \left\{ r \in \mathbb{N} \mid \forall z \in \mathbb{Z}_2,\ z \underset{2^k}{\equiv} x \Rightarrow \Phi\,(z) \underset{2^{k+r}}{\not\equiv} z \right\}$$

when the minimum exists, and ∞ otherwise.

That is, $\mathrm{fP}\,(x,k)$ is the smallest number of digits r needed so that no 2-adic agreeing with x on the first k digits will be a fixed point of $\Phi \bmod 2^{k+r}$. We call $\mathrm{fP}\,(x,k)$ the **farPoint of x on the first k digits**.

Example 2.3. *If $\Phi\,(x) \underset{2^k}{\not\equiv} x$ then $\mathrm{fP}\,(x,k) = 0$. The converse is also true.*

Example 2.4. $\mathrm{fP}\,(0,k) = \infty$ *for any k, since $\Phi\,(0) = 0$.*

Example 2.5. *Since $\Phi\,(1) \underset{2^2}{\equiv} 1$ and $\Phi\,(1) \underset{2^3}{\equiv} 5$ we have that $\mathrm{fP}\,(1,2) \neq 0$ and $\mathrm{fP}\,(1,2) = 1$.*

For $x \in \mathbb{Z}_2$, denote $L_k\,(x) = x\ Mod\ 2^k$ and $R_k\,(x) = \sigma^k\,(x)$. $L_k\,(x)$ and $R_k\,(x)$ will be called the **(k-)left** and **(k-)right** parts of x, respectively. Thus, if $x = d_0 d_1 \cdots d_{k-1} d_k \cdots$, then $L_k\,(x) = d_0 d_1 \cdots d_{k-1}$ and $R_k\,(x) = d_k d_{k+1} \cdots$. So, for example, $L_3\left(\frac{1}{3}\right) = 3$ and $R_3\left(\frac{1}{3}\right) = -\frac{1}{3}$, since $\frac{1}{3} = 1101010\cdots$ and $-\frac{1}{3} = 101010\cdots$. With these definitions we are able to state our first result.

Theorem 2.6. *Fix $H \geq 3$ and $q \in \mathbb{Q} \cap \mathbb{Z}_2$. Then the sequence $\{L_H R_n \Phi L_n\,(q)\}$ is eventually periodic.*

We call the property of Φ exhibited in this theorem the *periodicity of* Φ. We can use this property to make progress on the Fixed Point Conjecture by showing that certain sequences of farPoints are pseudoperiodic.

Theorem 2.7. *The sequence $\{\mathrm{fP}\,(2^n - 1, n+1)\}$ is finitely pseudoperiodic.*

If the Φ Fixed Point Conjecture is true, then all terms in the above sequence will be finite (and hence pseudoperiodic) except for when $n = 0, 2$. Any odd $x \in \mathbb{Z}_2 - \left\{\frac{1}{3}, -1\right\}$ that is a fixed point of Φ will agree with either -1 or $\frac{1}{3}$ on a finite number of digits. Direct calculation shows that such a fixed point x must agree with one of these fixed points on at least its first 100 digits. Theorem 2.7 corresponds to those potential fixed points whose initial segments agree with -1 on a finite number of digits. We have an analogous theorem for $\frac{1}{3}$:

Theorem 2.8. *The sequences $\left\{\mathrm{fP}\left(1 + \sum_{i=0}^{n} 2^{2i+1}, 2n+4\right)\right\}$ and $\left\{\mathrm{fP}\left(1 + \left(\sum_{i=0}^{n} 2^{2i+1}\right) + 2^{2n+2}, 2n+3\right)\right\}$ are finitely psuedoperiodic.*

Note that $\frac{1}{3} = 1 + \sum_{i=0}^{\infty} 2^{2i+1}$. Knowing that a particular value of farPoint is finite in any of the three sequences in Theorems 2.7 and 2.8 yields an infinite family of non-fixed points of Φ. In particular, the above theorems imply

Corollary 2.9. *If there are no fixed points of the form $\overbrace{11\cdots 1}^{n\ ones}0\cdots$ (respectively, $11\overbrace{010\cdots 10}^{n\ ones}0\cdots$ or $11\overbrace{010\cdots 1}^{n\ ones}1\cdots$), then there will be some h such that for every $r \in N$, there will be no fixed point of the form $\overbrace{1111\cdots 1}^{n+rh\ ones}0\cdots$ (respectively, $11\overbrace{010\cdots 10}^{n+rh\ ones}0\cdots$ or $11\overbrace{010\cdots 1}^{n+rh\ ones}1\cdots$).*

Interestingly, the h in Corollary 2.9 turns out to be a 2 power in every case. Some values of farPoint are listed in 1, from which we obtain an immediate example of the pseudoperiodicity from Theorem 2.7.

n	fP_n	n	fP_n	n	fP_n	n	fP_n	n	fP_n	n	fP_n	n	fP_n	n	fP_n
1	1	10	5	19	1	28	6	37	1	46	7	55	1	64	45
2	∞	11	1	20	8	29	1	38	11	47	1	56	37	65	1
3	1	12	18	21	1	30	12	39	1	48	13	57	1	66	10
4	7	13	1	22	6	31	1	40	10	49	1	58	5	67	1
5	1	14	11	23	1	32	62	41	1	50	8	59	1	68	7
6	10	15	1	24	11	33	1	42	5	51	1	60	6	69	1
7	1	16	83	25	1	34	13	43	1	52	9	61	1	70	12
8	29	17	1	26	5	35	1	44	10	53	1	62	13	71	1
9	1	18	75	27	1	36	15	45	1	54	6	63	1	72	9

TABLE 1. Values of $\mathrm{fP}_n = \mathrm{fP}\left(2^n - 1, n+1\right)$ for n from 1 to 72

Example 2.10. *We see that we can have no Φ fixed point of the form $1110\cdots$ from Table 1, and in this case we have that the h in Corollary 2.9 will be equal to 2. So there will be no fixed points of the form $111110\cdots$, $11111110\cdots$, or in general $\overbrace{1111\cdots1}^{1+2r \text{ ones}}0\cdots$ for any $r \in \mathbb{N}$, so that there is no fixed point of Φ whose digit expansion will have an odd number of ones followed by a zero.*

We will see why $h = 2$ is the period of this value of farPoint in the proof of Theorem 2.7 in the next section. Continuing the table further would illustrate the pseudoperiodicity of all of these farPoints. For example, $\mathrm{fP}\left(2^{10} - 1, 11\right)$ is pseudoperiodic with period $h = 2^{5-1} = 16$, as seen by the farPoint value of 5 for $n = 10, 26, 42, 58$ in the above table.

3. Pseudohomomorphism and Φ Periodicity

Here we discuss some general properties of Φ that will be used in the proofs of our main results. If $x \in \mathbb{N} \subseteq \mathbb{Z}_2$, then we define $\alpha(x) = \sum_{i=0}^{\infty} d_i$ where $x = d_0 d_1 \cdots$. The following theorem shows that, with sufficient restrictions, Φ acts on the right and left parts of a 2-adic nearly as an additive homomorphism.

Theorem 3.1. *Let $x \in \mathbb{Z}_2$, $k \in \mathbb{N}$ and $a \in \mathbb{N}$ with $a < 2^k$. Denote $m = \alpha(a)$. Then*

$$\Phi\left(a + 2^k x\right) = \Phi(a) + \frac{\Phi(x)}{3^m} 2^k$$

Proof. If $a = 2^{d_0} + 2^{d_1} + \cdots + 2^{d_{m-1}}$ and $x = 2^{d_m} + 2^{d_{m+1}} + \cdots$, with $0 \le d_0 < d_1 < \cdots < d_{m-1}$ and $0 \le d_m < d_{m+1} < \cdots$, then by (1.2),

$$\begin{aligned}
\Phi\left(a + 2^k x\right) &= -\left(\sum_{i=0}^{m-1} 2^{d_i} 3^{-i} + \sum_{i=0}^{\infty} 2^{d_{i+m}+k} 3^{-(i+m)}\right) \\
&= -\sum_{i=0}^{m-1} 2^{d_i} 3^{-i} + \frac{2^k}{3^m}\left(-\sum_{i=0}^{\infty} 2^{d_{i+m}} 3^{-i}\right) \\
&= \Phi(a) + \Phi(x) \frac{2^k}{3^m}
\end{aligned}$$

□

Thus, using the (k-)left and (k-)right notation, for any $x \in \mathbb{Z}_2$,

$$\Phi\left(x\right) = \Phi\left(L_k\left(x\right)\right) + \frac{\Phi\left(R_k\left(x\right)\right)}{3^m} 2^k \tag{3.1}$$

for any k, where $m = \alpha\left(L_k\left(x\right)\right)$. Whenever m is divisible by a positive power of 2, the factor 3^m acts as 1 for sufficiently low modulus, so that $\Phi\left(a + 2^k x\right) \equiv \Phi\left(a\right) + \Phi\left(x\right) 2^k$, as can be seen in the following Lemma.

Lemma 3.2. *Let $n \in \mathbb{N}^+$. Then $3^{2^n} \underset{2^{n+2}}{\equiv} 1$.*

Proof. The proof will be by induction. We have that $3^2 \underset{2^3}{\equiv} 1$. For $n \in \mathbb{N}^+$, assume $3^{2^n} \underset{2^{n+2}}{\equiv} 1$. Then for some $k \in \mathbb{Z}$, $2^{n+2}k+1 = 3^{2^n}$, and hence $\left(2^{n+2}\right)^2 k^2 + 2 \cdot 2^{n+2}k + 1 = 3^{2^{n+1}}$. The result follows. □

Recall that $\mathbb{Z}_2$ contains a subring of the rational numbers, namely those with odd denominator when in reduced form. Define $\mathbb{Q}_{odd} = \mathbb{Q} \cap \mathbb{Z}_2$ to be this subring, as in [4]. Then any $q \in \mathbb{Q}_{odd}$ must be of the form $q = d_0 d_1 d_2 \cdots d_{k-1} \overline{d_k d_{k+1} \cdots d_{k+v-1}}$. Hence for any $q \in \mathbb{Q}_{odd}$, there will be some $a, k, b, v \in \mathbb{N}$, with $a < 2^k$ and $b < 2^v$ such that $q = a + 2^k \left(\sum_{i=0}^{\infty} b 2^{vi}\right)$. To simplify this, we introduce the following notation. For $b, v, t \in \mathbb{N}$, we denote $\bar{b}_{v,t} = b \sum_{i=0}^{t-1} 2^{vi}$ and $\bar{b}_{v,\infty} = b \sum_{i=0}^{\infty} 2^{vi}$. In this notation, $q = a + 2^k \bar{b}_{v,\infty}$. There is a strong relationship between the action of Φ on a rational 2-adic and on a 2-adic that results from a truncation after a finite number of its repeating units, as is given by the following theorem.

Theorem 3.3. *Let $q \in \mathbb{Q}_{odd}$ and $a, k, b, v \in \mathbb{N}$ such that $a < 2^k, b < 2^v$, and $q = a + 2^k \bar{b}_{v,\infty}$. Define $m = \alpha\left(b\right)$. Let $t \in \mathbb{N}^+$. Then*

$$\Phi\left(a + 2^k \bar{b}_{v,t}\right) = \Phi\left(q\right) - \frac{\Phi\left(\bar{b}_{v,\infty}\right)}{3^{mt+\alpha(a)}} 2^{k+tv}$$

Proof. Clearly $L_{k+tv}\left(q\right) = a + 2^k \bar{b}_{v,t}$ and $R_{k+tv}\left(q\right) = \bar{b}_{v,\infty}$, so by again Theorem 3.1,

$$\Phi\left(q\right) = \Phi\left(a + 2^k \bar{b}_{v,t} + 2^{k+tv} \bar{b}_{v,\infty}\right) = \Phi\left(a + 2^k \bar{b}_{v,t}\right) + \frac{\Phi\left(\bar{b}_{v,\infty}\right)}{3^{mt+\alpha(a)}} 2^{k+tv}$$

□

We also state the analogous theorem for 2-adics whose digits are strictly repeating.

Corollary 3.4. *Let $v \in \mathbb{N}^+, t \in \mathbb{N}$ and let $b \in \mathbb{N}$ such that $b < 2^k$. Define $m = \alpha\left(b\right)$. Then*

$$\Phi\left(\bar{b}_{v,t}\right) = \Phi\left(\bar{b}_{v,\infty}\right) - \frac{\Phi\left(\bar{b}_{v,\infty}\right)}{3^{mt}} 2^{tv}$$

Proof. Let $q = \bar{b}_{v,\infty}$ in Theorem 3.3 □

Now we present the Φ periodicity result for rational 2-adics. It is well known that $\Phi\left(\mathbb{Q}_{odd}\right) \subseteq \mathbb{Q}_{odd}$, since if a 2-adic has a rational parity vector, then it is the solution of a linear equation with rational coefficients. Thus Φ of any rational number will have eventually repeating digits. The following result, which does not take into account fixed points, can be used to obtain the desired pseudoperiodicity of the farPoint sequences.

Theorem 3.5. *Let $q \in \mathbb{Q}_{odd}$, and $a, k, b, v \in \mathbb{N}$ such that $a < 2^k, b < 2^v$ and $q = a + 2^k \bar{b}_{v,\infty}$. Let*

$$\Phi(q) = r_0 r_1 \cdots r_{u-1} \overline{r_u r_{u+1} \cdots r_{u+p-1}}$$

and choose t such that $k + tv \geq u$. Then for all positive integers H,

$$R_{k+tv} \Phi L_{k+tv}(q) \underset{2^{H+2}}{\equiv} R_{k+(t+p2^H)v} \Phi L_{k+(t+p2^H)v}(q)$$

Proof. Define $m = \alpha(b)$. From Theorem 3.3 we see that

$$\begin{aligned} R_{k+tv}\left(\Phi\left(q \; Mod \; 2^{k+tv}\right)\right) &= R_{k+tv}\left(\Phi\left(a + 2^k \bar{b}_{v,t}\right)\right) \\ &= R_{k+tv}\left(\Phi(q) - \frac{\Phi\left(\bar{b}_{v,\infty}\right)}{3^{mt+\alpha(a)}} 2^{k+tv}\right) \end{aligned}$$

and also that

$$\begin{aligned} R_{k+(t+p2^H)v}\left(\Phi\left(q \; Mod \; 2^{k+\left(t+p2^H\right)v}\right)\right) &= R_{k+(t+p2^H)v}\left(\Phi\left(a + 2^k \bar{b}_{v,t+p2^H}\right)\right) \\ &= R_{k+(t+p2^H)v}\left(\Phi(q) - \frac{\Phi\left(\bar{b}_{v,\infty}\right)}{3^{m\left(t+p2^H\right)+\alpha(a)}} 2^{k+\left(t+p2^H\right)v}\right) \end{aligned}$$

But by Lemma 3.2,

$$\frac{\Phi\left(\bar{b}_{v,\infty}\right)}{3^{mt+\alpha(a)}} \underset{2^{H+2}}{\equiv} \frac{\Phi\left(\bar{b}_{v,\infty}\right)}{3^{mt+\alpha(a)}} \frac{1}{\left(3^{2^H}\right)^{mp}} = \frac{\Phi\left(\bar{b}_{v,\infty}\right)}{3^{m\left(t+p2^H\right)+\alpha(a)}}$$

Now, since $k + tv \geq u$,

$$R_{k+tv}(\Phi(q)) = R_{k+(t+p2^H)v}(\Phi(q))$$

Since for any $y, z \in \mathbb{Z}_2$ and $c \in \mathbb{N}$,

$$R_c(y + 2^c z) = R_c(y) + z \tag{3.2}$$

it follows that

$$\begin{aligned} R_{k+tv}\left(\Phi(q) - \tfrac{\Phi\left(\bar{b}_{v,\infty}\right)}{3^{mt+\alpha(a)}} 2^{k+tv}\right) &\underset{2^{H+2}}{\equiv} R_{k+tv}\Phi(q) - \tfrac{\Phi\left(\bar{b}_{v,\infty}\right)}{3^{mt+\alpha(a)}} \\ &\underset{2^{H+2}}{\equiv} R_{k+(t+p2^H)v}\Phi(q) - \tfrac{\Phi\left(\bar{b}_{v,\infty}\right)}{3^{m\left(t+p2^H\right)+\alpha(a)}} \\ &\underset{2^{H+2}}{\equiv} R_{k+(t+p2^H)v}\left(\Phi(q) - \tfrac{\Phi\left(\bar{b}_{v,\infty}\right)}{3^{m\left(t+p2^H\right)+\alpha(a)}} 2^{k+\left(t+p2^H\right)v}\right) \end{aligned}$$

where the first and third lines in particular follow from (3.2), and the result follows. □

There is a similar, simpler, result for strictly repeating 2-adics:

Corollary 3.6. *Let $v \in \mathbb{N}^+$, and $b \in \mathbb{N}$ such that $b < 2^v$. Define $m = \alpha(b)$. Let*

$$\Phi\left(\bar{b}_{v,\infty}\right) = r_0 r_1 \cdots r_{u-1} \overline{r_u r_{u+1} \cdots r_{u+p-1}}$$

and choose $t \in \mathbb{N}$ such that $tv \geq u$. Then for all positive integers H,

$$R_{tv}\left(\Phi\left(\bar{b}_{v,t}\right)\right) \underset{2^{H+2}}{\equiv} R_{(t+p2^H)v}\left(\Phi\left(\bar{b}_{v,t+p2^H}\right)\right)$$

Proof. Let $q = \bar{b}_{v,\infty}$ in Theorem 3.5. □

If q is a fixed point of Φ, then $k = u$ and $v = p$ in Theorem 3.5. We are in a position to prove Theorem 2.6.

4. Proofs of Main Results

Proof of Theorem 2.6. Let $q \in \mathbb{Q}_{odd}$ and $a', k', b', v \in \mathbb{N}$ such that $a' < 2^{k'}, b' < 2^v$, and $q = a' + 2^{k'}\overline{b'}_{v,\infty}$ and let $H \geq 3$ be fixed. Let t be the smallest positive natural number such that $k'+tv \geq u$, where $\Phi(q) = r_0 r_1 \cdots r_{u-1}\overline{r_u r_{u+1} \cdots r_{u+p-1}}$. Let $n \geq k'+tv$, and define $l = n-(k'+tv)$. By Theorem 3.5 with $k = k'+l$, $a = L_k(q)$, $b = L_v R_k(q)$,

$$\begin{aligned} R_n \Phi L_n(q) &= R_{k+tv}\Phi L_{k+tv}(q) \\ &\underset{2^H}{\equiv} R_{k+(t+p2^{H-2})v}\Phi L_{k+(t+p2^{H-2})v}(q) \\ &= R_{n+p2^{H-2}v}\Phi L_{n+p2^{H-2}v}(q) \end{aligned}$$

Hence $L_H R_n \Phi L_n(q) = L_H R_{n+p2^{H-2}v}\Phi L_{n+p2^{H-2}v}(q)$. Therefore the sequence $\{L_H R_n \Phi L_n(q)\}_{n \geq k'+tv}$ is periodic. □

From Corollary 3.6, the result that $\{\mathrm{fP}(2^n-1, n+1)\}$ is pseudoperiodic follows. To see this, we will consider 2-adics whose first n digits agree with those of the fixed point -1.

Proof of Theorem 2.7. Let $n \in \mathbb{N}$ and assume that the farPoint of 2^n-1 on the first $n+1$ digits is finite. Let $f = \mathrm{fP}(2^n-1, n+1)$ and $H \geq \max\{1, f-1\}$. So there is no $z \in \mathbb{Z}_2$ such that $z \underset{2^{n+1}}{\equiv} 2^n-1$ and $\Phi(z) \underset{2^{n+1+f}}{\equiv} z$, by definition of farPoint. Let $y \in \mathbb{Z}_2$ be arbitrary and define

$$z = 2^n - 1 + 2^{n+1}y \text{ and } z' = 2^{n+2^H} - 1 + 2^{n+1+2^H}y$$

Then z is not a fixed point of $\Phi \bmod 2^{n+1+f}$, and by Theorem 3.1, $\Phi(z) = \Phi(2^n-1) + \frac{\Phi(2y)}{3^n}2^n$. Similarly, $\Phi(z') = \Phi\left(2^{n+2^H}-1\right) + \frac{\Phi(2y)}{3^{n+2^H}}2^{n+2^H}$. Furthermore, since $3^{2^H} \underset{2^{H+2}}{\equiv} 1$ by Lemma 3.2 and $R_{n+2^H}\Phi\left(2^{n+2^H}-1\right) \underset{2^{H+2}}{\equiv} R_n\Phi(2^n-1)$ by Corollary 3.6, letting $b = v = 1$, again using (3.2),

$$\begin{aligned} R_{n+2^H}\Phi(z') &\underset{2^{H+2}}{\equiv} R_{n+2^H}\Phi\left(2^{n+2^H}-1\right) + \tfrac{\Phi(2y)}{3^{m+2^H}} \\ &\underset{2^{H+2}}{\equiv} R_n\Phi(2^n-1) + \tfrac{\Phi(2y)}{3^m} \\ &\underset{2^{H+2}}{\equiv} R_n\Phi(z) \end{aligned}$$

Furthermore, it is clear from how we have defined z and z' that $R_n(z) = 2y = R_{n+2^H}(z')$. Now, by definition of farPoint, $R_n(\Phi(z)) \underset{2^{f+1}}{\not\equiv} R_n(z)$, and hence we have that

$$R_{n+2^H}(\Phi(z')) \underset{2^{H+2}}{\equiv} R_n(\Phi(z)) \underset{2^{f+1}}{\not\equiv} R_n(z) \underset{2^{H+2}}{\equiv} R_{n+2^H}(z')$$

Thus $\mathrm{fP}\left(2^{n+2^H}-1, n+1+2^H\right) \leq f$.

To show the other inequality, we choose $y \in \mathbb{Z}_2$ so that if $z = 2^n-1+2^{n+1}y$, then $\Phi(z) \underset{2^{n+f}}{\equiv} z$. We know that such a y exists by definition of farPoint. As before, we will define $z' = 2^{n+2^H} - 1 + 2^{n+1+2^H}y$ for this choice of y, and from the above discussion we know that $R_{n+2^H}\Phi(z') \underset{2^f}{\equiv} R_n\Phi(z)$ and $R_n(z) \underset{2^f}{\equiv} R_{n+2^H}(z')$ since $f \leq H+1$. However, since $\Phi(z) \underset{2^{n+f}}{\equiv} z$, now $R_n(\Phi(z)) \underset{2^f}{\equiv} R_n(z)$, and therefore

$$R_{n+2^H}(\Phi(z')) \underset{2^f}{\equiv} R_n(\Phi(z)) \underset{2^f}{\equiv} R_n(z) \underset{2^f}{\equiv} R_{n+2^H}(z')$$

Clearly, $\Phi(z') \underset{2^{n+2^H}}{\equiv} z'$ by solenoidality since $\Phi(-1) = -1$, and therefore $\mathrm{fP}\left(2^{n+2^H}-1, n+1+2^H\right) = \mathrm{fP}(2^n-1, n+1)$. This completes the proof. □

The proof of Theorem 2.8 is similar, but relies instead upon Theorem 3.5.

Proof of Theorem 2.8. This time we will work with those 2-adic initial segments agreeing nontrivially with $\frac{1}{3} = \Phi\left(\frac{1}{3}\right)$, and so we apply Theorem 3.5 with $q = \frac{1}{3}$. Once again, for $n \in \mathbb{N}$, assume that $f = \mathrm{fP}\left(1 + \left(\sum_{i=0}^{n} 2^{2i+1}\right) + 2^{2n+2}, 2n+3\right)$ is finite and let $H \geq \max\{1, f-1\}$. Note that $1 + \sum_{i=0}^{n} 2^{2i+1} \equiv \frac{1}{3}\ Mod\ 2^{2n+2}$, and that there is no $z \in \mathbb{Z}_2$ with $z \underset{2^{2n+3}}{\equiv} 1 + \left(\sum_{i=0}^{n} 2^{2i+1}\right) + 2^{2n+2}$ and $\Phi(z) \underset{2^{2n+3+f}}{\equiv} z$. Let $y \in \mathbb{Z}_2$ and define

$$\begin{aligned} z &= 1 + \left(\sum_{i=0}^{n} 2^{2i+1}\right) + 2^{2n+2} + 2^{2n+3} y \\ z' &= 1 + \left(\sum_{i=0}^{n+2^{H+1}} 2^{2i+1}\right) + 2^{2\left(n+2^{H+1}\right)+2} + 2^{2\left(n+2^{H+1}\right)+3} y \end{aligned}$$

Here z is not a fixed point of $\Phi \bmod 2^{2n+3+f}$. By Theorem 3.1,

$$\Phi(z) = \Phi\left(1 + \left(\sum_{i=0}^{n} 2^{2i+1}\right)\right) + \frac{\Phi(1+2y)}{3^{n+2}} 2^{2n+2}$$

and also

$$\Phi(z') = \Phi\left(1 + \left(\sum_{i=0}^{n+2^{H+1}} 2^{2i+1}\right)\right) + \frac{\Phi(1+2y)}{3^{n+2^{H+1}+2}} 2^{2\left(n+2^{H+1}\right)+2}$$

Again, $3^{2^{H+1}} \underset{2^{H+2}}{\equiv} 1$ by Lemma 3.2, and

$$R_{2+2n+2^{H+2}} \Phi\left(1 + \left(\sum_{i=0}^{n+2^{H+1}} 2^{2i+1}\right) + 2^{2\left(n+2^{H+1}\right)+2}\right) \underset{2^{H+2}}{\equiv} R_{2+2n} \Phi\left(1 + \left(\sum_{i=0}^{n} 2^{2i+1}\right) + 2^{2n+2}\right)$$

by Theorem 3.5 with $b = k = v = 2$ and $a = 3$. Therefore,

$$\begin{aligned} R_{2+2n+2^{H+2}}\left(\Phi(z')\right) &\underset{2^{H+2}}{\equiv} R_{2+2n+2^{H+2}} \Phi\left(1 + \left(\sum_{i=0}^{n+2^{H+1}} 2^{2i+1}\right) + 2^{2\left(n+2^{H+1}\right)+2}\right) + \frac{\Phi(1+2y)}{3^{n+2^{H+1}+2}} \\ &\underset{2^{H+2}}{\equiv} R_{2+2n} \Phi\left(1 + \left(\sum_{i=0}^{n} 2^{2i+1}\right) + 2^{2n+2}\right) + \frac{\Phi(1+2y)}{3^{n+2}} \\ &\underset{2^{H+2}}{\equiv} \Phi(z) \end{aligned}$$

by (3.2). It is again clear from how z and z' are defined that $R_{2+2n}(z) = 1 + 2y = R_{2+2n+2^{H+2}}(z')$. By definition of farPoint, $R_{2+2n}(\Phi(z)) \underset{2^{f+1}}{\not\equiv} R_{2+2n}(z)$, and hence

$$R_{2+2n+2^{H+2}}\left(\Phi(z')\right) \underset{2^{H+2}}{\equiv} R_{2+2n}\left(\Phi(z)\right) \underset{2^{f+1}}{\not\equiv} R_{2+2n}(z) \underset{2^{H+2}}{\equiv} R_{2+2n+2^{H+2}}(z')$$

So we have shown that $\mathrm{fP}\left(1 + \left(\sum_{i=0}^{n+2^{H+1}} 2^{2i+1}\right) + 2^{2\left(n+2^{H+1}\right)+2}, 2\left(n + 2^{H+1}\right) + 3\right) \leq f$. Furthermore we can choose $y \in \mathbb{Z}_2$ so that if $z = 1 + \left(\sum_{i=0}^{n} 2^{2i+1}\right) + 2^{2n+2} + 2^{2n+3} y$, then $\Phi(z) \underset{2^{2n+f+2}}{\equiv} z$.

Similarly, we define $z' = 1 + \left(\sum_{i=0}^{n+2^{H+1}} 2^{2i+1}\right) + 2^{2\left(n+2^{H+1}\right)+2} + 2^{2\left(n+2^{H+1}\right)+3} y$ for this same y, and we can again obtain $R_{2+2n+2^{H+2}}\left(\Phi\left(z'\right)\right) \underset{2^f}{\equiv} R_{2+2n}\left(\Phi\left(z\right)\right)$ and $R_{2+2n}\left(z\right) \underset{2^f}{\equiv} R_{2+2n+2^{H+2}}\left(z'\right)$ since $f \leq H+1$. But since $\Phi\left(z\right) \underset{2^{2n+f+2}}{\equiv} z$, we have $R_{2+2n}\left(\Phi\left(z\right)\right) \underset{2^f}{\equiv} R_{2+2n}\left(z\right)$, and so

$$R_{2+2n+2^{H+2}}\left(\Phi\left(z'\right)\right) \underset{2^f}{\equiv} R_{2+2n}\left(\Phi\left(z\right)\right) \underset{2^f}{\equiv} R_{2+2n}\left(z\right) \underset{2^f}{\equiv} R_{2+2n+2^{H+2}}\left(z'\right)$$

And since $\frac{1}{3}$ is a fixed point of Φ, $\Phi\left(z'\right) \equiv z' \left(\operatorname{mod} 2+2n+2^{H+2}\right)$ by solenoidality, and thus we have shown that $\left\{\mathrm{fP}\left(1+\left(\sum_{i=0}^{n} 2^{2i+1}\right)+2^{2n+2}, 2n+3\right)\right\}$ is finitely pseudoperiodic. The other half of the proof, showing that the sequence $\left\{\mathrm{fP}\left(1+\sum_{i=0}^{n} 2^{2i+1}, 2n+4\right)\right\}$ is finitely pseudoperiodic is similar, also relying upon Theorem 3.5. □

Using the periodicity of Φ, once we have determined that there is one $x \in \mathbb{Z}_2$ that is not a fixed point of $\Phi \operatorname{mod} 2^k$ for some k, we obtain infinite families of 2-adics that are not fixed points of $\Phi \operatorname{mod} 2^{k+mr}$. While the applicability of the periodicity of Φ to the Fixed Points Conjecture is clear, it is not clear how it can be used to aid in the solution to the original $3x+1$ problem. One potentially interesting area for future study would be to apply this property to the map $\Omega = \Phi \circ V \circ \Phi^{-1}$, where $V\left(x\right) = -1 - x$ [4]. This map makes an intimate connection between Φ and the main conjectures associated with the $3x+1$ problem. Such investigations may shed light on the $3x+1$ problem itself.

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