The 3x+1 Problem

(OR HOW TO ASSIGN INTRACTABLE OPEN QUESTIONS TO UNDERGRADUATES)

The Conjecture

"Mathematics is not yet ready for such problems" - Erdos

The 3x+1 Map

Definition ***The 3x+1 map***:

$$T(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{3x+1}{2} & \text{if } x \text{ is odd} \end{cases}$$

Dynamical Systems Terminology

Orbit: Let X be a set, $f : X \to X$, and $x \in X$. The **f-orbit** of x is the infinite sequence

$$x, f(x), f^2(x), f^3(x), \ldots$$

where $f^k = f \circ f^{k-1}$ for all $k \geq 1$ and f^0 is the identity map.

Cycle: If $f^m(x) = x$ for some $m > 0$ we say $\{f^k(x) : k \in \mathbb{N}\}$ is a **cycle**.

Eventually Cyclic: If $f^m(x) = f^n(x)$ for some m, n with $m \neq n$ we say the orbit is **eventually cyclic**.

Divergent: An orbit that is not eventually cyclic is said to be **divergent**.

Conjecture (L. Collatz circa 1932) *The T-orbit of any positive integer contains 1.*

Example *Here are the T-orbits of the first 20 positive integers:*

$\overline{1, 2}$

$\overline{2, 1}$

$3, 5, 8, 4, 2, \overline{1, 2}$

$4,2,\overline{1,2}$

$5,8,4,2,\overline{1,2}$

$6,3,5,8,4,2,\overline{1,2}$

$7,11,17,26,13,20,10,5,8,4,2,\overline{1,2}$

$8,4,2,\overline{1,2}$

$9,14,7,11,17,26,13,20,10,5,8,4,2,\overline{1,2}$

$10,5,8,4,2,\overline{1,2}$

$11,17,26,13,20,10,5,8,4,2,\overline{1,2}$

$12,6,3,5,8,4,2,\overline{1,2}$

$13,20,10,5,8,4,2,\overline{1,2}$

$14,7,11,17,26,13,20,10,5,8,4,2,\overline{1,2}$

$15,23,35,53,80,40,20,10,5,8,4,2,\overline{1,2}$

$16,8,4,2,\overline{1,2}$

$17,26,13,20,10,5,8,4,2,\overline{1,2}$

$18,9,14,7,11,17,26,13,20,10,5,8,4,2,\overline{1,2}$

$19,29,44,22,11,17,26,13,20,10,5,8,4,2,\overline{1,2}$

$20,10,5,8,4,2,\overline{1,2}$

Example *The T-orbit of 27 is:*

27, 41, 62, 31, 47, 71, 107, 161, 242, 121, 182, 91, 137, 206, 103, 155, 233, 350, 175, 263, 395, 593, 890, 445, 668, 334, 167, 251, 377, 566, 283, 425, 638, 319, 479, 719, 1079, 1619, 2429, 3644, 1822, 911, 1367, 2051, 3077, 4616, 2308, 1154, 577, 866, 433, 650, 325, 488, 244, 122, 61, 92, 46, 23, 35, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1

More Well Known Open Problems

Conjecture ***Divergent Orbits Conjecture:*** *No positive integer has a divergent T-orbit.*

Conjecture ***Nontrivial Cycles Conjecture:*** *The only T-cycle of positive integers is:*

$$\{1,2\}$$

Conjecture ***Finite Cycles Conjecture:*** *The only T-cycles of integers are:*

$$\{1,2\}$$

$$\{0\}$$

$$\{-1\}$$

$$\{-5,-7,-10\}$$

$$\{-17,-25,-37,-55,-82,-41,-61,-91,-136,-68,-34\}$$

Background

What DO we know?

Literature

- Jan 1985 Lagarias, *The 3x+1 Problem and its Generalizations,* MAA Monthly
- 1991 Wirsching, *The Dynamical System Generated by the 3n+1 Function*
- Aug 1999 - Eichstät, Germany
  International Conference on the Collatz Problem and Related Topics
- Lagarias *3x+1 Problem Annotated Bibliography*: 95 mathematical publications since 1985

Verification

- Eric Roosendaal: Verified for

$$n \leq 184 \cdot 2^{50} = 207,165,582,859,042,816$$

- Crandall's Result: No nontrivial cycle can have less than $338,466,909$ elements!
- Conway: There are similar problems which are algorithmically undecidable!

Meanwhile at Scranton...

- 1991: Faculty Student Research Program (FSRP) formed at Scranton.
- Student Publications:
  - C. Farruggia, M. Lawrence, B. Waterhouse; **The Elimination of a Family of Periodic Parity Vectors in the $3x + 1$ Problem,** Pi Mu Epsilon Journal, 10 (4), Spring (1996), 275-280 (1996 Richard V. Andree award winner)
  - Fusaro, Marc, **A Visual Representation of Sequence Space**, Pi Mu Epsilon Journal, Pi Mu Epsilon Journal 10 (6), Spring 1997, 466-481 (1997 MAA EPADEL section student paper competition winner and 1997 Richard V. Andree award winner)
  - Joseph, J.; **A Chaotic Extension of the $3x + 1$ Function to $\mathbb{Z}_2[i]$,** Fibonacci Quarterly, 36.4 (Aug 1998), 309-316 (1996 MAA EPADEL section student paper competition winner)
  - Fraboni, M.;**Conjugacy and the $3x + 1$ Conjecture** (1998 MAA EPADEL section student paper competition winner)
  - Kucinski, G.; **Cycles for the $3x + 1$ Map on the Gaussian Integers**, to appear, Pi Mu Epsilon Journal
  - Yazinski, J.; **Elimination of Φ-fixed point candidates** (in preparation)
- Publications:
  - Monks, K.; **$3x + 1$ minus the $+$**, Discrete Mathematics and Theoretical Computer Science, 5, no. 1, (2002), 47-54
  - Monks, K. and Yazinski, J.; **The Autoconjugacy of the $3x + 1$ Function**, to appear in Discrete Math
  - Monks, K.; **A Category of Topological Spaces Classifying Acyclic Set Theoretic Dynamics**, in preparation

Possible Approaches

1. Extend T to other domains
2. Simplify T's iterations
3. Study T's cousins
4. Study T as its own cousin!
5. Study T's distant cousins

Extending the Domain

The OddRats:

$$\mathbb{Q}_{odd} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z},\ \gcd(a,b) = 1, \text{ and } b \text{ odd} \right\}$$

i.e. it is the set of all rational number having an odd denominator in reduced fraction form.

The 2-adic integers:

$$\mathbb{Z}_2 = \{a_0 a_1 a_2 \ldots_{(2)} : a_i \in \{0,1\}\}$$

with $+$ and $\cdot$ defined by the ordinary algorithms for binary arithmetic, i.e. we interpret each element as the formal sum:

$$a_0 a_1 a_2 \ldots_{(2)} = \sum_{i=0}^{\infty} a_i 2^i$$

Some Basic Facts about the 2-adics:

- $\mathbb{Z} \hookrightarrow \mathbb{Q}_{odd} \hookrightarrow \mathbb{Z}_2$
- a 2-adic is an (ordinary) integer iff its digits end with $\overline{0}$ or $\overline{1}$
- a 2-adic is an oddrat iff its digits are eventually repeating
- $a_0 a_1 a_2 \ldots_{(2)}$ is even $\Leftrightarrow a_0 = 0$
- We can define a metric on $\mathbb{Z}_2$by $d(x,x) = 0$ and $d(a_0 a_1 a_2 \ldots_{(2)}, b_0 b_1 b_2 \ldots_{(2)}) = \frac{1}{2^k}$ where $k = \min\{j : a_j \neq 0\}$ if $a_0 a_1 a_2 \ldots_{(2)} \neq b_0 b_1 b_2 \ldots_{(2)}$

Example

$$\begin{aligned} 13 &= 1011\overline{0}_{(2)} \\ -1 &= \overline{1}_{(2)} \\ 2 &= 01\overline{0}_{(2)} \\ 2/5 &= 0\overline{10110}_{(2)} \end{aligned}$$

Simplifying the Iteration

3x+1 minus the +

Results from: Monks, K.; $3x+1$ **minus the** $+$, Discrete Mathematics and Theoretical Computer Science, 5, no. 1, (2002), 47-54

- Define $T_0(x) = x/2$ and $T_1(x) = \frac{3x+1}{2}$ so that

$$T(x) = \begin{cases} T_0(x) & \text{if } x \underset{2}{\equiv} 0 \\ T_1(x) & \text{if } x \underset{2}{\equiv} 1 \end{cases}$$

- T is messy to iterate...

$$T^k(n) = T_{v_{k-1}} \circ T_{v_{k-2}} \circ \cdots \circ T_{v_0}(n) = \frac{3^m}{2^k} n + \sum_{i=0}^{k-1} v_i \frac{3^{v_{i+1}+\cdots+v_{k-1}}}{2^{k-i}}$$

where $m = \sum_{i=0}^{k-1} v_i$, $v_0, \ldots v_{k-1} \in \{0,1\}$, and $v_i \underset{2}{\equiv} T^i(n)$

- Compare with...

$$R_{v_{k-1}} \circ R_{v_{k-2}} \circ \cdots \circ R_{v_0}(n) = \frac{3^m}{2^k} n$$

where $R_0(n) = \frac{1}{2}n$ and $R_1(n) = \frac{3}{2}n$.

Q: Is there some function of the form

$$R(n) = \begin{cases} r_0 n & \text{if } n \underset{d}{\equiv} 0 \\ r_1 n & \text{if } n \underset{d}{\equiv} 1 \\ \vdots & \vdots \\ r_{d-1} n & \text{if } n \underset{d}{\equiv} d-1 \end{cases}$$

where $r_1, \ldots, r_{d-1} \in \mathbb{Q}$ and $d \geq 2$ such that knowledge of certain R-orbits would settle the $3x+1$ problem?

Theorem *There are infinitely many functions R of the form shown above having the property that the Collatz conjecture is true if and only if for all positive integers n the R-orbit of 2^n contains 2.*

In particular,

$$R(n) = \begin{cases} \frac{1}{11}n & \text{if } 11 \mid n \\ \frac{136}{15}n & \text{if } 15 \mid n \text{ and NOTA} \\ \frac{5}{17}n & \text{if } 17 \mid n \text{ and NOTA} \\ \frac{4}{5}n & \text{if } 5 \mid n \text{ and NOTA} \\ \frac{26}{21}n & \text{if } 21 \mid n \text{ and NOTA} \\ \frac{7}{13}n & \text{if } 13 \mid n \text{ and NOTA} \\ \frac{1}{7}n & \text{if } 7 \mid n \text{ and NOTA} \\ \frac{33}{4}n & \text{if } 4 \mid n \text{ and NOTA} \\ \frac{5}{2}n & \text{if } 2 \mid n \text{ and NOTA} \\ 7n & \text{otherwise} \end{cases}$$

(where NOTA means "None of the Above" conditions hold) is one such function.

Corollary *If $\{x_0, \ldots, x_{n-1}\}$ is a T-cycle of positive integers, $\mathcal{O} = \{i : x_i \text{ is odd}\}$, $\mathcal{E} = \{i : x_i \text{ is even}\}$, and $k = |\mathcal{O}|$ then*

$$\sum_{i \in \mathcal{E}} \left\lfloor \frac{x_i}{2} \right\rfloor = \sum_{i \in \mathcal{O}} \left\lfloor \frac{x_i}{2} \right\rfloor + k.$$

Relatives of T and Conjugacies

Definition *Maps $f : X \to X$ and $g : Y \to Y$ are* **conjugate with conjugacy** h *if and only if there exists a bijection h such that*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h \downarrow & & \downarrow h \\ Y & \underset{g}{\longrightarrow} & Y \end{array}$$

commutes.

If, in addition, X, Y are topological spaces and h is a homeomorphism then we say that h is a ***topological conjugacy***.

- Conjugacies preserve the dynamics of a map

Two Important Maps

Definition *The* **shift map,** $\sigma : \mathbb{Z}_2 \to \mathbb{Z}_2$, *is defined by*

$$\sigma(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x-1}{2} & \text{if } x \text{ is odd.} \end{cases}$$

Facts about the shift map:

- The effect of the shift map on a 2-adic is to erase the first digit, i.e. it shifts all digits one place to the left

$$\sigma(a_0a_1a_2\ldots_{(2)}) = a_1a_2a_3\ldots_{(2)}$$

- The σ-orbit of x is cyclic (resp. eventually cyclic) iff the 2-adic digits of x are periodic (resp. eventually periodic)

Example *The* σ*-orbit of* $-\frac{11}{33} = \overline{11010}_{(2)}$ *is a cycle of period five*

$$\overline{11010}_{(2)}, \overline{10101}_{(2)}, \overline{01011}_{(2)}, \overline{10110}_{(2)}, \overline{01101}_{(2)}, \overline{11010}_{(2)}, \ldots$$

Definition *(Lagarias) Define the* **parity vector map,** $\Phi^{-1} : \mathbb{Z}_2 \to \mathbb{Z}_2$ *by*

$$\Phi^{-1}(x) = v_0v_1v_2\ldots_{(2)}$$

where $v_i \in \{0,1\}$ *and* $v_i \underset{2}{\equiv} T^i(x)$ *for all* $i \in \mathbb{N}$, *i.e. the digits of the parity vector of* x *are obtained by concatenating the mod 2 values of the* T*-orbit of* x.

Example *Since the* T*-orbit of* 3 *is*

$$3, 5, 8, 4, \overline{2, 1}$$

the parity vector of 3 *is*

$$\Phi^{-1}(3) = 1100\overline{01}_{(2)} = -\frac{23}{3}$$

Facts about Φ^{-1}

- Φ^{-1} is a topological conjugacy between T and σ ! (Lagarias)
- Bernstein gave an explicit formula for the inverse map Φ, namely,

$$\Phi(2^{d_0} + 2^{d_1} + 2^{d_2} + \cdots) = -\sum_i \frac{1}{3^{i+1}} 2^{d_i}$$

whenever $0 \leq d_0 < d_1 < d_2 < \cdots$ is a finite or infinite sequence of natural numbers.

- (Lagarias) Φ^{-1} and Φ are ***solenoidal***, that is to say that to say that for all $a, b \in \mathbb{Z}_2$ and any $k \in \mathbb{Z}^+$

$$a \underset{2^k}{\equiv} b \Leftrightarrow \Phi(a) \underset{2^k}{\equiv} \Phi(b)$$

Even More Open Problems...

Conjecture (Lagarias) ***Periodicity Conjecture:***

$$\Phi^{-1}\left(\mathbb{Q}_{odd}\right) \subseteq \mathbb{Q}_{odd}$$

- Bernstein and Lagarias: Periodicity Conjecture $\Rightarrow$ Divergent Orbits Conjecture.

Conjecture (Bernstein-Lagarias)
Φ***-Fixed Point Conjecture:*** *The only odd fixed points of* Φ *are* $\frac{1}{3}$ *and* -1.

In Search of Interesting Conjugacies

Fraboni - Classification of Linear Conjugacies

Q: What other functions are there analogous to the shift map and parity vector map?

Definition *A function* $f_{a,b,c,d} : \mathbb{Z}_2 \to \mathbb{Z}_2$ *is* ***modular*** *if it is of the form*

$$f_{a,b,c,d}(x) = \begin{cases} \frac{ax+b}{2} & \text{if } x \text{ even} \\ \frac{cx+d}{2} & \text{if } x \text{ odd} \end{cases}$$

with $a, b, c, d \in \mathbb{Z}_2$.

Definition *Let* $\mathcal{F}$ *be the set of modular functions,* $f_{a,b,c,d}$*, such that* a, c *and* d *are odd and* b *is even.*

Example $T = f_{1,0,3,1}$ *and* $\sigma = f_{1,0,1,-1}$ *are both in* $\mathcal{F}$

Theorem (Fraboni)
(1) A modular function f *is conjugate to* T *if and only if* $f \in \mathcal{F}$.
(2) Every element of $\mathcal{F}$ *is topologically conjugate to* T.
(3) Every function that is conjugate to T *by a linear map is in* $\mathcal{F}$.

The Nontrivial Autoconjugacy of T

Results from: Monks, K. and Yazinski, J.; **The Autoconjugacy of the $3x+1$ Function**, to appear in Discrete Math

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- Hedlund (1969): $\mathrm{Aut}(\sigma) = \{id, V\}$ where $V(x) = -1 - x$ and id is the identity map
- $V(x)$ is the 2-adic whose digits are the bit-complement of the digits of x

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Example $V(\overline{11010}_{(2)}) = \overline{00101}_{(2)}$

Q: What is $\mathrm{Aut}(T)$?

- Notice

$$\begin{array}{ccc} \mathbb{Z}_2 & \xrightarrow{T} & \mathbb{Z}_2 \\ \Phi^{-1}\downarrow & & \downarrow\Phi^{-1} \\ \mathbb{Z}_2 & \xrightarrow[\sigma]{} & \mathbb{Z}_2 \\ \mathrm{V}\downarrow & & \downarrow\mathrm{V} \\ \mathbb{Z}_2 & \xrightarrow[\sigma]{} & \mathbb{Z}_2 \\ \Phi\downarrow & & \downarrow\Phi \\ \mathbb{Z}_2 & \xrightarrow[T]{} & \mathbb{Z}_2 \end{array}$$

commutes.

Definition *Define*

$$\boxed{\Omega := \Phi \circ V \circ \Phi^{-1}}$$

*We call Ω the **nontrivial autoconjugacy** of T.*

Answer:

$$\mathrm{Aut}(T) = \{id, \Omega\}$$

Facts about Ω:

- $\Omega^2 = id$ and $\Omega \circ T = T \circ \Omega$
- Ω maps a 2-adic integer x to the unique 2-adic integer $\Omega(x)$ whose parity vector is the one's complement of the parity vector of x, i.e. all corresponding terms in the T-orbits of x and $\Omega(x)$ have opposite parity.

Example *The T-orbit of $-11/3$ is*

$$-\frac{11}{3}, \overline{-5, -7, -10}$$

and the T-orbit of $8/5$ is

$$\frac{8}{5}, \overline{\frac{4}{5}, \frac{2}{5}, \frac{1}{5}}.$$

By uniqueness we conclude that $\Omega(-11/3) = 8/5$.

Example *Suppose we wish to compute $\Omega(3)$. The T-orbit of 3 is*

$$3, 5, 8, 4, \overline{2, 1}$$

so that

$$\Phi^{-1}(3) = 11000\overline{01}$$

and its one's complement is

$$V \circ \Phi^{-1}(3) = 00111\overline{10}$$

By Bernstein's formula for Φ we obtain

$$\Omega(3) = \Phi \circ V \circ \Phi^{-1}(3) = \Phi(00111\overline{10}) = -\frac{4}{9}$$

whose T-orbit is

$$-\frac{4}{9}, -\frac{2}{9}, -\frac{1}{9}, \frac{1}{3}, \overline{1, 2}.$$

Parity Neutral Collatz

Definition *Let $\xi : \mathbb{Z}_2 \to \mathbb{Z}_2$ by*

$$\xi(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \Omega(x) & \text{if } x \text{ is odd} \end{cases}$$

for all $x \in \mathbb{Z}_2$.

Example

$$\begin{array}{ccc} 3 & \overset{\Omega}{\leftrightarrow} & -4/9 \\ T\downarrow & & \downarrow T \\ 5 & \overset{\Omega}{\leftrightarrow} & -2/9 \\ T\downarrow & & \downarrow T \\ 8 & \overset{\Omega}{\leftrightarrow} & -1/9 \\ T\downarrow & & \downarrow T \\ 4 & \overset{\Omega}{\leftrightarrow} & 1/3 \\ T\downarrow & & \downarrow T \\ 2 & \overset{\Omega}{\leftrightarrow} & 1 \end{array}$$

Definition *Define $\sim$ on $\mathbb{Z}_2$ by*

$$x \sim y \Leftrightarrow \left(x = y \text{ or } x = \Omega(y)\right)$$

for all $x, y \in \mathbb{Z}_2$

- $\sim$ is an equivalence relation on $\mathbb{Z}_2$
- $\mathbb{Z}_2/\sim \; = \left\{\{x, \Omega(x)\} : x \in \mathbb{Z}_2 \text{ and } x \text{ is odd}\right\}$

Example $[3] = [-4/9] = \{3, -4/9\}$

Definition $\Psi : \mathbb{Z}_2/\sim \; \to \mathbb{Z}_2/\sim$ *by* $\Psi([x]) = [T(x)]$ *for all* $x \in \mathbb{Z}_2$.

Theorem *The following are equivalent.*

(a) The Collatz Conjecture.

(b) The ξ-orbit of any positive integer contains 1.

(c) The Ψ-orbit of the class of any positive integer contains $[1]$.

Example *The T-orbit of 3 is*

$$3, 5, 8, 4, \overline{2, 1}$$

while the ξ-orbit of 3 is

$$3, -4/9, -2/9, -1/9, 8, 4, \overline{2, 1}$$

and the Ψ-orbit of $[3]$ is

$$\left\{3, -\frac{4}{9}\right\}, \left\{-\frac{2}{9}, 5\right\}, \left\{-\frac{1}{9}, 8\right\}, \left\{4, \frac{1}{3}\right\}, \overline{\{2, 1\}}$$

Application to Divergent Orbits

Conjecture ***Autoconjugacy Conjecture:***

$$\Omega\left(\mathbb{Q}_{odd}\right) \subseteq \mathbb{Q}_{odd}$$

Theorem *The following are equivalent.*

(a) The Periodicity Conjecture.

(b) The Autoconjugacy Conjecture.

(c) No oddrat has a divergent T-orbit.

Furthermore, the statement $\Omega(\mathbb{Z}^{+}) \subseteq \mathbb{Q}_{odd}$ is equivalent to the Divergent Orbits Conjecture.

Application to Cycles

Definition *T-cycle C is self conjugate if $\Omega(C) = C$.*

Example *$\{1,2\}$ is a self-conjugate T-cycle.*

Theorem *A T-cycle C is self conjugate if and only if C is the set of iterates of x where*

$$x = \Phi(\overline{v_0 v_1 \cdots v_k v_0^* v_1^* \cdots v_k^*})$$

for some $v_0, v_1, \ldots, v_k \in \{0,1\}$ (note $0^* = 1$ and $1^* = 0$)

Example *To illustrate the theorem, start with any finite binary sequence, e.g. 11, and catenate its one's complement:*

$$111^*1^* = 1100.$$

Extend this to a periodic sequence, $\overline{1100}$, and compute $x = \Phi(\overline{1100}) = 5/7$. Then by the previous theorem the T-orbit of $5/7$ is self conjugate. Indeed the T-orbit of $\frac{5}{7}$ is

$$\overline{\frac{5}{7}, \frac{11}{7}, \frac{20}{7}, \frac{10}{7}}$$

and $\Omega(5/7) = 20/7$.

Self Conjugate T-cycles with ten elements or less
$\{1,2\}$
$\{\frac{5}{7},\frac{11}{7},\frac{20}{7},\frac{10}{7}\}$
$\{\frac{19}{37},\frac{47}{37},\frac{89}{37},\frac{152}{37},\frac{76}{37},\frac{38}{37}\}$
$\{\frac{17}{25},\frac{38}{25},\frac{19}{25},\frac{41}{25},\frac{74}{25},\frac{37}{25},\frac{68}{25},\frac{34}{25}\}$
$\{\frac{13}{35},\frac{37}{35},\frac{73}{35},\frac{127}{35},\frac{208}{35},\frac{104}{35},\frac{52}{35},\frac{26}{35}\}$
$\{\frac{211}{781},\frac{707}{781},\frac{1451}{781},\frac{2567}{781},\frac{4241}{781},\frac{6752}{781},\frac{3376}{781},\frac{1688}{781},\frac{844}{781},\frac{422}{781}\}$
$\{\frac{373}{781},\frac{950}{781},\frac{475}{781},\frac{1103}{781},\frac{2045}{781},\frac{3458}{781},\frac{1729}{781},\frac{2984}{781},\frac{1492}{781},\frac{746}{781}\}$
$\{\frac{383}{781},\frac{965}{781},\frac{1838}{781},\frac{919}{781},\frac{1769}{781},\frac{3044}{781},\frac{1522}{781},\frac{761}{781},\frac{1532}{781},\frac{766}{781}\}$

- One immediate consequence is that any self conjugate cycle must have an even number of elements.

Theorem *If C is a self conjugate T-cycle then $C \subseteq \mathbb{Q}^+_{odd}$, i.e. any self conjugate T-cycle contains only positive rational entries.*

Q: Are there self conjugate cycles integer cycles other than $\{1,2\}$?

Theorem *For any self conjugate T-cycle C*

$$0 < min(C) \leq 1 < max(C).$$

Hence, the only self conjugate T-cycle of integers is $\{1,2\}$.

Proofs

Definition *Let $\kappa_n(x)$ be the number of ones in the first n digits of the parity vector of x.*

Facts about $\kappa_n(x)$

- $\kappa_n(x) + \kappa_n(\Omega(x)) = n$
- Dividing by n,

$$\frac{\kappa_n(x)}{n} + \frac{\kappa_n(\Omega(x))}{n} = 1$$

Theorem *Let $x \in \mathbb{Z}_2$. Then*

$$\underline{lim}\frac{\kappa_n(x)}{n} + \overline{lim}\frac{\kappa_n(\Omega(x))}{n} = 1.$$

The following theorem is a generalization of results of Lagarias and Eliahou.

Theorem *Let* $x \in \mathbb{Q}_{odd}$.

(a) If the orbit of x *is eventually cyclic then* $\lim_{n\to\infty}\frac{\kappa_n(x)}{n}$ *exists and*

$$\frac{ln\,2}{ln(3+\frac{1}{m})} \leq \lim_{n\to\infty}\frac{\kappa_n(x)}{n} \leq \frac{ln\,2}{ln(3+\frac{1}{M})}$$

where m, M *are the least and greatest cyclic elements in* $\mathcal{O}(x)$.

(b) If the orbit of x *is divergent then*

$$\frac{ln\,2}{ln\,3} \leq \underline{lim}\frac{\kappa_n(x)}{n}.$$

Distant Cousins - Changing Categories

Results from: Monks, K.; **A Category of Topological Spaces Classifying Acyclic Set Theoretic Dynamics**, in preparation

Definition *A* **set theoretic discrete dynamical system** *is a pair* (X,f) *where* X *is a set and* $f : X \to X$.

Definition *Let* $f : X \to X$ *and* $g : Y \to Y$. *Then* $h : X \to Y$ *is a* ***semi-conjugacy*** *if and only if*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h\downarrow & & \downarrow h \\ Y & \xrightarrow[g]{} & Y \end{array}$$

commutes.

Definition *Let* $f : X \to X$. *Define*

$$\tau_f = \{A \subseteq X : f(A) \subseteq A\}$$

Theorem τ_f *is a topology on* X.

Remark *We call* τ_f *the* ***topology induced by*** f.

Theorem *Semiconjugacies are continuous with respect to the induced topologies. Conjugacies are homeomorphisms.*

Theorem *The Collatz conjecture is true if and only if the topological space* $(\mathbb{Z}^+, \tau_T)$ *is connected.*

Yazinski - Work on the Φ-fixed point conjecture

Theorem *Let* $b \in \mathbb{Z}_{(2)}$, $a, t \in \mathbb{N}$ *with* $2^t > a$, *and* m *the number of ones in the binary digits of* a. *Then*

$$\Phi(a + b2^t) = \Phi(a) + \frac{\Phi(b)}{3^m} 2^t$$

- $3^{2^i k} \underset{2^{i+2}}{\equiv} 1$ for all $i \geq 1$, so for $m = 2^i k$ with $i \geq 1$ we have

$$\Phi(a + b2^t) \underset{2^{t+i+2}}{\equiv} \Phi(a) + \Phi(b)2^t$$

Corollary *There is no* Φ*-fixed point of the form*

$$\overbrace{11\cdots11}^{2k+1 \text{ ones}}\ 0\ldots_{(2)}$$

or

$$\overbrace{11010\cdots101011}^{2k+1 \text{ ones}}\ 0\ldots_{(2)}$$

where $k \in \mathbb{Z}^+$.

In Search of the "Collatz Fractal"

Joseph's Extension

- Extension to $\mathbb{Z}_2[i]$
- Even and odd correspond to equivalence classes in $\mathbb{Z}/2\mathbb{Z}$.
- $\mathbb{Z}_2[i]/2\mathbb{Z}_2[i] = \{[0], [1], [i], [1 + i]\}$

Definition *Let*

$$\tilde{T} : \mathbb{Z}_2[i] \to \mathbb{Z}_2[i]$$

by

$$\widetilde{T}(x) = \begin{cases} \frac{x}{2} & \text{if } x \in [0] \\ \frac{3x+1}{2} & \text{if } x \in [1] \\ \frac{3x+i}{2} & \text{if } x \in [i] \\ \frac{3x+1+i}{2} & \text{if } x \in [1+i] \end{cases}$$

Kucinski - Cycles in $\widetilde{T}|\mathbb{Z}[i]$

Theorem (Kucinski) *$\widetilde{T}|\mathbb{Z}[i]$ has exactly 77 distinct cycles of period less than or equal to 400 distributed as follows:*

Period	*Number $T|\mathbb{Z}$ Cycles*	*Number $\widetilde{T}|\mathbb{Z}[i]$ Cycles*
1	2	4
2	1	3
3	1	9
5	0	2
8	0	10
11	1	5
19	0	30
46	0	2
84	0	10
103	0	2

Conjecture *Further computations will make it more plausible that we should make a finite cycles conjecture for $\widetilde{T}$.*

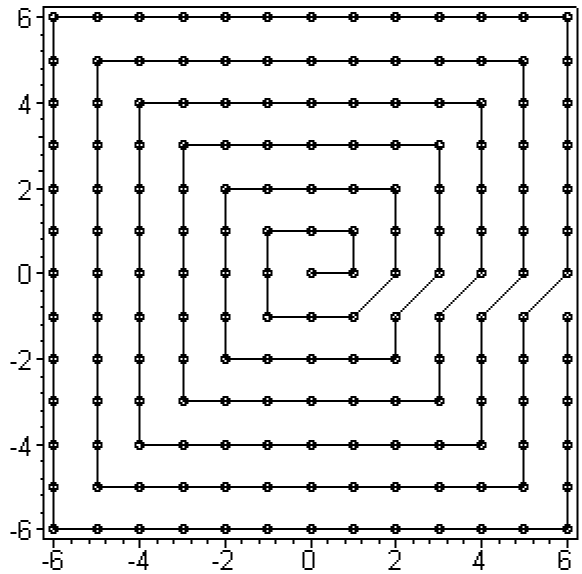
- Wanted: a continuous (preferably entire) function that interpolates $T|\mathbb{Q}_{odd}$ or $\widetilde{T}|\mathbb{Q}_{odd}[i]$
- No way!
- M. Chamberland:

$$f(x) = \frac{x}{2}\cos^2\left(\frac{\pi}{2}x\right) + \frac{3x+1}{2}\sin^2\left(\frac{\pi}{2}x\right)$$

is entire and extends $T|\mathbb{Z}$.

An analytic extension of $\widetilde{T}|\mathbb{Z}[i]$

Definition: Let $\{a_0, a_1, a_2, \dots\} = \mathbb{Z}[i]$ be the enumeration of the points of $\mathbb{Z}[i]$ as shown:



Theorem (Joseph, Monks) Let $F : \mathbb{C} \to \mathbb{C}$ by

$$f_0(z) = 0, \text{ and for } n > 0$$

$$f_n(z) = \pi_n(z)\left(\frac{z}{a_n}\right)^{m_n}\left(\widetilde{T}^n(a_n) - \sum_{k=0}^{n-1} f_k(a_n)\right),$$

$$\pi_n(z) = \prod_{k=1}^{n} \frac{(z - a_k)}{(a_n - a_k)},$$

$$p_n = \left\lfloor \frac{\sqrt{n} + 1}{2} \right\rfloor,$$

$$K_n = \left|\widetilde{T}^n(a_n) - \sum_{k=0}^{n-1} f_k(a_n)\right|,$$

$$m_n = \left\lceil \log_2\left(\left(1 + 2\sqrt{2}\right)^{n-1} p_n^{n-1}\right) K_n \right\rceil$$

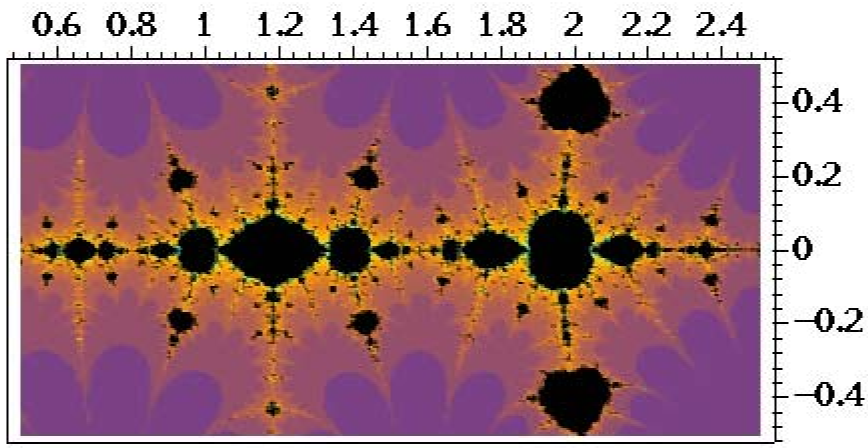
$$F(z) = \sum_{n=0}^{\infty} f_n(z).$$

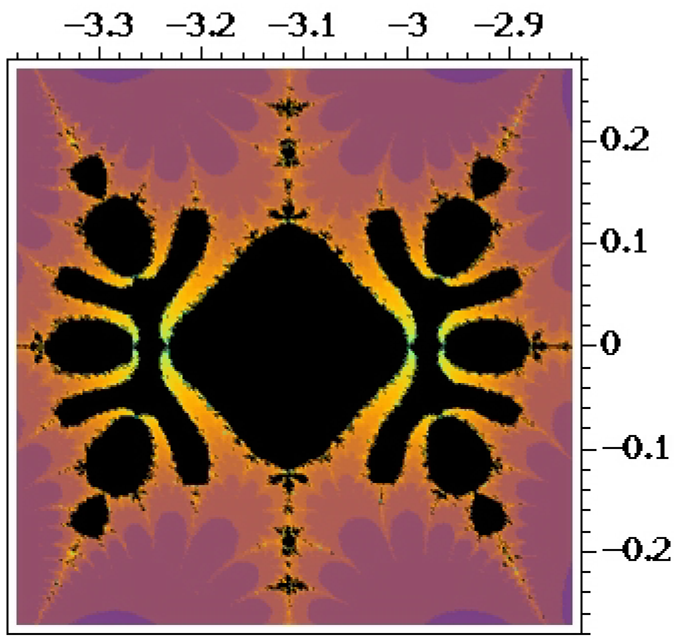
F is an entire function which extends $\widetilde{T}|\mathbb{Z}[i]$.

Remark *Not quite the kind of formula you want to use to make a fractal!*

A Collatz Julia set

Using Chamberland's map we get the following Julia set:


0.6 0.8 1 1.2 1.4 1.6 1.8 2 2.2 2.4
0.4
0.2
0
−0.2
−0.4


−3.3 −3.2 −3.1 −3 −2.9
0.2
0.1
0
−0.1
−0.2