The 3x+1 Problem

(or how to assign intractable open questions to undergraduates)

The Conjecture

“Mathematics is not yet ready for such problems” - Erdos

The 3x+1 Map

Definition The 3x+1 map:

\[ T(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{3x+1}{2} & \text{if } x \text{ is odd} \end{cases} \]

Dynamical Systems Terminology

Orbit: Let \( X \) be a set, \( f : X \to X \), and \( x \in X \). The \( f \)-orbit of \( x \) is the infinite sequence \( x, f(x), f^2(x), f^3(x), \ldots \)

where \( f^k = f \circ f^{k-1} \) for all \( k \geq 1 \) and \( f^0 \) is the identity map.

Cycle: If \( f^m(x) = x \) for some \( m > 0 \) we say \( \{ f^k(x) : k \in \mathbb{N} \} \) is a cycle.

Eventually Cyclic: If \( f^m(x) = f^n(x) \) for some \( m, n \) with \( m \neq n \) we say the orbit is eventually cyclic.

Divergent: An orbit that is not eventually cyclic is said to be divergent.

Conjecture (L. Collatz circa 1932) The \( T \)-orbit of any positive integer contains \( 1 \).

Example Here are the \( T \)-orbits of the first 20 positive integers:

\( 1, 2 \)
\( 2, 1 \)
\( 3, 5, 8, 4, 2, 1, 2 \)
Example  The $T$-orbit of 27 is:


More Well Known Open Problems

**Conjecture**  *Divergent Orbits Conjecture*: No positive integer has a divergent $T$-orbit.

**Conjecture**  *Nontrivial Cycles Conjecture*: The only $T$-cycle of positive integers is: {$1, 2$}

**Conjecture**  *Finite Cycles Conjecture*: The only $T$-cycles of integers are:

- {$1, 2$}
- {$0$}
- {$-1$}
- {$-5, -7, -10$}
- {$-17, -25, -37, -55, -82, -41, -61, -91, -136, -68, -34$}
Background

What DO we know?

Literature
- Jan 1985 Lagarias, *The 3x+1 Problem and its Generalizations*, MAA Monthly
- 1991 Wirsching, *The Dynamical System Generated by the 3n+1 Function*
- Aug 1999 - Eichstät, Germany
*International Conference on the Collatz Problem and Related Topics*
- Lagarias *3x+1 Problem Annotated Bibliography*: 95 mathematical publications since 1985

Verification
- Eric Roosendaal: Verified for
  \[ n \leq 184 \cdot 2^{50} = 207,165,582,859,042,816 \]
- Crandall’s Result: No nontrivial cycle can have less than 338,466,909 elements!
- Conway: There are similar problems which are algorithmically undecidable!

Meanwhile at Scranton...
- 1991: Faculty Student Research Program (FSRP) formed at Scranton.
- Student Publications:
  - C. Farruggia, M. Lawrence, B. Waterhouse; *The Elimination of a Family of Periodic Parity Vectors in the 3x + 1 Problem*, Pi Mu Epsilon Journal, 10 (4), Spring (1996), 275-280 (1996 Richard V. Andree award winner)
  - Fraboni, M.; *Conjugacy and the 3x + 1 Conjecture* (1998 MAA EPADEL section student paper competition winner)
  - Kucinski, G.; *Cycles for the 3x + 1 Map on the Gaussian Integers*, to appear, Pi Mu Epsilon Journal
  - Yazinski, J.; *Elimination of \( \Phi \)-fixed point candidates* (in preparation)
- Publications:
  - Monks, K.; *3x + 1 minus the +*, Discrete Mathematics and Theoretical Computer Science, 5, no. 1, (2002), 47-54
  - Monks, K. and Yazinski, J.; *The Autoconjugacy of the 3x + 1 Function*, to appear in Discrete Math
  - Monks, K.; *A Category of Topological Spaces Classifying Acyclic Set Theoretic Dynamics*, in preparation
Possible Approaches

1. Extend \( T \) to other domains
2. Simplify \( T \)'s iterations
3. Study \( T \)'s cousins
4. Study \( T \) as its own cousin!
5. Study \( T \)'s distant cousins

Extending the Domain

The OddRats:

\[
\mathbb{Q}_{\text{odd}} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, \gcd(a, b) = 1, \text{ and } b \text{ odd} \right\}
\]
i.e. it is the set of all rational number having an odd denominator in reduced fraction form.

The 2-adic integers:

\[
\mathbb{Z}_2 = \left\{ a_0a_1a_2\ldots(2) : a_i \in \{0, 1\} \right\}
\]
with + and \( \cdot \) defined by the ordinary algorithms for binary arithmetic, i.e. we interpret each element as the formal sum:

\[
a_0a_1a_2\ldots(2) = \sum_{i=0}^{\infty} a_i2^i
\]

Some Basic Facts about the 2-adics:

- \( \mathbb{Z} \leftrightarrow \mathbb{Q}_{\text{odd}} \leftrightarrow \mathbb{Z}_2 \)
- a 2-adic is an (ordinary) integer iff its digits end with \( \overline{0} \) or \( \overline{1} \)
- a 2-adic is an oddrat iff its digits are eventually repeating
- \( a_0a_1a_2\ldots(2) \) is even \( \iff a_0 = 0 \)
- We can define a metric on \( \mathbb{Z}_2 \) by \( d(x, x) = 0 \) and \( d(a_0a_1a_2\ldots(2), b_0b_1b_2\ldots(2)) = \frac{1}{2^k} \) where \( k = \min\{j : a_j \neq 0\} \) if \( a_0a_1a_2\ldots(2) \neq b_0b_1b_2\ldots(2) \)

Example
\[ 13 = 1011_{(2)} \]
\[ -1 = \overline{T}_{(2)} \]
\[ 2 = 01\overline{0}_{(2)} \]
\[ 2/5 = 01\overline{1}10_{(2)} \]

Simplifying the Iteration

3x+1 minus the +

Results from: Monks, K.; 3x + 1 minus the +, Discrete Mathematics and Theoretical Computer Science, 5, no. 1, (2002), 47-54

- Define \( T_0(x) = x/2 \) and \( T_1(x) = \frac{3x+1}{2} \) so that

\[
T(x) = \begin{cases} 
T_0(x) & \text{if } x \equiv 0 \\
T_1(x) & \text{if } x \equiv 1 
\end{cases}
\]

- \( T \) is messy to iterate...

\[
T^k(n) = T_{v_{k-1}} \circ T_{v_{k-2}} \circ \cdots \circ T_{v_0}(n) = \frac{3^m}{2^k} n + \sum_{i=0}^{k-1} v_i \frac{3^{v_{i+1} + \cdots + v_{k-1}}}{2^{k-i}}
\]

where \( m = \sum_{i=0}^{k-1} v_i \), \( v_0, \ldots, v_{k-1} \in \{0, 1\} \), and \( v_i \equiv T^i(n) \)

- Compare with...

\[
R_{v_{k-1}} \circ R_{v_{k-2}} \circ \cdots \circ R_{v_0}(n) = \frac{3^m}{2^k} n
\]

where \( R_0(n) = \frac{1}{2} n \) and \( R_1(n) = \frac{3}{2} n \).

Q: Is there some function of the form

\[
R(n) = \begin{cases} 
r_0n & \text{if } n \equiv 0 \\
r_1n & \text{if } n \equiv 1 \\
& \vdots \\
r_{d-1}n & \text{if } n \equiv d-1 
\end{cases}
\]

where \( r_1, \ldots, r_{d-1} \in \mathbb{Q} \) and \( d \geq 2 \) such that knowledge of certain \( R \)-orbits would settle the 3x + 1 problem?
Theorem There are infinitely many functions $R$ of the form shown above having the property that the Collatz conjecture is true if and only if for all positive integers $n$ the $R$-orbit of $2^n$ contains $2$.

In particular,

$$R(n) = \begin{cases} \frac{1}{11}n & \text{if } 11 \mid n \\ \frac{136}{15}n & \text{if } 15 \mid n \text{ and NOTA} \\ \frac{5}{17}n & \text{if } 17 \mid n \text{ and NOTA} \\ \frac{4}{5}n & \text{if } 5 \mid n \text{ and NOTA} \\ \frac{26}{21}n & \text{if } 21 \mid n \text{ and NOTA} \\ \frac{13}{13}n & \text{if } 13 \mid n \text{ and NOTA} \\ \frac{7}{7}n & \text{if } 7 \mid n \text{ and NOTA} \\ \frac{33}{4}n & \text{if } 4 \mid n \text{ and NOTA} \\ \frac{5}{2}n & \text{if } 2 \mid n \text{ and NOTA} \\ 7n & \text{otherwise} \end{cases}$$

(where NOTA means “None of the Above” conditions hold) is one such function.

Corollary If $\{x_0, \ldots, x_{n-1}\}$ is a $T$-cycle of positive integers, $O = \{i : x_i \text{ is odd}\}$, $E = \{i : x_i \text{ is even}\}$, and $k = |O|$ then

$$\sum_{i \in E} \left\lfloor \frac{x_i}{2} \right\rfloor = \sum_{i \in O} \left\lfloor \frac{x_i}{2} \right\rfloor + k.$$ 

Relatives of $T$ and Conjugacies

Definition Maps $f : X \to X$ and $g : Y \to Y$ are conjugate with conjugacy $h$ if and only if there exists a bijection $h$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h \downarrow & & \downarrow h \\ Y & \xrightarrow{g} & Y \end{array}$$

commutes.

If, in addition, $X$, $Y$ are topological spaces and $h$ is a homeomorphism then we say that $h$ is a topological conjugacy.
Conjugacies preserve the dynamics of a map

Two Important Maps

**Definition** *The shift map, \( \sigma : \mathbb{Z}_2 \to \mathbb{Z}_2 \), is defined by*

\[
\sigma(x) = \begin{cases} 
\frac{x}{2} & \text{if } x \text{ is even} \\
\frac{x-1}{2} & \text{if } x \text{ is odd.}
\end{cases}
\]

**Facts about the shift map:**

- The effect of the shift map on a 2-adic is to erase the first digit, i.e. it shifts all digits one place to the left.
  \[
  \sigma(a_0a_1a_2\ldots(2)) = a_1a_2a_3\ldots(2)
  \]
- The \( \sigma \)-orbit of \( x \) is cyclic (resp. eventually cyclic) iff the 2-adic digits of \( x \) are periodic (resp. eventually periodic).

**Example** The \( \sigma \)-orbit of \( -\frac{11}{33} = \overline{10101}_2 \) is a cycle of period five:

\[
\overline{10101}_2, \overline{10110}_2, \overline{01101}_2, \overline{11010}_2, \overline{01011}_2, \overline{10101}_2, \ldots
\]

**Definition** *(Lagarias) Define the parity vector map, \( \Phi^{-1} : \mathbb{Z}_2 \to \mathbb{Z}_2 \) by*

\[
\Phi^{-1}(x) = v_0v_1v_2\ldots(2)
\]

where \( v_i \in \{0, 1\} \) and \( v_i = T^i(x) \) for all \( i \in \mathbb{N} \), i.e. the digits of the parity vector of \( x \) are obtained by concatenating the mod 2 values of the \( T \)-orbit of \( x \).

**Example** Since the \( T \)-orbit of \( 3 \) is

\[
3, 5, 8, 4, \Sigma, 1
\]

the parity vector of \( 3 \) is

\[
\Phi^{-1}(3) = 1100\Sigma(2) = -\frac{23}{3}
\]

**Facts about \( \Phi^{-1} \):**

- \( \Phi^{-1} \) is a topological conjugacy between \( T \) and \( \sigma \) ! *(Lagarias)*
- Bernstein gave an explicit formula for the inverse map \( \Phi \), namely,

\[
\Phi(2^{d_0} + 2^{d_1} + 2^{d_2} + \cdots) = -\sum_i \frac{1}{3^{i+1}} 2^{d_i}
\]

whenever \( 0 \leq d_0 < d_1 < d_2 < \cdots \) is a finite or infinite sequence of natural numbers.
- *(Lagarias) \( \Phi^{-1} \) and \( \Phi \) are solenoidal*, that is to say that for all \( a, b \in \mathbb{Z}_2 \) and any \( k \in \mathbb{Z}^+ \).
Even More Open Problems...

Conjecture (Lagarias) **Periodicity Conjecture:**

\[ \Phi^{-1}(Q_{\text{odd}}) \subseteq Q_{\text{odd}} \]

- Bernstein and Lagarias: Periodicity Conjecture \( \Rightarrow \) Divergent Orbits Conjecture.

Conjecture (Bernstein-Lagarias) **\( \Phi \)-Fixed Point Conjecture:** The only odd fixed points of \( \Phi \) are \( \frac{1}{3} \) and \(-1\).

In Search of Interesting Conjugacies

Fraboni - Classification of Linear Conjugacies

Q: What other functions are there analogous to the shift map and parity vector map?

**Definition** A function \( f_{a,b,c,d} : \mathbb{Z}_2 \to \mathbb{Z}_2 \) is **modular** if it is of the form

\[
f_{a,b,c,d}(x) = \begin{cases} \frac{ax+b}{2} & \text{if } x \text{ even} \\ \frac{cx+d}{2} & \text{if } x \text{ odd} \end{cases}
\]

with \( a, b, c, d \in \mathbb{Z}_2 \).

**Definition** Let \( \mathcal{F} \) be the set of modular functions, \( f_{a,b,c,d} \), such that \( a, c \) and \( d \) are odd and \( b \) is even.

**Example** \( T = f_{1,0,3,1} \) and \( \sigma = f_{1,0,1,-1} \) are both in \( \mathcal{F} \)

**Theorem (Fraboni)**

1. A modular function \( f \) is conjugate to \( T \) if and only if \( f \in \mathcal{F} \).
2. Every element of \( \mathcal{F} \) is topologically conjugate to \( T \).
3. Every function that is conjugate to \( T \) by a linear map is in \( \mathcal{F} \).

The Nontrivial Autoconjugacy of \( T \)
Results from: Monks, K. and Yazinski, J.; The Autoconjugacy of the $3x + 1$ Function, to appear in Discrete Math

- Hedlund (1969): $\text{Aut}(\sigma) = \langle id, V \rangle$ where $V(x) = -1 - x$ and $id$ is the identity map
- $V(x)$ is the 2-adic whose digits are the bit-complement of the digits of $x$

**Example** $V(11010_2) = 00101_2$

Q: What is $\text{Aut}(T)$?

- Notice

```
\[
\begin{array}{ccc}
\mathbb{Z}_2 & \xrightarrow{r} & \mathbb{Z}_2 \\
\phi^{-1} \downarrow & \quad & \downarrow \phi^{-1} \\
\mathbb{Z}_2 & \xrightarrow{\sigma} & \mathbb{Z}_2 \\
V \downarrow & \quad & \downarrow V \\
\mathbb{Z}_2 & \xrightarrow{\sigma} & \mathbb{Z}_2 \\
\phi \downarrow & \quad & \downarrow \phi \\
\mathbb{Z}_2 & \xrightarrow{r} & \mathbb{Z}_2 \\
\end{array}
\]
```

commutes.

**Definition** Define

\[
\Omega := \Phi \circ V \circ \Phi^{-1}
\]

We call $\Omega$ the nontrivial autoconjugacy of $T$.

Answer:

$\text{Aut}(T) = \langle id, \Omega \rangle$

**Facts about $\Omega$:**

- $\Omega^2 = id$ and $\Omega \circ T = T \circ \Omega$
- $\Omega$ maps a 2-adic integer $x$ to the unique 2-adic integer $\Omega(x)$ whose parity vector is the one’s complement of the parity vector of $x$, i.e. all corresponding terms in the $T$-orbits of $x$ and $\Omega(x)$ have opposite parity.

**Example** The $T$-orbit of $-11/3$ is
and the $T$-orbit of $8/5$ is

$$\frac{8}{5}, \frac{4}{5}, \frac{2}{5}, \frac{1}{5}.$$ 

By uniqueness we conclude that $\Omega(11/3) = 8/5$.

**Example** Suppose we wish to compute $\Omega(3)$. The $T$-orbit of $3$ is $3, 5, 8, 4, 2, 1$ so that

$$\Phi^{-1}(3) = 11001101$$

and its one’s complement is

$$V \circ \Phi^{-1}(3) = 00111010$$

By Bernstein’s formula for $\Phi$ we obtain

$$\Omega(3) = \Phi \circ V \circ \Phi^{-1}(3) = \Phi(00111010) = -\frac{4}{9}$$

whose $T$-orbit is

$$-\frac{4}{9}, -\frac{2}{9}, -\frac{1}{9}, \frac{1}{3}, 1.$$ 

**Parity Neutral Collatz**

**Definition** Let $\xi : \mathbb{Z}_2 \to \mathbb{Z}_2$ by

$$\xi(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \Omega(x) & \text{if } x \text{ is odd} \end{cases}$$

for all $x \in \mathbb{Z}_2$.

**Example**
Definition Define $\sim$ on $\mathbb{Z}_2$ by

$$x \sim y \iff (x = y \text{ or } x = \Omega(y))$$

for all $x, y \in \mathbb{Z}_2$

- $\sim$ is an equivalence relation on $\mathbb{Z}_2$
- $\mathbb{Z}_2/\sim = \{\{x, \Omega(x)\} : x \in \mathbb{Z}_2 \text{ and } x \text{ is odd}\}$


Definition $\Psi : \mathbb{Z}_2/\sim \to \mathbb{Z}_2/\sim$ by $\Psi([x]) = [T(x)]$ for all $x \in \mathbb{Z}_2$.

Theorem The following are equivalent.

(a) The Collatz Conjecture.
(b) The $\xi$-orbit of any positive integer contains 1.
(c) The $\Psi$-orbit of the class of any positive integer contains $[1]$.

Example The $T$-orbit of 3 is

$$3, 5, 8, 4, \underline{2, 1}$$

while the $\xi$-orbit of 3 is

$$3, -4/9, -2/9, -1/9, 8, 4, \underline{2, 1}$$

and the $\Psi$-orbit of $[3]$ is

$$\{3, -4/9\}, \{-2/9, 5\}, \{-1/9, 8\}, \{4, 1/3\}, \{\underline{2, 1}\}$$
Application to Divergent Orbits

**Conjecture** Autoconjugacy Conjecture: \( \Omega(Q_{\text{odd}}) \subseteq Q_{\text{odd}} \)

**Theorem** The following are equivalent.

(a) The Periodicity Conjecture.

(b) The Autoconjugacy Conjecture.

(c) No oddrat has a divergent \( T \)-orbit.

Furthermore, the statement \( \Omega(\mathbb{Z}^+) \subseteq Q_{\text{odd}} \) is equivalent to the Divergent Orbits Conjecture.

Application to Cycles

**Definition** \( T \)-cycle \( C \) is self conjugate if \( \Omega(C) = C \).

**Example** \( \{1, 2\} \) is a self-conjugate \( T \)-cycle.

**Theorem** A \( T \)-cycle \( C \) is self conjugate if and only if \( C \) is the set of iterates of \( x \) where

\[
x = \Phi(v_0v_1\ldots v_kv_0^*v_1^*\ldots v_k^*)
\]

for some \( v_0, v_1, \ldots, v_k \in \{0, 1\} \) (note \( 0^* = 1 \) and \( 1^* = 0 \)).

**Example** To illustrate the theorem, start with any finite binary sequence, e.g. 11, and catenate its one’s complement:

\[ 11^*1^* = 1100. \]

Extend this to a periodic sequence, \( 1100^* \), and compute \( x = \Phi(1100^*) = 5/7 \). Then by the previous theorem the \( T \)-orbit of 5/7 is self conjugate. Indeed the \( T \)-orbit of \( 5/7 \) is

\[
\frac{5}{7}, \frac{11}{7}, \frac{20}{7}, \frac{10}{7}
\]

and \( \Omega(5/7) = 20/7 \).
One immediate consequence is that any self conjugate cycle must have an even number of elements.

Theorem If \( C \) is a self conjugate \( T \)-cycle then \( C \subseteq \mathbb{Q}_{\text{odd}}^+ \), i.e. any self conjugate \( T \)-cycle contains only positive rational entries.

Q: Are there self conjugate cycles integer cycles other than \( \{1, 2\} \)?

Theorem For any self conjugate \( T \)-cycle \( C \)
\[
0 < \text{min}(C) \leq 1 < \text{max}(C).
\]
Hence, the only self conjugate \( T \)-cycle of integers is \( \{1, 2\} \).

Proofs

Definition Let \( \kappa_n(x) \) be the number of ones in the first \( n \) digits of the parity vector of \( x \).

Facts about \( \kappa_n(x) \)
- \( \kappa_n(x) + \kappa_n(\Omega(x)) = n \)
- Dividing by \( n \),
\[
\frac{\kappa_n(x)}{n} + \frac{\kappa_n(\Omega(x))}{n} = 1
\]

Theorem Let \( x \in \mathbb{Z}_2 \). Then

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<table>
<thead>
<tr>
<th>Self Conjugate ( T )-cycles with ten elements or less</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1, 2}</td>
</tr>
<tr>
<td>( {\frac{5}{7}, \frac{11}{7}, \frac{20}{7}, \frac{10}{7}} )</td>
</tr>
<tr>
<td>( {\frac{19}{37}, \frac{47}{37}, \frac{89}{37}, \frac{152}{37}, \frac{76}{37}, \frac{38}{37}} )</td>
</tr>
<tr>
<td>( {\frac{17}{25}, \frac{38}{25}, \frac{19}{25}, \frac{41}{25}, \frac{74}{25}, \frac{37}{25}, \frac{68}{25}, \frac{34}{25}} )</td>
</tr>
<tr>
<td>( {\frac{13}{35}, \frac{37}{35}, \frac{73}{35}, \frac{127}{35}, \frac{208}{35}, \frac{104}{35}, \frac{52}{35}, \frac{26}{35}} )</td>
</tr>
<tr>
<td>( {\frac{211}{781}, \frac{707}{781}, \frac{1451}{781}, \frac{2567}{781}, \frac{4241}{781}, \frac{6752}{781}, \frac{3376}{781}, \frac{1688}{781}, \frac{844}{781}, \frac{422}{781}} )</td>
</tr>
<tr>
<td>( {\frac{373}{781}, \frac{950}{781}, \frac{475}{781}, \frac{1103}{781}, \frac{2045}{781}, \frac{3458}{781}, \frac{1729}{781}, \frac{2984}{781}, \frac{1492}{781}, \frac{746}{781}} )</td>
</tr>
<tr>
<td>( {\frac{383}{781}, \frac{965}{781}, \frac{1838}{781}, \frac{919}{781}, \frac{1769}{781}, \frac{3044}{781}, \frac{1522}{781}, \frac{761}{781}, \frac{1532}{781}, \frac{766}{781}} )</td>
</tr>
</tbody>
</table>
\[
\lim \frac{\kappa_n(x)}{n} + \lim \frac{\kappa_n(\Omega(x))}{n} = 1.
\]

The following theorem is a generalization of results of Lagarias and Eliahou.

**Theorem** Let \( x \in Q_{\text{odd}} \).

(a) If the orbit of \( x \) is eventually cyclic then \( \lim_{n \to \infty} \frac{\kappa_n(x)}{n} \) exists and

\[
\frac{\ln 2}{\ln (3 + \frac{1}{m})} \leq \lim_{n \to \infty} \frac{\kappa_n(x)}{n} \leq \frac{\ln 2}{\ln (3 + \frac{1}{M})}
\]

where \( m, M \) are the least and greatest cyclic elements in \( O(x) \).

(b) If the orbit of \( x \) is divergent then

\[
\frac{\ln 2}{\ln 3} \leq \lim_{n \to \infty} \frac{\kappa_n(x)}{n}.
\]

**Distant Cousins - Changing Categories**

Results from: Monks, K.; *A Category of Topological Spaces Classifying Acyclic Set Theoretic Dynamics*, in preparation

**Definition** A set theoretic discrete dynamical system is a pair \((X,f)\) where \( X \) is a set and \( f : X \to X \).

**Definition** Let \( f : X \to X \) and \( g : Y \to Y \). Then \( h : X \to Y \) is a semi-conjugacy if and only if

\[
\begin{align*}
X & \xrightarrow{f} X \\
h \downarrow & \quad \downarrow h \\
Y & \xrightarrow{g} Y
\end{align*}
\]

commutes.

**Definition** Let \( f : X \to X \). Define

\[
\tau_f = \{ A \subseteq X : f(A) \subseteq A \}
\]

**Theorem** \( \tau_f \) is a topology on \( X \).

**Remark** We call \( \tau_f \) the topology induced by \( f \).
**Theorem** Semiconjugacies are continuous with respect to the induced topologies. Conjugacies are homeomorphisms.

**Theorem** The Collatz conjecture is true if and only if the topological space \((\mathbb{Z}^+, \tau_T)\) is connected.

**Yazinski - Work on the \(\Phi\)-fixed point conjecture**

**Theorem** Let \(b \in \mathbb{Z}_{(2)}, a, t \in \mathbb{N}\) with \(2^t > a\), and \(m\) the number of ones in the binary digits of \(a\). Then
\[
\Phi(a + b2^t) = \Phi(a) + \frac{\Phi(b)}{3^m}2^t
\]

- \(3^{2k} = 1\) for all \(i \geq 1\), so for \(m = 2^k\) with \(i \geq 1\) we have
\[
\Phi(a + b2^t) = \Phi(a) + \Phi(b)2^t
\]

**Corollary** There is no \(\Phi\)-fixed point of the form
- \(\frac{2k+1}{2^{n+2}}\) ones
  \[
  \overbrace{11\cdots11}^{2k+1 \text{ ones}} \overbrace{0\cdots0}^{(2)}
  \]
  or
- \(\frac{2k+1}{11010\cdots101011} \) ones
  \[
  \overbrace{11010\cdots101011}^{2k+1 \text{ ones}} \overbrace{0\cdots0}^{(2)}
  \]
where \(k \in \mathbb{Z}^+\).

**In Search of the “Collatz Fractal”**

**Joseph’s Extension**

- Extension to \(\mathbb{Z}_2[i]\)
- Even and odd correspond to equivalence classes in \(\mathbb{Z}/2\mathbb{Z}\).
- \(\mathbb{Z}_2[i]/2\mathbb{Z}_2[i] = \{0, [1], [i], [1+i]\}\)

**Definition** Let \(\tilde{T} : \mathbb{Z}_2[i] \rightarrow \mathbb{Z}_2[i]\) by
Theorem (Kucinski) \( \tilde{T}|\mathbb{Z}[i] \) has exactly 77 distinct cycles of period less than or equal to 400 distributed as follows:

| Period | Number \( \tilde{T}|\mathbb{Z} \) Cycles | Number \( \tilde{T}|\mathbb{Z}[i] \) Cycles |
|--------|----------------------------------------|----------------------------------------|
| 1      | 2                                      | 4                                      |
| 2      | 1                                      | 3                                      |
| 3      | 1                                      | 9                                      |
| 5      | 0                                      | 2                                      |
| 8      | 0                                      | 10                                     |
| 11     | 1                                      | 5                                      |
| 19     | 0                                      | 30                                     |
| 46     | 0                                      | 2                                      |
| 84     | 0                                      | 10                                     |
| 103    | 0                                      | 2                                      |

Conjecture Further computations will make it more plausible that we should make a finite cycles conjecture for \( \tilde{T} \).

- Wanted: a continuous (preferably entire) function that interpolates \( T|\mathbb{Q}_{\text{odd}} \) or \( T|\mathbb{Q}_{\text{odd}}[i] \)
- No way!
- M. Chamberland:

\[
f(x) = \frac{x}{2} \cos^2\left(\frac{\pi}{2} x \right) + \frac{3x + 1}{2} \sin^2\left(\frac{\pi}{2} x \right)
\]

is entire and extends \( T|\mathbb{Z} \).

An analytic extension of \( \tilde{T}|\mathbb{Z}[i] \)

Definition: Let \( \{a_0, a_1, a_2, \ldots\} = \mathbb{Z}[i] \) be the enumeration of the points of \( \mathbb{Z}[i] \) as shown:
Theorem (Joseph, Monks) Let $F : \mathbb{C} \to \mathbb{C}$ by

$$f_0(z) = 0, \text{ and for } n > 0$$

$$f_n(z) = \pi_n(z) \left( \frac{z}{a_n} \right)^{m_n} \left( \mathcal{T}^n(a_n) - \sum_{k=0}^{n-1} f_k(a_n) \right),$$

$$\pi_n(z) = \prod_{k=1}^{n} \frac{(z - a_k)}{(a_n - a_k)},$$

$$p_n = \left\lfloor \frac{\sqrt{n} + 1}{2} \right\rfloor,$$

$$K_n = \left\lfloor \mathcal{T}^n(a_n) - \sum_{k=0}^{n-1} f_k(a_n) \right\rfloor,$$

$$m_n = \left\lfloor \log_2 \left( \left( 1 + 2\sqrt{2} \right)^{n-1} p_n^{n-1} \right) K_n \right\rfloor.$$